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OF

HENRY J. S. SMITH

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THE COLLECTED

MATHEMATICAL PAPERS

OF

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CONTENTS OF VOLUME II

PAGE

XXIII. Mémoire sur Quelques Problèmes Cubiques et Biquadratiques . . 1

Annali di Matematica, Ser. II. Vol. III. pp. 112–165, 218–242. Mémoire Couronné par l'Académie Royale des Sciences de Berlin, avec une moitié du prix Steiner en Juillet 1868.

XXIV. Arithmetical Notes 67

Proceedings of the London Mathematical Society, Vol. IV. pp. 236–253. The three papers which form these Notes were read on Jan. 9 and Feb. 13, 1873.

XXV. On the Integration of Discontinuous Functions 86

Proceedings of the London Mathematical Society, Vol. VI. pp. 140–153. Read June 10, 1875.

XXVI. On the Higher Singularities of Plane Curves 101

Proceedings of the London Mathematical Society, Vol. VI. pp. 153–182. Read June 10, 1875.

XXVII. Mathematical Notes 132

Proceedings of the London Mathematical Society, Vol. VII. pp. 237, 238. Read Dec. 9, 1875. First printed in the Messenger of Mathematics, Vol. V. pp. 143, 144 (Jan. 1876).

XXVIII. Note on Continued Fractions 135

Messenger of Mathematics, Ser. II. Vol. VI. pp. 1–14 (May, 1876).

XXIX. Note on the Theory of the Pellian Equation, and of Binary Quadratic Forms of a Positive Determinant 148

Proceedings of the London Mathematical Society, Vol. VII. pp. 199–208. Read May 11, 1876.

PAGE

XXX. On the Value of a Certain Arithmetical Determinant 161

Proceedings of the London Mathematical Society, Vol. VII. pp. 208–212. Read May 11, 1876.

XXXI. On the Present State and Prospects of Some Branches of Pure Mathematics 166

Proceedings of the London Mathematical Society, Vol. VIII. pp. 6–29. Read Nov. 9, 1876.

XXXII. On the Conditions of Perpendicularity in a Parallelepipedal System . 191

Proceedings of the London Mathematical Society, Vol. VIII. pp. 83–103. Read Dec. 14, 1876.

XXXIII. On the Conditions of Perpendicularity in a Parallelepipedal System . 213

Philosophical Magazine, Ser. V. Vol. IV. pp. 18–25. Read before the Crystallological Society, June 14, 1876.

XXXIV. Sur les Intégrales Elliptiques Complètes 221

Atti della R. Accademia dei Lincei. Transunti, Ser. III. Vol. I. pp. 42–44. Read Jan. 7, 1877.

XXXV. Mémoire sur les Équations Modulaires 224

Atti della R. Accademia dei Lincei. Memorie della classe di Scienze fisiche, matematiche e naturali. Ser. III. Vol. I. pp. 136–149. Read Feb. 4, 1877.

XXXVI. On the Singularities of the Modular Equations and Curves 242

Proceedings of the London Mathematical Society, Vol. IX. pp. 242–272. Read Feb. 14 and April 11, 1878.

XXXVII. Note on a Modular Equation of the Transformation of the Third Order 274

Proceedings of the London Mathematical Society, Vol. X. pp. 87–91. Read Feb. 13, 1879.

XXXVIII. Note on the Formula for the Multiplication of Four Theta Functions . 279

Proceedings of the London Mathematical Society, Vol. X. pp. 91–100. Read Feb. 13, 1879.

XXXIX. De Fractionibus Quibusdam Continuis 287

Collectanea Mathematica (in memoriam Dominici Chelini), Milan, 1881, pp. 117–143. The paper is dated 1879.

XL. On some Discontinuous Series considered by Riemann 312

Messenger of Mathematics, Ser. II. Vol. XI. pp. 1–11 (May, 1881).

XLI. Notes on the Theory of Elliptic Transformation 321

Messenger of Mathematics, Ser. II. Vol. XII. pp. 49–99 (August–November, 1882).

PAGE

XLII. Notes on the Theory of Elliptic Transformation 368

Messenger of Mathematics, Ser. II. Vol. XIII. pp. 1–54 (May–August, 1883).

XLIII. Memoir on the Theta and Omega Functions 415

XLIV. Mémoire sur la Représentation des Nombres par des Sommes de Cinq Carrés 623

From the Mémoires présentés par divers savants à l'Académie des Sciences de l'Institut National de France, Vol. XXIX.

APPENDIX.

I. Address to the Mathematical and Physical Section of the British Association at Bradford in 1873 681

II. Arithmetical Instruments 691

III. Geometrical Instruments and Models. 698

IV. Introduction to the Mathematical Papers of William Kingdon Clifford . 711

XXIII.

MÉMOIRE SUR QUELQUES PROBLÈMES CUBIQUES ET BIQUADRATIQUES.

[Annali di Matematica, Ser. II. vol. iii. pp. 112–165, 218–242. Mémoire Couronné par l'Académie Royale des Sciences de Berlin, avec une moitié du prix Steiner en Juillet 1868.]

On admet qu'on peut résoudre par de simples intersections de lignes droites tout problème de géométrie plane, qui n'a qu'une seule solution, et qui n'est pas transcendental; ou, pour nous exprimer avec plus de précision, tout problème dont la solution unique peut s'obtenir par des intersections de courbes géométriques d'un ordre quelconque fini. On admet de même, que tout problème quadratique, qui n'est pas transcendental, peut se résoudre par des intersections de lignes droites, et de sections coniques; et, plus généralement, que tout problème qui n'a que n solutions, et qui n'est pas transcendental, peut se résoudre par des intersections de droites, et de courbes de l'ordre n. Et en effet, la vérité de ces théorèmes paraît découler des premiers principes de l'Algèbre. Mais quand il s'agit de problèmes d'un degré supérieur au second il se présente un choix de méthodes, puisqu'on peut se servir, soit des intersections de droites par des courbes dont l'ordre n est égal au nombre de solutions du problème, soit des intersections de courbes dont l'ordre est intermédiaire entre les deux nombres 1 et $n-1$. Ainsi, l'on peut faire dépendre la solution de tout problème cubique ou biquadratique, soit des intersections de courbes du troisième ou du quatrième ordre par des droites, soit des intersections mutuelles de courbes du second ordre, puisqu'on a la même évidence algébrique de la généralité absolue des deux

méthodes. Or, c'est la dernière de ces deux méthodes qui paraît la plus simple, et qui à été, à juste titre, préférée par les géomètres. Ainsi, l'on a ramené la recherche des points d'intersection d'une droite par une courbe du troisième ou du quatrième ordre à la recherche plus simple des points d'intersection de deux coniques, tandis que personne, que nous sachions, n'a suivi la marche inverse, qui à la vérité serait peu naturelle. Il nous a donc paru qu'il fallait avant tout apporter quelque perfectionnement théorique aux moyens dont on se sert pour trouver les intersections de deux coniques données. Pour cela, nous avons cru devoir imiter, autant que possible, les méthodes dont on s'est servi si heureusement pour les problèmes quadratiques. En effet, on sait que tout problème, qui n'admet que deux solutions, et qui n'est pas transcendental, peut être résolu par les intersections de lignes droites, et d'une seule circonférence de cercle, tracée d'avance dans le plan. On doit ce beau résultat, base véritable de la partie opérative de la science, aux travaux des illustres fondateurs de la géométrie moderne ; il suffira de citer à cet égard le *Traité des Propriétés Projectives de* M. Poncelet, et l'ouvrage élémentaire de Steiner, ayant pour titre : *Die geometrischen Konstructionen ausgeführt mittelst der geraden Linie und eines festen Kreises.* On suppose ordinairement que le centre du cercle tracé est donné, afin d'avoir en même temps la ligne droite à l'infini, et les deux points imaginaires conjugués où cette droite est coupée par une circonférence quelconque ; éléments dont la connaissance est indispensable pour la solution d'un grand nombre de problèmes dont les données en dépendent, soit explicitement, soit implicitement. Nous ajouterons, qu'au lieu d'une circonférence de cercle, on peut prendre pour courbe auxiliaire une section conique quelconque complètement décrite, dont on doit connaître, non seulement le centre (ou deux diamètres, si la courbe est parabolique) mais aussi le foyer.

En passant aux problèmes d'un ordre plus élevé, nous supposerons qu'une seule section conique (qui d'ailleurs ne doit pas être un cercle) soit complètement donnée et décrite, et nous démontrerons qu'en se servant de cette courbe auxiliaire on résout tous les problèmes cubiques et biquadratiques avec la règle et le compas seulement, en les ramenant, pour ainsi dire, dans les limites de la géométrie élémentaire. Nous ferons aussi remarquer que pour chaque problème il suffira de tracer une seule circonférence de cercle, et qu'en supposant connus cinq points seulement de la section conique, on peut remettre la description de cette courbe jusqu'au moment où l'on veut en déterminer les points d'intersection par le cercle qu'on doit tracer. Il est bien entendu que, lorsqu'on veut opérer de cette manière, on doit supposer que la ligne droite à l'infini, et les points cycliques

sur cette droite soient connus; c'est à dire que doivent être données (1) un parallélogramme quelconque, (2) deux angles droites, ou trois angles égaux quelconques, ces angles dans les deux cas pouvant avoir ou le même sommet, ou des sommets différents, mais étant assujettis à ne pas former un parallélogramme. Sans ces données, on ne pourrait déterminer ni le centre du cercle, qu'on aura à tracer, ni son rayon. La solution de chaque problème cubique sera purement linéaire, sauf le tracé du cercle; mais pour les problèmes biquadratiques il y aura en général une construction quadratique, qu'on ne pourra éviter, mais qu'on pourra effectuer, soit avant, soit après le tracé du cercle. Mais puisqu'au point de vue pratique on abrège beaucoup d'opérations linéaires en se servant d'une section conique, nous n'insisterons point sur la linéarité absolue des constructions, et, le plus ordinairement, nous ferons usage en chaque problème de la conique auxiliaire dès le commencement même de la solution.

La méthode que nous venons d'esquisser n'a rien de bien nouveau, puisqu'on la trouve déjà, comme résultat analytique, dans la *Géométrie* de Descartes *. Mais nous ne connaissons pas un seul problème qui ait été résolu géométriquement par cette méthode. Les auteurs qui on traité de cette matière, parmi lesquels nous signalerons De la Hire, Maclaurin, et Joachimsthal †, nous paraissent ne pas avoir cherché à démontrer par des considérations de géométrie pure la proposition si remarquable de Descartes, ni à s'en servir pour la solution géométrique des problèmes. Ainsi Maclaurin ne s'occupe-t-il guère que de la construction géométrique des racines des équations algébriques; De la Hire, qui s'est proposé de trouver les normales à une section conique qui passent par un point donné, n'a rattaché sa solution de ce problème à aucune théorie géométrique; enfin, Joachimsthal, qui a donné une autre solution, d'une élégance admirable, de ce même problème biquadratique, s'est encore servi de formules analytiques, dont, comme il l'avoue lui-même, sa construction géométrique n'est pas entièrement affranchie. Il restait donc à trouver les liens qui rattachent la proposition de Descartes aux théories modernes de la science, et à montrer le parti qu'on peut en tirer pour les constructions actuelles de la géométrie pure. Nous nous empressons d'ajouter que dans ces développements nous n'aurons qu'à nous occuper de considérations très élémentaires. En effet, nous admettrons que la solution de tout problème cubique ou biquadratique peut se réduire à la détermination des points d'intersection de deux coniques, qui ne sont pas décrites, mais dont chacune est déterminée par un nombre suffisant d'éléments; et nous

* Note I (p. 50). † Note II (p. 50).

ramènerons ce dernier problème à celui de trouver les points d'intersection d'une circonférence de cercle par la section conique décrite d'avance, au moyen de simples transformations homographiques ou corrélatives, dont on pourra se servir en plusieurs manières différentes.

Les problèmes que nous aurons à résoudre, et surtout le problème cubique dont la solution est spécialement demandée par l'Académie, impliquent nécessairement des considérations relatives aux éléments imaginaires. Nous jugeons donc à propos, avant de venir au sujet qui doit nous occuper principalement, d'entrer dans quelques détails sur cette matière. Nous sommes loin de penser que nous pourrions ajouter à cet égard quelque chose qui fût inconnue aux géomètres ; et nous n'avons aucune prétention de pouvoir éclaircir la vraie nature de ce phénomène singulier qu'on nomme l'imaginaire en géométrie. Mais nous avons remarqué que les questions théoriques qui concernent les imaginaires ont été traitées avec beaucoup plus de détail que les questions pratiques. On a bien donné les moyens nécessaires pour déterminer les éléments réels, qui dépendent pour ainsi dire immédiatement d'éléments imaginaires donnés. On sait, par exemple, trouver les axes de symptose réels de deux couples de droites imaginaires conjuguées, etc. ; mais il s'en faut beaucoup que de telles déterminations suffisent aux besoins actuels de la science. Il paraît résulter de considérations algébriques très élémentaires que tout problème qui peut se résoudre lorsq'on en suppose les données réelles, devra également rester résoluble lorsqu'on substitue deux éléments imaginaires conjugués à deux éléments quelconques réels qui entrent symétriquement dans la question. Mais pour opérer généralement cette extension de la solution d'un problème aux données réelles à la solution d'un problème aux données imaginaires, il faut souvent, du moins dans l'état actuel de la science, isoler l'un de l'autre les deux éléments d'un même couple d'éléments imaginaires conjugués, afin de faire sur chacun d'eux une suite d'opérations plus ou moins longue, ayant pour résultat des couples d'éléments imaginaires conjugués, dont la combinaison donne enfin les éléments réels qu'on cherche. Cette isolation des éléments imaginaires n'a rien d'absolu ; elle est relative à un système de deux éléments imaginaires conjugués arbitrairement choisis. En effet, étant donnée une seule couple d'éléments imaginaires conjugués, il paraît tout aussi impossible de distinguer entre ces éléments que de distinguer entre les racines de l'équation

$$x^2 + 1 = 0.$$

Mais, de même qu'en distinguant hypothétiquement entre les deux racines de cette équation (ce qu'on fait, par exemple, quand on les désigne par $+i$ et $-i$),

on parvient à distinguer en même temps entre les deux racines imaginaires de toute autre équation quadratique ; de même, en créant une fois pour toutes une distinction fictive entre les deux éléments d'une couple quelconque d'éléments imaginaires conjugués on arrive à établir en même temps une distinction pareille entre les deux éléments de toute autre couple ; puis, en admettant cette isolation relative, on peut opérer sur les éléments imaginaires tout aussi bien que sur les éléments réels ; quoiqu'il faut avouer que, dans le cas des premiers, les opérations auxquelles on se trouve conduit sont presque toujours d'une longueur rebutante. L'illustre et regrettable auteur de la *Géométrie de Situation* a savamment effectué cette isolation hypothétique des éléments imaginaires, en rattachant chaque élément d'une même couple à l'un des deux sens opposés qu'on peut observer en toute formation géométrique qui contient des éléments imaginaires. Mais cette manière de considérer la chose, quoique très utile pour les développements théoriques, nous semble se prêter aux constructions avec moins de simplicité que celle que nous avons dû préférer pour notre but actuel.

D'après ce qui vient d'être dit, nous diviserons ce mémoire en trois parties. Dans la première nous traiterons des constructions dont les données sont imaginaires ; dans la seconde nous démontrerons la théorème de Descartes ; dans la troisième nous étudierons divers problèmes cubiques ou biquadratiques, et notamment celui qui a été signalé par l'Académie.

Première Partie.

1. Dans cette partie de notre travail nous ferons usage du mot grec *dyade* pour exprimer l'ensemble de deux éléments conjugués imaginaires. Par chaque dyade de points imaginaires il passe une seule droite réelle ; elle sera pour nous l'*axe* de la dyade. Dans l'espace, il existe, comme on sait, des dyades de droites imaginaires qui ne se coupent pas, et qui ne passent par aucun point réel. Mais les deux rayons d'une dyade de droites imaginaires, qui se trouvent dans un plan réel, se coupent en un point qui est toujours réel, et que nous nommerons le *centre* de la dyade de lignes droites. Pour plus de simplicité, nous considérerons chaque dyade comme formée par les éléments doubles d'une involution réelle, dont les segments, ou les angles, empiètent les uns sur les autres. Cette définition d'une dyade aura toute la généralité nécessaire, puisqu'on sait que toute détermination quadratique se réduit, en dernière analyse, à la recherche des éléments doubles de deux systèmes homographiques, éléments qui sont en même temps les éléments doubles d'une involution, qu'on déduit linéairement des deux systèmes homographiques, en prenant dans chacun des deux systèmes,

l'élément conjugué d'un même élément P, dont le conjugué harmonique par rapport à ces deux éléments sera aussi le conjugué dans l'involution cherchée. Nous dirons qu'une dyade est *donnée*, quand on a deux couples d'éléments réels de l'involution dont la dyade répresente les éléments doubles.

Maintenant, si l'on prend une première dyade $a_1 a_2$ sur l'axe A, et une seconde dyade $b_1 b_2$ sur l'axe B, différent de A, qu'il coupe au point (A_1, B_1), les deux couples de droites imaginaires $a_1 b_1$, $a_2 b_2$, et $a_1 b_2$, $a_2 b_1$, seront évidemment des dyades de droites. Soient P et Q les centres de ces dyades; nous les appellerons les *centres d'homologie* des dyades $a_1 a_2$, $b_1 b_2$. On trouve linéairement la droite PQ, en joignant les deux points A_2, B_2 qui sont conjugués de (A_1, B_1) dans les deux involutions qui déterminent les dyades $a_1 a_2$, $b_1 b_2$. Mais la construction des points P, Q eux-mêmes est essentiellement quadratique. Soient $a_1 a_2$, $\beta_1 \beta_2$ des couples de points appartenant aux deux involutions respectivement; les points P, Q seront les points doubles d'une involution, dont $A_2 B_2$ est une première couple, et dont on trouve deux autres couples en prenant les intersections de PQ par les deux couples de droites $a_1 \beta_1$, $a_2 \beta_2$ et $a_1 \beta_2$, $a_2 \beta_1$. Cependant, on voit facilement que les segments de cette involution ne peuvent pas empiéter les uns sur les autres; donc les points PQ seront toujours réels, comme on peut voir *à priori*. L'un des points PQ étant donné, l'autre s'en déduit linéairement, puisque PQ, $A_2 B_2$ sont quatre points harmoniques. Pour avoir une image nette des deux dyades et de leurs deux centres d'homologie, on peut remarquer que si $a_1 a_2$ est la dyade cyclique à l'infini, $b_1 b_2$ est la dyade commune à un système de cercles, ayant B pour axe radical, et PQ pour points limites. Nous ne nous arrêterons pas à la construction correlative des *axes d'homologie*, ou de *symptôse*, de deux dyades de droites imaginaires, qui n'ont pas le même centre.

En considérant deux dyades de points imaginaires, qui n'ont pas le même axe, comme homologiques par rapport à l'un de leurs centres d'homologie, on établit une certaine correspondance entre les points des deux dyades, en telle sorte que si l'on échange entr'eux les points imaginaires de l'une, il faut en même temps échanger entr'eux les points imaginaires de l'autre. Puisqu'il y a deux centres d'homologie, cette correspondance peut s'établir de deux manières différentes, qui se rapportent à ces deux centres respectivement. Ainsi nous dirons que l'homologie des deux dyades est donnée par le centre P, ou par le centre Q, selon que l'on a choisi le premier ou le second de ces deux centres pour établir la correspondance. Pour les dyades de droites imaginaires dont les centres sont différents, on a une définition correlative.

Si l'on a deux dyades de même espèce, ayant le même centre, ou le même

axe, on doit établir leur homologie d'une manière indirecte, en prenant une troisième dyade, dont le centre ou l'axe soit différent, mais dont l'homologie avec chacune des deux premières soit donnée.

L'homologie d'une dyade de points, et d'une dyade de droites, est donnée immédiatement, lorsque les droites de la seconde dyade passent par les points correspondants de la première. En tout autre cas on établit indirectement l'homologie des deux dyades; soit en déterminant le centre d'homologie de la première dyade, et de la dyade de points qui résulte de l'intersection de la seconde dyade par un axe quelconque réel, soit en déterminant l'axe d'homologie de la seconde dyade et d'une dyade quelconque de droites imaginaires passant par la première dyade *.

2. Quand on ne considère que deux dyades, indépendantes l'une de l'autre, on peut choisir arbitrairement entre les deux manières de déterminer leur homologie. Mais en considérant un plus grand nombre de dyades, on voit qu'on peut disposer comme on veut de l'homologie de la seconde avec la première, de la troisième avec la première, et ainsi de suite, mais qu'alors l'homologie de deux dyades quelconques de cette série sera complètement déterminée. Nous aurons donc à résoudre le problème très général, mais très élémentaire, que voici.

Etant donnée l'homologie des dyades $a_1 a_2$, $b_1 b_2$, *et aussi l'homologie des dyades* $a_1 a_2$, $c_1 c_2$, *trouver l'homologie des dyades* $b_1 b_2$, $c_1 c_2$.

Ce problème est toujours résoluble linéairement. Il y a plusieurs cas à considérer, mais pour abréger nous n'en considérons que la moitié, d'où l'on pourra déduire la solution des autres en s'appuyant sur le principe de dualité.

I. Soient A, B, C les axes des trois dyades de points $a_1 a_2$, $b_1 b_2$, $c_1 c_2$; nous supposerons en premier lieu que ces droites soient toutes différentes entr'elles. Soient Q et R les centres d'homologie de $a_1 a_2$, $c_1 c_2$ et de $a_1 a_2$, $b_1 b_2$ respectivement. Il s'agit de trouver P le centre d'homologie de $b_1 b_2$, $c_1 c_2$. On se rappelle que si P' est le centre d'homologie opposé à P, la droite PP' peut se construire linéairement †.

Première solution.—Du point Q projetons les points de l'axe A sur la droite C, et du point R projetons les mêmes points sur la droite B. Soient $a_1 a_2$ deux points conjugués de l'involution déterminant la dyade $a_1 a_2$; $\beta_1 \beta_2$ et $\gamma_1 \gamma_2$ les projections de ces points sur B et C respectivement. Les droites $\beta_1 \gamma_1$, $\beta_2 \gamma_2$, etc., envelopperont une section conique, tangente aux deux droites B et C; de plus, les couples de tangentes, telles que $\beta_1 \gamma_1$, $\beta_2 \gamma_2$, formeront un système de tangentes

* Note III (p. 51). † Note IV (p. 54).

en involution de cette conique ; donc les points d'intersection des deux tangentes de chaque couple seront en ligne droite. Le pôle de cette droite, par rapport à la conique, est le point P cherché. Pour le déterminer, il suffira de trouver le point de contact sur chacune des deux tangentes $\beta_1\gamma_1$, $\beta_2\gamma_2$: la corde joignant ces points ira couper PP' au point P.

Si les trois axes A, B, C se coupent au même point O, les deux droites B, C, qui sont déjà homographiques par rapport aux points $\beta_1\beta_2\ldots$, $\gamma_1\gamma_2\ldots$, deviendront homologiques par rapport à ces mêmes points, puisque le point O se correspondra à lui-même sur chacune des deux droites. Dans ce cas le centre d'homologie des deux droites est précisément le point cherché P, qui se trouvera à l'intersection des droites PP', QR. Il y aurait aussi une simplification, si les trois droites QR, B, C concouraient en un même point.

Seconde solution.—On déterminera, comme nous avons fait dans la première solution, deux divisions homographiques sur les droites B et C. L'axe * de ces deux divisions coupera PP' an point P'. Ce point trouvé, la position du point P s'en déduira linéairement.

Cette solution se simplifie (1) si les divisions homographiques sur les deux droites B et C deviennent homologiques, (2) si l'axe de ces deux divisions se confond avec PP'. En ce dernier cas, on échange entr'eux, sur l'un des deux droites B et C, les deux points conjugués de chaque couple de l'involution qui détermine la dyade que l'on considère sur cette droite. Alors les deux droites deviennent homologiques, et leur centre d'homologie est le point P qu'on cherche.

Si l'axe A se confond avec l'un des axes B et C, on pourrait encore trouver l'homologie des dyades b_1b_2, c_1c_2, en faisant de légères modifications dans les solutions précédentes sans rien changer à leur principe. Mais on peut aussi opérer de la manière suivante. Soit z_1z_2 la dyade auxiliare qui établit l'homologie de a_1a_2, b_1b_2, dont on suppose les axes coincidents. L'homologie des dyades a_1a_2, z_1z_2 sera donnée, et aussi celle des dyades a_1a_2, c_1c_2 ; on en déduira l'homologie des dyades z_1z_2, c_1c_2. Mais l'homologie des dyades z_1z_2, b_1b_2 est connue ; donc on pourra trouver l'homologie des dyades b_1b_2, c_1c_2. Cette solution suppose seulement, que l'axe de la dyade z_1z_2 ne se confonde pas avec C.

Si les trois axes A, B, C coïncident, soit y_1y_2 la dyade auxiliaire qui établit l'homologie de a_1a_2, c_1c_2 ; on trouvera par le cas précédent l'homologie

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* L'axe des deux divisions homographiques $a_1a_2\ldots$, $\beta_1\beta_2\ldots$ est la droite lieu des points d'intersection des couples de droites telles que $a_1\beta_2$, $a_2\beta_1$.

de $b_1 b_2$, $y_1 y_2$. Mais alors $y_1 y_2$ sera une dyade auxiliaire qui établira l'homologie de $b_1 b_2$, $c_1 c_2$.

II. Ce qui précède suffit pour tous les cas où l'on ne considère que des dyades de la même espèce : nous allons maintenant supposer que les trois dyades $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ soient d'espèces différentes. Nous ne considérerons que les deux cas suivants, dans lesquels nous supposerons toujours que l'homologie de $a_1 a_2$, $b_1 b_2$, et de $a_1 a_2$, $c_1 c_2$, soit donnée, et que l'on cherche l'homologie de $b_1 b_2$, $c_1 c_2$.

(i.) Soient $a_1 a_2$, $b_1 b_2$ des dyades de points, $c_1 c_2$ une dyade de droites : soit aussi $y_1 y_2$ la dyade de points auxiliaires qui sert à établir l'homologie de $a_1 a_2$, $c_1 c_2$. Pour déterminer l'homologie de $b_1 b_2$, $c_1 c_2$, on n'aura qu'à trouver l'homologie de $y_1 y_2$, $b_1 b_2$, ce qu'on pourra faire, puisqu'on connaîtra l'homologie de $a_1 a_2$, $b_1 b_2$, et aussi celle de $a_1 a_2$, $y_1 y_2$.

(ii.) Soit $a_1 a_2$ une dyade de droites, $b_1 b_2$, $c_1 c_2$ des dyades de points. Désignons par $y_1 y_2$, $z_1 z_2$ les dyades de points qui établissent l'homologie de $a_1 a_2$ avec $c_1 c_2$ et $b_1 b_2$ respectivement. Dans la série de dyades $b_1 b_2$, $z_1 z_2$, $y_1 y_2$, $c_1 c_2$, on connaîtra l'homologie de chacune avec celle qui la précède, donc on pourra trouver l'homologie de la première avec la dernière.

On peut donner un énoncé plus général du problème de cet article.

'*Soient* $a_1 a_2$, $b_1 b_2$, ..., $x_1 x_2$ *des dyades en nombre quelconque, de même espèce ou d'espèces différentes ; étant donnée l'homologie de chacune d'elles avec celle qui la suit immédiatement, on peut trouver l'homologie de la première avec la dernière.*'

3. Maintenant, prenons arbitrairement une dyade fixe, qui doit nous servir comme terme de comparaison pour toutes les autres dyades que nous aurons à considérer. Nous dirons qu'un élément imaginaire est donné, quand la dyade à laquelle cet élément appartient est donnée, l'homologie de cette dyade avec la dyade fixe étant aussi donnée. On voit que pour connaître un élément imaginaire, il faut connaître (1) l'axe, ou le centre, de la dyade à laquelle l'élément appartient, (2) deux couples d'éléments réels de l'involution qui détermine cette dyade, (3) l'homologie de cette même dyade avec la dyade fixe. Il faut convenir que de cette manière on n'isole pas l'un des deux éléments imaginaires d'une même dyade, et que, par conséquent, l'expression 'élément imaginaire donné,' n'est pas rigoureuse. Cependant on voit que si l'on échange un élément imaginaire donné avec son conjugué, il faut en même temps échanger tout autre élément donné avec son conjugué. Et c'est là tout ce qu'il faut pour qu'on puisse opérer avec des données imaginaires, de la même manière qu'avec des données réelles. En effet, nous pourrons maintenant résoudre les deux problèmes correlatifs que voici.

Trouver le point d'intersection de deux droites données, dont l'une au moins est supposée imaginaire.

Trouver la droite qui passe par deux points donnés, dont l'un au moins est supposé imaginaire.

Il suffira de considérer le second de ces deux problèmes. Soient donc a_1, b_1, deux points imaginaires donnés, dont les axes ne se confondent pas. Puisque les points a_1, b_1 sont donnés, les dyades $a_1 a_2$, $b_1 b_2$, auxquelles ils appartiennent, sont aussi données, de même que l'homologie de chacune d'elles avec la dyade fixe. On peut donc trouver linéairement l'homologie de ces deux dyades. Soit P leur centre d'homologie ; P sera le centre de la dyade à laquelle appartient la droite cherchée (a_1, b_1). Les involutions qui correspondent aux dyades $a_1 a_2$, $b_1 b_2$ déterminent la même involution au point P ; cette involution déterminera la dyade $Pa_1 b_1$, $Pa_2 b_2$. Enfin l'homologie de cette dyade de droites avec chacune des dyades $a_1 b_1$, $a_2 b_2$ est donnée immédiatement ; on peut donc trouver son homologie avec la dyade fixe ; ce qui achève la détermination de la droite (a_1, b_1).

4. Il résulte de la solution de ces deux problèmes, que lorsqu'on peut résoudre linéairement un problème quelconque, dont les données ne contiennent que des points et des droites réelles, on pourra encore résoudre linéairement ce même problème, après qu'on aura substitué, en tout ou en partie, des droites et des points imaginaires donnés, aux données réelles du problème. On n'aura qu'à suivre de pas en pas la solution du problème aux données réelles pour en conclure la solution du problème correspondant aux données imaginaires. Seulement, puisque les déterminations des droites joignant des points donnés, et des points d'intersection des droites données, qui sont des *Postulata* pour les cas réels, exigent des constructions détournées pour les cas imaginaires, on conçoit que la méthode, quoique parfaitement générale, doit conduire à des opérations assez longues.

Supposons qu'on ait résolu linéairement un problème dont les données soient toutes réelles ; si, parmi ces données, il y en a deux qui entrent symétriquement dans la question, on pourra substituer une dyade donnée à ces éléments symétriques, sans que le problème cesse d'être résoluble linéairement. En effet, après cette substitution il n'y aura qu'une seule dyade indépendante ; en la prenant pour la dyade fixe, on trouvera linéairement l'homologie de toute dyade dérivée qu'on aura à considérer, et l'on n'aura qu'à opérer sur des éléments imaginaires donnés. Mais il en serait autrement, si l'on voulait substituer à la fois plusieurs dyades à la place d'un nombre égal de couples de données réelles. Toutes ces dyades seraient indépendantes, et pour qu'on pût appliquer la méthode précé-

dente, il faudrait qu'on en connût les centres, ou les axes d'homologie. Dans les applications, on fera bien de déterminer actuellement ces centres ou ces axes par la construction que nous avons rappelée ci-dessus. Mais, puisque cette construction est quadratique, il importe de faire voir qu'on pourrait s'en passer à la rigueur. Pour cela, il faut seulement qu'au lieu d'introduire toutes les dyades à la fois, on les introduise l'une après l'autre. Toutes les fois que l'on substituera une seule dyade à deux données symétriques réelles, on aura un nouveau problème, dont on pourra trouver la solution par ce qui précède. Mais on peut considérer ce nouveau problème comme n'impliquant que des données réelles, puisque la dyade que nous y avons fait entrer, peut être remplacée par deux couples de l'involution qui la détermine. On pourra donc substituer une seconde dyade à une seconde couple de données réelles, et ainsi de suite, sans qu'on ait besoin d'aucune construction quadratique.

5. Nous allons maintenant indiquer quelques problèmes dont on puisse trouver les solutions par les principes précédents. Tout élément, qui n'est pas dit expressément être réel, pourra être imaginaire.

(i.) *Étant donnés trois points d'une même droite, ou trois droites passant par un même point, trouver le conjugué harmonique d'un de ces éléments, par rapport aux deux autres.*

Soient a_1, b_1, c_1 des points donnés d'une même droite ω_1, et proposons nous de trouver le point d_1, conjugué harmonique de c_1, par rapport à $a_1 b_1$. Soient A, B, C les axes dyades $a_1 a_2$, $b_1 b_2$, $c_1 c_2$; α, β, γ, les trois sommets du triangle ABC. La droite A sera l'axe d'homologie des dyades $(\alpha a_1, \alpha a_2)$ et $\omega_1 \omega_2$; pareillement, B sera l'axe d'homologie des dyades $(\beta b_1, \beta b_2)$ et $\omega_1 \omega_2$; on pourra donc déterminer l'axe d'homologie X des dyades $(\alpha a_1, \alpha a_2)$, $(\beta b_1, \beta b_2)$; soit $x_1 x_2$ la dyade de points determinée sur X par ces deux dyades, et D l'axe d'homologie des dyades $(\gamma x_1, \gamma x_2)$ et $\omega_1 \omega_2$; D sera l'axe de la dyade cherchée $d_1 d_2$, dont l'homologie avec $\omega_1 \omega_2$ sera évidemment connue.

(ii.) *Étant donnés, dans deux séries homographiques, trois éléments de l'une, et les trois éléments correspondants de l'autre, trouver l'élément de l'une des deux séries qui correspond à un élément quelconque de l'autre.*

On résoudra linéairement ce problème général, en imitant, de la manière que nous avons expliquée, les solutions qu'on a données du problème dans le cas particulier où les données sont réelles. Le principe de ces solutions (auxquelles on peut donner des formes très variées) consiste, comme on sait, à trouver une troisième série qui soit homologique avec chacune des deux séries données.

On aura en même temps la solution du problème important,

Étant donnés cinq éléments de même espèce qui déterminent une conique, trouver autant d'éléments de cette courbe qu'on voudra,
qui est au fond le même que le précédent.

Il y a deux cas très particuliers qui méritent quelque attention.

(1.) *Étant données deux dyades de droites $a_1 a_2$, $b_1 b_2$ ayant le même centre P, et deux autres dyades $\alpha_1 \alpha_2$, $\beta_1 \beta_2$, ayant le même centre Π, et vérifiant l'équation* *

$$\Pi . [\alpha_1 \alpha_2, \beta_1 \beta_2] = P . [a_1 a_2, b_1 b_2]$$

trouver la droite d'un des deux faisceaux qui correspond à une droite donnée de l'autre faisceau.

Pour que ce problème n'admette qu'une seule solution il faut qu'on connaisse l'axe d'homologie A des dyades $a_1 a_2$, $\alpha_1 \alpha_2$, et l'axe d'homologie B des dyades $b_1 b_2$, $\beta_1 \beta_2$. Il est vrai que l'un des deux axes étant donné, on pourrait trouver l'autre linéairement, mais, pour abréger, nous supprimerons la démonstration de cette assertion, et nous supposerons que les axes A et B soient tous les deux donnés †. Soient $p_1 p_2$, $q_1 q_2$ les dyades qu'on aura sur les droites A et B. En faisant passer une conique par le points $p_1 p_2$, $q_1 q_2$, $P\Pi$ on ramènera le problème à celui de déterminer autant de points qu'on voudra d'une conique dont on connaît deux dyades et deux points réels. Mais, comme on sait, le theorème de Carnot conduit à une solution linéaire de ce dernier problème, même en supposant qu'un seul point réel de la conique soit donné.

(2.) *Par un point P d'une conique dont on connaît cinq points réels il passe une droite imaginaire donnée ; trouver le second point d'intersection de cette droite par la conique.*

Soit (1°) P un point réel. On sait trouver l'axe de symptôse réel de la conique et de la dyade à laquelle appartient la droite imaginaire : le point d'intersection de cet axe par la droite imaginaire donnée sera le point cherché.

Soit (2°) P un point imaginaire. L'axe de la dyade à laquelle ce point appartient sera l'un des axes de symptôse de la conique et de la dyade à laquelle appartient la droite imaginaire donnée. L'autre axe se trouvera linéairement et fera connaître la solution du problème.

(iii.) *Étant donné le tracé complet d'une conique réelle, trouver les points d'intersection d'une droite ω_1 par une conique C, réelle ou imaginaire, dont on donnait cinq points.*

Soit C réelle. On trouvera par une construction quadratique connue, pour

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* Voir la note Art. 3, troisième partie (p. 36).

† Note V (p. 54).

laquelle on se servira de la conique tracée, les deux axes de symptôse de C et de la dyade de droites $\omega_1 \omega_2$; les points d'intersection de ces deux axes par ω_1 seront les points cherchés.

Soit C imaginaire. On ramènera la question à la recherche des points doubles de deux divisions homographiques sur une même droite. On projetera ces deux divisions sur la conique tracée ; et l'on cherchera l'axe des deux divisions projetées. Cet axe sera en général imaginaire ; on déterminera les points où il est coupé par la conique tracée ; ces points feront connaître la solution du problème.

(iv.) *Étant donnés cinq points imaginaires, trouver des coniques réelles appartenant au faisceau qui contient les coniques imaginaires déterminées par ces cinq points, et par leurs cinq points conjugués.*

Soit P un point réel, et proposons-nous de déterminer la conique du faisceau qui passe par le point P.

Il importe d'observer que si l'homologie des cinq dyades était inconnue (hypothèse que nous excluons, puisque nous supposons que les cinq points sont donnés) il y aurait seize solutions correspondant à chaque point P, puisque les cinq dyades déterminent trente deux coniques dont chacune passe par l'un de deux points de chaque dyade. Cette remarque montre clairement l'importance qu'on doit attacher à l'homologie des dyades.

Pour trouver la conique passant par P, on prendra un point quelconque réel q, et l'on en déterminera la polaire relativement à l'une des coniques imaginaires. Cette détermination sera toujours possible puisqu'elle n'exige que des opérations linéaires que nous avons déjà enseigné à faire. Soit Q le centre de la dyade à laquelle appartient la polaire de q ; toute conique du faisceau divisera harmoniquement le segment réel qQ. En prenant quatre points q_1, q_2, q_3, q_4, et leurs quatre points *réciproques* Q_1, Q_2, Q_3, Q_4, on aura la solution du problème, puisqu'on sait construire la conique qui passe par un point réel donné, et coupe harmoniquement quatre segments réels.

On résoudrait de la même manière le problème suivant.

Étant donnés deux systèmes polaires réels, dont les coniques peuvent être imaginaires, trouver la conique qui passe par un point réel donné, et qui appartient au faisceau déterminé par les deux coniques.

Nous ferons remarquer à cette occasion qu'il y a deux espèces différentes de coniques imaginaires qu'il importe de distinguer entr'elles. Une conique imaginaire de la première espèce n'a aucun point réel ; elle est coupée par toute transversale réelle en deux points imaginaires conjugués ; elle a un système polaire

réel. Une conique imaginaire de la seconde espèce a une base formée, soit de quatre points réels, soit de deux points réels et d'une dyade, soit de deux dyades. Elle coupe toute transversale réelle (autre que celles qui joignent deux des points de la base) en deux points qui ne sont ni tous les deux réels, ni des points imaginaires conjugués. La polaire de tout pôle réel qui n'est pas un sommet du triangle harmonique de la base, est imaginaire.

(v.) *Étant données deux dyades* $a_1 a_2$, $\beta_1 \beta_2$, *ayant le même axe, et deux autres dyades* $a_1 a_2$, $b_1 b_2$, *ayant des axes A et B différents l'un de l'autre; trouver la conique* Σ *qui passe par les deux dyades* $a_1 a_2$, $b_1 b_2$, *et qui satisfait à l'équation*

$$[a_1, a_2, b_1, b_2] = [a_1, a_2, \beta_1, \beta_2].$$

Pour que ce problème soit linéaire il faut que l'homologie des dyades $a_1 a_2$, $a_1 a_2$, et aussi celle des dyades $\beta_1 \beta_2$, $b_1 b_2$, soient données. On prendra le centre d'homologie de $a_1 a_2$, $a_1 a_2$; de ce centre on projetera $\beta_1 \beta_2$ sur A; et l'on déterminera le centre d'homologie de $b_1 b_2$, et de la projection de $\beta_1 \beta_2$ sur A. Ce dernier centre sera un point de la conique cherchée Σ, qui sera dès lors complètement déterminée.

Si l'on supposait que les dyades $a_1 a_2$, $\beta_1 \beta_2$, n'ont pas le même axe, mais qu'elles appartiennent à une conique réelle Γ, l'expression $[a_1 a_2, \beta_1 \beta_2]$ étant relative à cette courbe, on pourrait encore déterminer la conique Σ par la construction précédente. On n'aurait qu'à projeter les dyades $a_1 a_2$, $\beta_1 \beta_2$ sur une droite L, en prenant pour centre de projection un point quelconque de Γ.

On voit que les deux coniques Γ et Σ seront homographiques par rapport aux points $a_1 a_2$, $\beta_1 \beta_2$, et $a_1 a_2$, $b_1 b_2$. Soit ξ un point donné réel de Γ; on trouvera de la manière suivante le point correspondant x de la conique Σ. Soient $a_1' a_2'$, $\beta_1' \beta_2'$, ξ' les projections de $a_1 a_2$, $\beta_1 \beta_2$, ξ sur la droite L: du centre d'homologie des dyades $a_1' a_2'$, $a_1 a_2$ on projetera $\beta_1' \beta_2'$, ξ' sur A: soient β_1'', β_2'', ξ'' les projections de ces points. On trouvera le centre d'homologie ρ des dyades $\beta_1'' \beta_2''$, $b_1 b_2$; ce centre appartiendra à la conique Σ, et le point x cherché sera le second point d'intersection de la droite $\rho \xi''$ par Σ.

Si, au lieu d'un point réel ξ, une dyade $\xi_1 \xi_2$, appartenant à Γ, était donnée, on trouverait $x_1 x_2$, la dyade correspondante de Σ, par une construction toute semblable. On aurait sur la droite A une dyade $\xi_1'' \xi_2''$, dont l'homologie avec $\xi_1 \xi_2$ serait connue; puis on déterminerait l'axe de symptôse de Σ et de la dyade des droites $\rho \xi_1''$ $\rho \xi_2''$; ce qui suffirait pour faire connaître la dyade $x_1 x_2$, et l'homologie de cette dyade avec la dyade donnée $\xi_1 \xi_2$.

On peut énoncer le problème que nous venons de résoudre de la manière suivante.

Étant données deux dyades d'une première figure correspondant à deux dyades d'une seconde figure, transformer la première figure homographiquement en la seconde.

(vi.) *Étant données deux dyades* $a_1 a_2$, $b_1 b_2$, *ayant des axes différents* A *et* B, *et quatre points* 1, 2, 3, 4 *formant la base d'un faisceau de coniques réelles, trouver la conique qui passe par les deux dyades* $a_1 a_2$, $b_1 b_2$ *et qui satisfait à l'équation*

$$(1, 2, 3, 4)\,[a_1, a_2, b_1, b_2] = [a_1, a_2, b_1, b_2].$$

Soit Γ une conique quelconque passant par les deux dyades $a_1 a_2$, $b_1 b_2$. Qu'on prenne un point réel γ de cette conique et que de ce point on projette sur la conique les involutions qui sont déterminées par le faisceau (1, 2, 3, 4) sur les droites A, B. Soient p et q les pôles des deux involutions qu'on aura maintenant sur la conique ; les droites des deux faisceaux (p) et (q) correspondront anharmoniquement aux coniques du faisceau (1, 2, 3, 4) : donc ces deux faisceaux de droites seront homographiques ; soit C la conique, lieu des points d'intersection des rayons correspondants. Qu'on détermine les axes de symptôse des dyades (pa_1, pa_2), (qb_1, qb_2) avec la conique C ; soient $\alpha_1 \alpha_2$, $\beta_1 \beta_2$ les points d'intersection de ces axes par C ; les points p et q seront les centres d'homologie des dyades $a_1 a_2$, $\alpha_1 \alpha_2$ et $b_1 b_2$, $\beta_1 \beta_2$ respectivement. Dans le faisceau (p) les droites imaginaires pa_1, pa_2 correspondront aux coniques $(1, 2, 3, 4, a_1)$, $(1, 2, 3, 4, a_2)$; de même dans le faisceau (q) les droites imaginaires qb_1, qb_2 correspondront aux coniques $(1, 2, 3, 4, b_1)$, $(1, 2, 3, 4, b_2)$. Donc le rapport anharmonique des quatre points α_1, α_2, β_1, β_2 appartenant à la conique C, sera égal au rapport anharmonique

$$(1, 2, 3, 4)\,.\,[a_1, a_2, b_1, b_2].$$

On déterminera la conique qui passe par les deux dyades a_1, a_2, b_1, b_2, et qui satisfait à l'équation

$$[a_1, a_2, b_1, b_2] = [\alpha_1, \alpha_2, \beta_1, \beta_2] ;$$

cette conique sera celle qu'on cherche.

Cette solution est purement linéaire, puisqu'on ne trace pas les coniques dont il y est question. On pourra l'abréger en opérant de la manière suivante. Soit ω le point d'intersection des droites A et B ; ξ, η les points où ces mêmes droites coupent pour la seconde fois la conique $(1, 2, 3, 4, \omega)$. Soit γ un point quelconque de la droite $\xi\eta$; en prenant pour Γ la conique $(a_1, a_2, b_1, b_2, \gamma)$ les faisceaux (p) et (q) deviendront homologiques, et les dyades $\alpha_1 \alpha_2$, $\beta_1 \beta_2$ seront connues, dès que l'on aura déterminé l'axe d'homologie de ces deux faisceaux.

Si l'on se permettait une construction quadratique, on pourrait encore raccourcir la solution. On trouverait un centre d'homologie des deux involutions déterminées par le faisceau (1, 2, 3, 4) sur les axes A et B ; en prenant ce centre pour le point γ, les deux points p et q viendraient se confondre en un seul point, qui appartiendrait à la conique cherchée. À la vérité, cette méthode ne serait pas applicable, si les centres d'homologie des deux involutions étaient imaginaires.

Nous ajouterons quelques corollaires de ce problème, qui nous seront utiles plus tard.

(1.) *Étant données quatre dyades* $a_1 a_2$, $b_1 b_2$, $c_1 c_2$, $d_1 d_2$, *et un point réel* p, *trouver le point* ω *opposé à deux de ces dyades* $c_1 c_2$, $d_1 d_2$, *par rapport à la courbe cubique* $(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, p)$.

On déterminera la conique Σ qui passe par les points a_1, a_2, b_1, b_2, et qui satisfait à l'équation

$$(c_1, c_2, d_1, d_2) \cdot [a_1, a_2, b_1, b_2] = [a_1, a_2, b_1, b_2].$$

Ensuite on trouvera le point p' de cette conique pour lequel on a

$$(c_1, c_2, d_1, d_2)\,[a_1, a_2, b_1, b_2, p] = [a_1, a_2, b_1, b_2, p'].$$

Pour cela, on remarquera que dans la solution générale précédente (où l'on peut remplacer les points 1, 2, 3, 4 par les points c_1, c_2, d_1, d_2) on trouve immédiatement le point π de la conique auxiliaire C, qui correspond à la conique (c_1, c_2, d_1, d_2, p). Mais ce point trouvé, on en déduira le point p' par une construction qui a été déjà indiquée (v). Enfin, le point où la droite pp' rencontre pour la seconde fois la conique (a_1, a_2, b_1, b_2, p') sera le point ω cherché.

Ainsi on pourra trouver linéairement autant de points qu'on voudra d'une cubique dont on ne connaîtra que quatre dyades de points imaginaires, et un seul point réel. Ce cas nous paraît avoir été omis par les auteurs qui ont traité des constructions des courbes cubiques.

Si, au lieu d'un point réel p, une dyade $p_1 p_2$ de points imaginaires était donnée, on pourrait encore trouver la dyade $\omega_1 \omega_2$, dont les éléments sont les points opposés au système des quatre points c_1, c_2, d_1, d_2, par rapport aux deux courbes cubiques imaginaires

$$(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, p_1) \quad \text{et} \quad (a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, p_2)\,*.$$

On commencera par trouver la dyade $\pi_1 \pi_2$, qui appartient à la conique auxiliaire

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* Note VI (p. 55).

C, et dont les éléments correspondent aux coniques imaginaires $(c_1, c_2, d_1, d_2, p_1)$ et $(c_1, c_2, d_1, d_2, p_2)$. Le centre d'homologie de cette dyade et de $p_1 p_2$ sera un point connu. Puis on déterminera la dyade $p_1' p_2'$, qui appartient à la conique Σ, et qui vérifie l'équation

$$(c_1, c_2, d_1, d_2).[a_1, a_2, b_1, b_2, p_1, p_2] = [a_1, a_2, b_1, b_2, p_1', p_2'].$$

Soit P le centre d'homologie des dyades $p_1' p_2'$, $p_1 p_2$. Un des axes de symptôse de la dyade de droites (Pp_1, Pp_2), et de la conique Σ, sera l'axe de la dyade $p_1' p_2'$; l'autre pourra se déterminer linéairement; les points d'intersection des droites Pp_1, Pp_2 par ce second axe de symptôse seront les points $\omega_1 \omega_2$ cherchés.

(2.) *Étant données quatre dyades de points imaginaires* $a_1 a_2$, $b_1 b_2$, $c_1 c_2$, $d_1 d_2$, *trouver le neuvième point* θ, *appartenant à toute courbe cubique qui passe par les quatre dyades.*

C'est ce qui se fera par une construction connue. Soient x, y deux points réels; soient ξ, η les points opposés au système $c_1 c_2$, $d_1 d_2$ par rapport aux deux courbes $(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, x)$ et $(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, y)$ respectivement, ξ', η' les points opposés au système $a_1 a_2$, $b_1 b_2$ par rapport à ces mêmes courbes; les droites $\xi\xi'$, $\eta\eta'$ se couperont au point θ.

Seconde Partie.

1. Soit Σ une section conique, qui ne soit ni évanescente ni un cercle, et que nous supposerons complètement tracée; nous reviendrons plus tard sur le cas où l'on n'aurait qu'une partie de la courbe. Toutes les autres coniques dont nous aurons à nous occuper ne seront point tracées (à moins qu'elles ne soient des cercles, ou des couples de lignes droites); elles seront seulement définies par un nombre suffisant de conditions. Voici le problème que nous aurons à résoudre.

'*Étant données deux coniques* S_1, S_2, *trouver leurs quatre points d'intersection, ou, ce qui revient au même, leur triangle harmonique commun, avec la règle et le compas seulement, mais en se servant de la conique tracée* Σ.'

Les coniques S_1, S_2 peuvent être imaginaires de première espèce, ou imaginaires de seconde espèce et conjuguées l'une à l'autre. Mais puisque, dans ces deux cas, on peut construire autant de points qu'on voudra sur les coniques réelles du faisceau (S_1, S_2), nous pourrons supposer, sans perte de généralité, que S_1, S_2 soient elles-mêmes réelles.

Il y a trois réseaux de coniques qu'on est conduit naturellement à considérer relativement au faisceau (S_1, S_2), ou plutôt relativement à son triangle harmonique $\alpha\beta\gamma$; ce sont (1) le réseau des coniques circonscrites au triangle $\alpha\beta\gamma$, (2) le

réseau des coniques inscrites à ce triangle, et (3) le réseau harmonique, c'est à dire le réseau des coniques dont ce même triangle est un triangle harmonique. Une conique du premier réseau est déterminée par deux points ; une conique du second réseau par deux tangentes ; une conique du troisième réseau, soit par deux points, soit par deux tangentes. Le second réseau contient quatre cercles, dont la construction est déjà un problème biquadratique. Mais chacun des autres réseaux ne contient qu'un seul cercle ; ce sont le cercle circonscrit, et le cercle polaire, du triangle $\alpha\beta\gamma$. On peut se servir de l'un ou de l'autre de ces deux cercles pour résoudre notre problème, et l'on est ainsi conduit à deux méthodes de solution différentes. D'après la première méthode on transforme homographiquement une conique quelconque du premier réseau dans la conique Σ, ou bien on transforme corrélativement une conique du second réseau en cette même conique. D'après la seconde méthode, c'est sur une conique du troisième réseau que l'on opère, et la transformation peut être de l'une ou de l'autre espèce. Dans tous les cas on cherche le cercle unique du réseau transformé : les intersections de ce cercle avec la conique Σ font connaître la solution du problème.

2. Avant de donner les détails nécessaires sur ces deux méthodes, nous allons résoudre quelques problèmes préliminaires, qui se rapportent aux coniques des trois réseaux : surtout, nous ferons voir comment on peut construire, dans tous les cas possibles, les cercles du premier et du troisième réseau.

Les coniques du réseau circonscrit sont précisément les coniques réciproques de droites, qui ont été considérées par MM. Poncelet et Chasles. Ainsi chaque conique K de ce réseau est le lieu des points qui sont réciproques des points d'une certaine droite k relativement au faisceau (S_1, S_2) ; K est aussi le lieu des pôles de k, relativement aux coniques du faisceau (S_1, S_2) ; enfin, l'ensemble des lignes K, k est la courbe Jacobienne du réseau déterminé par S_1, S_2 et la droite k, prise deux fois. Pour construire la conique du réseau circonscrit qui passe par deux points $p_1 p_2$, on construit les points $P_1 P_2$ réciproques de $p_1 p_2$; la conique réciproque de la droite $P_1 P_2$ est la conique cherchée. De même, on pourra trouver la conique du réseau circonscrit qui soit tangente en un point donné p à une droite L. On prendra la conique réciproque de L ; sa tangente, au point réciproque de p, sera réciproque de la conique cherchée. Pour les coniques du second réseau on a des constructions corrélatives ; mais la construction de la conique σ appartenant au réseau harmonique, et passant par deux points $p_1 p_2$, est un peu moins facile. Soient $x_1 x_2$, $y_1 y_2$ deux couples de points qui divisent harmoniquement le segment $p_1 p_2$; construisons les points réciproques $X_1 X_2, Y_1 Y_2$, et menons les droites $X_1 X_2$, $Y_1 Y_2$ se coupant au point q. Soient encore abc les

points où la droite $p_1 p_2$ rencontre les côtés du triangle $\alpha\beta\gamma$; et désignons par $a'b'c'$ les points conjugués harmoniques de abc par rapport à $p_1 p_2$. En considérant l'involution dont les points doubles sont $p_1 p_2$, on voit que le quatrième point commun aux coniques $(\alpha, \beta, \gamma, x_1, x_2)$, $(\alpha, \beta, \gamma, y_1, y_2)$, (c'est-à-dire le point Q réciproque de q) est le point d'intersection des trois droites $\alpha a'$, $\beta b'$, $\gamma c'$; mais ces trois droites sont évidemment les polaires de a, b, c, relativement à σ ; donc le point Q réciproque de q est le pôle de la droite $p_1 p_2$, relativement à cette conique. Soit $(\alpha, \beta, \gamma, x, y)$ une conique du réseau circonscrit qui passe par deux points quelconques xy de la droite $p_1 p_2$; cette conique est aussi une conique du réseau circonscrit appartenant au faisceau (S_1, σ) ; par conséquent les points réciproques des points de $(\alpha, \beta, \gamma, x, y)$, considérés relativement au faisceau (S_1, σ), seront en ligne droite. Pour avoir cette droite, on déterminera les points XY, réciproques de xy, qu'on sait trouver, puisqu'on connaît les polaires de x, y relativement aux coniques σ et S_1. Soit θ un point quelconque de $(\alpha, \beta, \gamma, x, y)$; les polaires de θ, relativement à σ et S_1, se croisent sur la droite XY ; on connaît aussi le point où la polaire de θ relativement à σ coupe la droite $p_1 p_2$; on pourra donc construire cette polaire, qui déterminera complètement la conique σ. On ne doit pas prendre pour xy les points doubles de l'involution déterminée par (S_1, σ) sur la droite $p_1 p_2$; ce qui ferait coïncider la droite XY avec $p_1 p_2$; de même, on ne doit pas prendre pour θ le point réciproque du point d'intersection de XY, $p_1 p_2$.

De ce que nous avons dit on tire la construction suivante pour les deux cercles. Soient $x_1 x_2$, $y_1 y_2$ deux couples de points rectangulaires à l'infini ; $X_1 X_2$, $Y_1 Y_2$ leurs points réciproques relativement au faisceau (S_1, S_2). Le centre Q du cercle polaire est le point réciproque de l'intersection q de $X_1 X_2$, $Y_1 Y_2$. Soient $r_1 r_2$ les deux autres points diagonaux du quadrilatère $X_1 X_2'$, $Y_1 Y_2$. Le cercle circonscrit est réciproque de la droite $r_1 r_2$; de plus les points $R_1 R_2$, réciproques de $r_1 r_2$, sont les extrémités opposées d'un même diamètre de ce cercle. En effet, la droite $r_1 r_2$ coupe la conique C, réciproque de la droite à l'infini, en deux points imaginaires $\Omega_1 \Omega_2$, qui sont réciproques des deux points $\omega_1 \omega_2$ de la dyade cyclique. Car les points d'une conique réciproque d'une droite correspondent anharmoniquement aux points de cette droite ; de sorte qu'aux points doubles de l'involution $x_1 x_2$, $y_1 y_2$ sur la droite à l'infini correspondent les points doubles de l'involution $X_1 X_2$, $Y_1 Y_2$ sur la conique C. De plus, les deux points $r_1 r_2$ sont harmoniquement conjugués aux points $\Omega_1 \Omega_2$; donc, sur le cercle réciproque de la droite $r_1 r_2$, les points $R_1 R_2$ sont harmoniquement conjugués à $\omega_1 \omega_2$; ce qui veut dire que $R_1 R_2$ est un diamètre du cercle. On connaît donc le cercle circonscrit ;

quant au cercle polaire on achevera sa détermination, soit en observant qu'il coupe orthogonalement le cercle directeur du cercle circonscrit, soit en trouvant par la méthode que nous avons indiquée la polaire d'un point quelconque. Dans les applications, on pourra toujours remplacer la conique non tracée S, par la conique tracée Σ. On prendra pour x, y dans la construction précédente les points $x_1 x_2$ dont on s'est déjà servi, et dont on aura construit les polaires relativement à Σ. On abaissera de Q des perpendiculaires sur ces polaires ; la droite qui joindra leurs pieds sera réciproque de l'hyperbole équilatère $(\alpha, \beta, \gamma, x_1, x_2)$ par rapport au faisceau (Σ, σ). Soit θ un point quelconque de l'hyperbole, θ' le point d'intersection de la droite réciproque de cette courbe par la polaire de θ relativement à Σ ; la perpendiculaire abaissée de θ' sur $Q\theta$ sera la polaire de θ relativement au cercle polaire. On arriverait aussi à des constructions assez simples, en prenant pour xy, soit les deux points à l'infini sur Σ, soit les deux points de la dyade cyclique. Il est inutile d'ajouter qu'on ne peut déterminer linéairement aucun point du cercle polaire, ce cercle pouvant être imaginaire.

On sait que le cercle circonscrit à un triangle harmonique de la conique S coupe orthogonalement le cercle directeur de S ; et que le cercle polaire d'un triangle circonscrit à S coupe aussi orthogonalement ce même cercle directeur. On pourrait donc trouver le cercle circonscrit en construisant les cercles directeurs de trois coniques quelconque du réseau harmonique, et le cercle polaire en construisant les cercles directeurs de trois coniques quelconques du réseau inscrit. On aura aussi les résultats particuliers que voici, dont la plupart étaient connus :

'Le lieu des centres des hyperboles équilatères du réseau circonscrit, est le "cercle des neuf points" du triangle harmonique fondamental.

'Le lieu des centres des hyperboles équilatères du réseau harmonique est le cercle circonscrit. Ces hyperboles se coupent aux quatre centres des cercles inscrits au triangle harmonique.

'Le lieu des foyers des paraboles du réseau inscrit est le cercle circonscrit.

'Les droites directrices des paraboles du réseau inscrit se coupent au centre du cercle polaire.

'Les droites directrices des paraboles du réseau harmonique se coupent au centre du cercle circonscrit.'

3. Ce qui précède suffit pour notre but actuel ; cependant nous croyons devoir ajouter quelques observations élémentaires qui pourraient être utiles en d'autres occasions. On peut considérer les sommets du triangle harmonique commun à deux coniques comme représentant d'une certaine manière les trois

racines d'une équation cubique. Ainsi au problème analytique 'Trouver les fonctions symétriques des racines d'une équation cubique' correspondra le problème géométrique 'Trouver les lieux et les enveloppes qui dépendent symétriquement des trois éléments du triangle harmonique commun à deux coniques non tracées.' D'après ce qui a été dit, on aura des moyens assez simples pour trouver les points, les droites et les coniques qui dépendent symétriquement des éléments du triangle, considéré par rapport au système de deux points ou de deux droites quelconques. Mais nous ne nous sommes pas occupé des lieux et des enveloppes qui se rapportent au triangle considéré relativement à un seul point ou à une seule droite. Parmi tous ceux qu'on pourrait imaginer ce sont surtout les dérivées polaires du triangle (considéré comme ligne soit du troisième ordre, soit de la troisième classe) qui offrent quelque intérêt. Pour abréger, nous ne parlerons que du centre des distances moyennes du triangle, et de la droite à l'infini considérée relativement au triangle, puisqu'on tirera facilement de ce que nous allons dire tout ce qui est relatif à un point ou à une droite quelconque. En revenant donc sur la notation de l'article 2, nous nous proposerons de trouver (1) le centre des distances moyennes, (2) l'ellipse *minima* circonscrite, (3) l'ellipse *maxima* inscrite.

Supposons que le point q' soit harmoniquement conjugué à q par rapport aux points $X_1 X_2$; si Q' est le point réciproque de q' relativement au faisceau (S_1, S_2), il est évident que QQ' sera un diamètre de l'hyperbole équilatère $(\alpha, \beta; \gamma, x_1, x_2)$. Soit O le point milieu de $R_1 R_2$, O' le point milieu de OQ; O et O' seront les centres du cercle circonscrit, et du cercle des neuf points. Ce dernier cercle sera complètement déterminé, puisque le point milieu de QQ' sera un point de sa circonférence. Soit Γ le point harmoniquement conjugué à Q par rapport à OO': Γ sera le centre des distances moyennes du triangle; on sait, en effet, que le centre des distances moyennes et le centre du cercle polaire sont les deux centres de similitude du cercle circonscrit et du cercle des neuf points, et que les deux centres de similitude divisent harmoniquement le segment OO'. On voit en même temps, qu'en généralisant convenablement cette construction, on pourra trouver avec la règle seule le pôle d'une droite quelconque, et la polaire d'un point quelconque par rapport au triangle $\alpha\beta\gamma$. Les deux ellipses *maxima* et *minima* sont les coniques polaires du centre des distances moyennes et de la droite à l'infini, par rapport au triangle. Mais elles sont aussi les coniques du premier et du second réseau pour lesquelles la droite à l'infini est la polaire du centre des distances moyennes. Il suffira donc de résoudre ce problème 'Trouver la conique du premier réseau pour laquelle un point donné Γ est le pôle d'une droite donnée Λ.'

Soit x_1 un point quelconque de Λ; on trouvera la conique du troisième réseau qui est tangente aux droites $x_1\Gamma$ et Λ, et l'on mènera du point Γ la seconde tangente à cette conique, qui coupera Λ au point x_2. Les deux points $x_1 x_2$ seront conjugués par rapport à la conique qu'on cherche. Soient $y_1 y_2$ une seconde paire de points conjugués sur la même droite; prenons $X_1 X_2 Y_1 Y_2$ les points réciproques de $x_1 x_2 y_1 y_2$; la diagonale du quadrilatère $X_1 X_2 Y_1 Y_2$, qui ne passe pas par le point d'intersection de $X_1 X_2$, $Y_1 Y_2$ sera la droite réciproque de la conique cherchée. On peut aussi opérer de la manière suivante. Soit x un point quelconque de Λ; toutes les coniques du réseau circonscrit, qui coupent harmoniquement le segment Γx, passent par le même quatrième point, dont nous allons trouver le point réciproque. Soient Γ', X les points réciproques de Γ, x; les coniques dont il s'agit auront pour droites réciproques des droites conjuguées à la droite $\Gamma' X$ par rapport à la conique réciproque de Γx, puisque quatre points harmoniques en ligne droite ont pour réciproques quatre points harmoniques sur la conique réciproque de la droite. Soient $a_1 a_2$, $b_1 b_2$ deux couples de points harmoniquement conjugués aux points Γ, x; $A_1 A_2$, $B_1 B_2$ les points réciproques correspondants; le point ξ où les cordes $A_1 A_2$, $B_1 B_2$ vont concourir sera le point réciproque du quatrième point d'intersection. En prenant un second point y sur la droite Λ, on trouvera un autre point η tel que ξ; la droite $\xi\eta$ sera réciproque de la conique cherchée.

On trouverait la conique du second réseau, qui aurait un pôle et une polaire donnés, par la construction corrélative. Si c'était la conique du troisième réseau qu'on voulait trouver, on prendrait la conique réciproque de la droite Λ, puis la polaire de Γ' par rapport à cette conique, et enfin la conique réciproque de la polaire. Cette dernière conique coupe la droite Λ aux mêmes points où elle est coupée par la conique harmonique qu'on cherche. On pourra donc trouver avec la règle seule, autant de pôles et de polaires relativement à cette courbe que l'on voudra; puisque pour cela il n'est pas nécessaire de connaître les deux points où la courbe est coupée par la droite Λ, mais seulement l'involution qui détermine ces deux points *.

Nous ajouterons encore un exemple de l'application des principes précédents. Soit proposé de déterminer le cercle qui passe par les pieds des perpendiculaires abaissées d'un point donné P sur les côtes du triangle $\alpha\beta\gamma$. On déterminera la conique du réseau inscrit qui a pour foyer le point P. Le cercle décrit sur l'axe focal de cette conique comme diamètre sera le cercle cherché.

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* Note VII (p. 55).

Quoiqu'on puisse représenter les racines d'une équation cubique par les trois éléments du triangle harmonique commun à deux sections coniques, il arrive ordinairement que la solution d'un problème cubique ne conduit pas à la recherche d'un tel triangle, mais à la recherche de trois des points d'intersection de deux sections coniques, dont un des points d'intersection est donné d'avance; ou à la recherche des trois points communs à un réseau de coniques circonscrites au même triangle. Or, le second de ces deux cas se réduit au premier, puisqu'en considérant les coniques d'un faisceau déterminé par deux coniques du réseau, on voit qu'elles correspondent anharmoniquement aux points où elles coupent une troisième conique du réseau qui n'appartient pas au faisceau que l'on considère. On pourra donc déterminer linéairement le quatrième point commun à une conique quelconque du faisceau et à la troisième conique. De plus, étant donné un des points d'intersection P de deux coniques, le triangle $\alpha\beta\gamma$ est homologique au triangle abc, dont les sommets sont les trois autres points d'intersection; il est aussi inscrit à ce même triangle. Le centre d'homologie est le point P; quant à l'axe d'homologie, c'est l'axe de symptôse d'une conique circonscrite à $P\alpha\beta\gamma$ et d'une conique du réseau harmonique qui est tangente à la première conique au point P. L'homologie des deux triangles pourra donc se déterminer linéairement; et l'on aura ainsi une méthode générale, dont on pourra se servir pour déduire les lieux et les enveloppes relatives à l'un de ces triangles des lieux et des enveloppes relatives à l'autre.

4. *Première méthode.*—On déterminera cinq points a, b, c, d, e d'une conique quelconque σ du réseau circonscrit appartenant au système des deux coniques (S_1, S_2). On choisira à volonté trois points $a'b'c'$ de la conique Σ, qu'on fera correspondre homographiquement à abc; et l'on en déterminera un quatrième d' de manière que le rapport anharmonique des quatre points $a'b'c'd'$ de la conique Σ soit égal à celui des points $abcd$ de σ. Puis on transformera homographiquement les coniques (S_1, S_2) de manière qu'aux points $abcd$ de la figure donnée correspondent les points $a'b'c'd'$ de la figure nouvelle. La conique σ se transforme en Σ, laquelle sera ainsi une conique du réseau circonscrit appartenant aux transformées de S_1 et S_2. Le cercle appartenant à ce réseau coupera Σ en quatre points, dont un qui est réel sera connu d'avance, et pourra se construire linéairement, sans tracer ni le cercle, ni la conique Σ. Les trois autres points (dont un sera toujours réel) seront les points de la nouvelle figure qui correspondent aux sommets $\alpha\beta\gamma$ du triangle harmonique commun à S_1 et S_2. On pourra donc construire ce triangle; ce qui permettra de trouver par une construction quadratique les points d'intersection de S_1 et S_2.

On voit que cette construction est purement linéaire, jusqu'au moment où l'on veut déterminer les intersections du cercle et de Σ. Mais on peut l'abréger de beaucoup, en se servant dès le commencement du tracé de la conique Σ. On prendra deux points $p_1 p_2$ de cette conique, et l'on choisira pour σ la conique $(\alpha, \beta, \gamma, p_1, p_2)$. Puis on considérera la droite $p_1 p_2$ comme axe d'homologie de σ et Σ; en se servant du tracé de Σ on pourra trouver les deux centres d'homologie des deux courbes, appartenant à cet axe. On voit qu'on fera bien de prendre pour $p_1 p_2$ deux points imaginaires conjugués appartenant à Σ; si l'on prenait deux points réels, il pourrait arriver que les points d'intersection de Σ et σ fussent tous réels, sans qu'il y eût aucune tangente commune; mais alors les deux centres d'homologie appartenant à l'axe $p_1 p_2$ pouraient devenir imaginaires. En supposant donc que $p_1 p_2$ soient imaginaires, on fera avec deux couples de l'involution qui détermine ces points les mêmes opérations que nous avons faites avec les deux couples de points rectangulaires $x_1 x_2$, $y_1 y_2$ de l'article 2; et, de même que nous y avons trouvé les deux extrémités d'un même diamètre du cercle circonscrit, on trouvera ici les deux extrémités réelles $\alpha_1 \alpha_2$ d'une corde de σ passant par le pôle de la droite $p_1 p_2$ relativement à σ. On joindra le point d'intersection de $\alpha_1 \alpha_2$, $p_1 p_2$, au pôle de $p_1 p_2$ relativement à Σ, par une droite qui coupera Σ en deux points $b_1 b_2$, qui seront toujours réels, puisque le pôle de $p_1 p_2$ est intérieur à Σ; les intersections de $\alpha_1 b_1$, $\alpha_2 b_2$, et de $\alpha_1 b_2$, $\alpha_2 b_1$ seront les deux centres d'homologie de σ et Σ, appartenant à l'axe d'homologie $(p_1 p_2)$. Avec l'un de ces centres d'homologie, on déterminera la droite λ et la dyade $\theta_1 \theta_2$, homologues de la droite à l'infini et de la dyade cyclique; puis on trouvera le segment réel intercepté par la conique $(\alpha, \beta, \gamma, \theta_1, \theta_2)$ sur une droite passant par le pôle de λ relativement à cette conique: enfin on déterminera le quatrième point d'intersection de $(\alpha, \beta, \gamma, \theta_1, \theta_2)$ avec $(\alpha, \beta, \gamma, p_1, p_2)$ ou σ. En passant à la figure homologique, la première de ces deux coniques deviendra un cercle, dont on connaîtra un diamètre et une des intersections avec Σ. On n'aura donc qu'à tracer le cercle pour avoir les trois points homologues des points cherchés.

Si Σ est une ellipse on prendra pour $p_1 p_2$ les deux points imaginaires où cette courbe est coupée par la droite à l'infini. Dans ce cas, on déterminera un diamètre du cercle circonscrit au triangle $\alpha\beta\gamma$, et un diamètre de σ, qui sera l'ellipse circonscrite au même triangle et homothétique avec Σ. On trouvera les deux centres de similitude des deux ellipses, en déterminant les points extrêmes d'un diamètre de Σ parallèle au diamètre de σ; ce qui exigerait une construction quadratique, si Σ n'était point tracée. Alors, pour avoir la solution du problème, on n'aura qu'à tracer le cercle correspondant au cercle circonscrit à $\alpha\beta\gamma$. Nous

croyons que cette solution est peut-être la plus simple qu'on puisse trouver; cependant elle ne laisse pas d'être fort longue. Pour l'abréger autant que possible, on prendra les deux points à l'infini sur les axes principaux de Σ pour l'un des couples de points dont on se servira pour trouver un diamètre, soit du cercle circonscrit, soit de l'ellipse σ; le second couple ne pourra pas être le même pour les deux courbes, mais on fera en sorte que les deux couples aient un point commun. Ainsi il faudra chercher les points réciproques, par rapport au faisceau donné, de dix points différents, afin d'avoir un diamètre de chacune des deux courbes, et leur quatrième point d'intersection. Cette recherche sera un peu pénible, puisque la détermination de chaque point réciproque exige la construction de la polaire d'un même point par rapport à chacune des deux coniques non tracées S_1 et S_2.

On pourrait éviter toute construction quadratique, sans cesser de se servir d'une conique homologue de Σ, en prenant pour σ la conique du réseau circonscrit qui est tangente à Σ en un point donné, puisqu'en ce cas, un des centres d'homologie des deux coniques étant connu d'avance, on peut trouver l'autre linéairement. C'est ce qu'on serait conduit naturellement à faire si la conique Σ était une parabole. On prendrait pour σ la parabole homothétique du réseau circonscrit; on en trouverait trois points, dont un serait son quatrième point d'intersection par le cercle circonscrit. En menant des tangentes à Σ parallèles aux tangentes de σ en deux de ces trois points, et en joignant les points de contact des tangentes homologues, on aurait le centre de similitude. En ce cas on n'aurait à trouver que [neuf] points réciproques, mais, par compensation, la détermination du centre de similitude serait un peu plus compliquée.

5. Cette première méthode va nous fournir une démonstration bien simple de ce beau theorème de Descartes, qu'il n'y a pas un arc de section conique si petit qu'il ne suffise pas pour résoudre géométriquement tout problème cubique ou biquadratique. Prenons deux points a, b sur une conique quelconque σ du réseau circonscrit; faisons correspondre à ces deux points deux points $a'b'$ de la partie tracée de Σ. Soit α le sommet réel, ou l'un des sommets réels, du triangle $\alpha\beta\gamma$. La conique σ est divisée par la corde ab en deux parties; en déterminant, avec une approximation même très grossière, la position du point α (ce qui se fera dans tous les cas au moyen de quelques points qu'on prendra sur une conique quelconque du réseau circonscrit, autre que σ) on distinguera l'arc $a\alpha b$, qui passe par α, de l'autre partie de la courbe. Selon que l'on prend pour c un point de l'arc $a\alpha b$, ou de l'arc opposé, on prendra pour c' un point de l'arc tracé $a'b'$, ou un point de la conique Σ qui ne soit pas compris entre ces deux points. D'après

cela, les points de l'arc tracé $a'b'$ correspondront sur la conique Σ aux points de l'arc aab de la conique σ; donc le point correspondant de a tombera entre les points a', b' de l'arc tracé, et y sera déterminé par le cercle circonscrit, appartenant à la figure transformée, qui viendra couper l'arc $a'b'$ en ce point. Mais on connaît un second point d'intersection de ce cercle et de Σ; c'est celui qu'on peut construire linéairement; donc on pourra trouver par une construction linéaire la sécante commune, dont les intersections avec le cercle feront connaître les deux autres sommets du triangle $\alpha\beta\gamma$.

6. *Seconde méthode.*—Lorsqu'on se sert du cercle polaire pour trouver les intersections des coniques données S_1 et S_2, on peut opérer d'un grand nombre de manières différentes. Nous n'indiquerons ici que les constructions que nous croyons les plus simples, mais nous n'osons pas affirmer qu'on ne pourrait en trouver d'autres, dépendant du même principe général, qui auraient quelque avantage sur celles que nous allons exposer.

(1.) Supposons que $\alpha\beta\gamma$, le triangle harmonique commun de S_1 et S_2, n'ait qu'un sommet réel. En ce cas, deux coniques quelconques réelles du réseau harmonique se coupent en deux points réels, et en deux points imaginaires conjugués; de plus, ce réseau ne contient aucune conique imaginaire de la première espèce. D'après cela, on peut opérer avec une conique quelconque de ce réseau de la même manière qu'avec une conique circonscrite. Soit σ la conique qu'on choisit; on la transformera soit homographiquement, soit corrélativement, en Σ, et après avoir déterminé le cercle appartenant au réseau transformé, on trouvera les deux intersections réelles du cercle et de Σ. On aura ainsi le triangle harmonique commun à ces deux coniques, triangle qui est le correspondant dans la nouvelle figure du triangle $\alpha\beta\gamma$ dans l'ancienne. On pourra aussi employer une transformation homologique, au lieu de la transformation homographique générale; seulement on remarquera que la détermination des centres d'homologie de σ et Σ est loin d'être aussi facile dans le cas actuel que dans le cas d'une conique circonscrite. En effet, il convient de prendre pour σ une conique du réseau harmonique ayant avec Σ une sécante idéale commune. Or, on ne peut déterminer linéairement aucun point d'une telle conique, quoiqu'on sache trouver son système polaire. On aura donc besoin d'une construction quadratique pour trouver les points extrêmes d'une corde de σ passant par le pôle de l'axe d'homologie relativement à cette conique; points dont on fait usage pour trouver directement les deux centres d'homologie. Il est vrai qu'on pourrait opérer avec une conique σ tangente à Σ en un point donné, mais la construction d'une telle conique serait un peu pénible.

La méthode que nous venons d'indiquer s'applique aussi au cas que nous avons exclu, mais elle conduit quelquefois à un résultat inutile. En effet, quand le triangle $\alpha\beta\gamma$ est réel, les coniques du réseau harmonique qu'on détermine par deux points imaginaires conjugués peuvent être imaginaires; ainsi, le cercle du réseau transformé peut être imaginaire, ou, tout en restant réel, il peut ne recontrer Σ en aucun point réel. D'ailleurs, quand même cette méthode réussirait, elle conduirait à des opérations assez prolixes.

7. (2.) Supposons que les points d'intersection de S_1, S_2 ne soient pas tous imaginaires; nous allons voir qu'on pourra déterminer ces points directement, sans chercher préalablement le triangle harmonique.

Soit (1°) Σ une ellipse, A_1A_2, B_1B_2 ses deux axes principaux. Qu'on mène une transversale L qui coupe en deux points imaginaires une conique quelconque réelle S du faisceau (S_1, S_2). Ce faisceau détermine une involution sur la transversale L, dont les points doubles $\mu_1\mu_2$ sont nécessairement réels. Des points μ_1, μ_2 menons des tangentes à S; soient a_1a_2, b_1b_2 les points de contact des tangentes issues de μ_1 et μ_2 respectivement; ces points, qui seront tous réels, diviseront la conique S harmoniquement. Qu'on transforme la figure homographiquement de manière qu'aux deux couples a_1a_2, b_1b_2 correspondent les deux couples A_1A_2, B_1B_2; ce qui peut se faire de huit manières différentes. La conique S deviendra Σ, puisqu'elle doit devenir une conique divisée harmoniquement par les deux couples de points A_1A_2, B_1B_2. De plus les quatre points d'intersection de S_1 et S_2 se transformeront en quatre points de l'ellipse Σ, situés sur une même circonférence de cercle. Car les points $\mu_1\mu_2$, qui sont des points conjugués par rapport à toutes les coniques du faisceau (S_1, S_2), se transformeront en les points situés à l'infini sur les axes principaux de Σ; donc toutes les coniques du faisceau transformé auront leurs axes principaux parallèles à ceux de Σ, et par conséquent se couperont sur une circonférence de cercle Ω. Pour avoir ce cercle, on cherchera son centre par la construction ci-dessus; puis on prendra un couple de points réciproques P_1P_2 par rapport au faisceau transformé; le cercle Ω coupera orthogonalement le cercle dont P_1P_2 est un diamètre. On peut aussi déterminer Ω en prenant trois couples de points réciproques par rapport au faisceau transformé: ce qui donnera trois cercles que Ω coupera orthogonalement.

Soit (2°) Σ une hyperbole ayant A_1A_2 pour axe réel, et passant par les points $\Phi_1\Phi_2$ à l'infini. On coupera le faisceau (S_1S_2) par une transversale sur laquelle on ait une involution aux points doubles réels $\mu_1\mu_2$. Soit S une conique du faisceau qui coupe la transversale en des points réels $\phi_1\phi_2$. L'un des points $\mu_1\mu_2$ sera extérieur à S; soit μ_1 ce point, et désignons par a_1a_2 les points de

contact de tangentes à S issues de μ_1 ; la conique S sera divisée harmoniquement par les couples de points $\phi_1 \phi_2$, $a_1 a_2$. En transformant la figure de manière que a_1, a_2 deviennent A_1, A_2, et que ϕ_1, ϕ_2 deviennent $\Phi_1 \Phi_2$, S deviendra Σ, et aux points $\mu_1 \mu_2$ correspondront respectivement les points à l'infini sur l'axe imaginaire et sur l'axe réel de Σ. Donc les points d'intersection du faisceau transformé appartiendront à une circonférence de cercle, qu'on pourra déterminer comme précédemment.

Soit enfin (3°) Σ une parabole, ayant A pour sommet et tangente au point M à la ligne droite à l'infini. On coupera toujours le faisceau $(S_1 S_2)$ par une transversale sur laquelle on ait des points doubles réels $\mu_1 \mu_2$. Soit S la conique du faisceau qui touche la transversale au point μ_1 ; de μ_2 menons la seconde tangente à S ; soit a son point de contact. Choisissons à volonté deux points x et X sur les coniques S et Σ respectivement ; soient y et Y deux autres points sur ces mêmes coniques, tels que le rapport anharmonique des points μ_1, a, x, y de S soit égal à celui des points M, A, X, Y de Σ. En faisant correspondre homographiquement les points M $A X Y$ aux points $\mu a x y$, on oura la même construction que dans les deux cas précédents.

Nous remarquerons qu'après avoir mené la transversale on est plus restreint, quant au choix de la conique qui doit être transformée en Σ, quand cette courbe est une parabole, que quand elle est une conique centrale. Mais, dans le premier cas, la transformation de la conique en Σ peut être opérée d'une infinité de manières différentes.

Cette méthode offre de grands avantages. On n'opère que sur les coniques du faisceau donné, sans avoir besoin d'aucune autre conique du réseau harmonique ; ce qui facilite beaucoup la construction. De plus, on arrive directement à la solution du problème biquadratique des intersections, et on ne la fait pas dépendre du problème cubique du triangle harmonique. Quand Σ est une parabole on peut prendre une droite tangente à l'une des courbes S_1 et S_2, pour la transversale qui coupe le faisceau ; en opérant ainsi, on n'aura qu'une seule construction quadratique à effectuer ; c'est celle qui est inévitable (à ce qu'il paraît) quand on veut trouver le rayon du cercle polaire au moyen duquel on résout le problème.

8. (3.) Quand aucun des points d'intersection de S_1 et S_2 n'est réel, la détermination du triangle harmonique précède nécessairement la détermination des points d'intersection. En ce cas on peut faire usage de l'un ou de l'autre des procédés suivants.

(i.) Prenons une droite tangente à S_1 en un point quelconque x_1 ; ce point

sera un des points doubles de l'involution déterminée par le faisceau donné sur la droite ; soit x_2 le second point double, qui pourra être trouvé linéairement ; en désignant par S_2 la conique du faisceau qui passe par x_2, on aura deux coniques du faisceau tangentes à la même droite réelle. Puisque ces coniques n'ont aucun point réel commun, et qu'elles ont une tangente réelle commune, elles en ont quatre. Soient σ_1 et σ_2 les coniques réciproques polaires de S_1 et S_2 par rapport à Σ ; σ_1 et σ_2 se couperont en quatre points réels qu'on déterminera par la méthode précédente, dont l'application sera très facile, puisqu'un des quatre points d'intersection étant connu d'avance, il suffira de trouver le centre du cercle polaire pour déterminer complètement ce cercle. Les polaires des points d'intersection de σ_1 et σ_2 seront les tangentes communes de S_1 et S_2, et feront connaître le triangle harmonique du faisceau.

Au lieu de prendre les coniques réciproques polaires de S_1 et S_2, il sera plus facile, en beaucoup de cas, de faire usage de la transformation corrélative suivante. En supposant, pour abréger, que Σ soit une ellipse, soit L un point intérieur à S_1, soient μ_1, μ_2 les rayons doubles réels du faisceau en involution déterminé au point L par les coniques S_1, S_2 considérées comme formant un faisceau tangentiel. Soient enfin a_1, b_1 et a_2, b_2 les deux couples de tangentes à S_1 aux points où cette conique coupe les droites μ_1, μ_2. En faisant correspondre corrélativement les deux couples de sommets de Σ aux deux couples de droites $a_1 b_1$, $a_2 b_2$, on transformera S_1 en Σ, et les quatre tangentes communes de S_1 et S_2 deviendront quatre points de Σ, situés sur une même circonférence de cercle, dont un sera connu d'avance. On déduira la démonstration de celle de l'article 7. Cette méthode pourra servir à trouver directement les tangentes communes de deux coniques données, en supposant toutefois qu'il y en a de réelles.

(ii.) Deux coniques quelconques du réseau harmonique, qui se coupent en un point réel, se coupent aussi, dans le cas qui nous occupe, en trois autres points réels. On pourra donc considérer, au lieu de S_1 et S_2, deux des coniques du réseau harmonique qui passent par un même point, et l'on appliquera la méthode de l'article 7. Ici encore il y aura l'avantage qu'on connaîtra d'avance un des points communs à Σ et au cercle polaire du réseau transformé.

9. (4.) Les cas du problème général, où il s'agit de trouver les intersections d'une conique tracée Σ avec une conique non tracée S, mérite quelque attention. On pourra en ce cas se servir des principes de l'article 7, et l'on transformera le faisceau (S, Σ) en un autre faisceau qui contienne à la fois la conique Σ et un cercle, soit en transformant S en Σ, soit en transformant Σ en elle-même. Quand le faisceau (S, Σ) détermine, sur la ligne droite à l'infini, une involution ayant

des points doubles réels (ce qui arrivera toujours si Σ est une ellipse ou une parabole), on pourra opérer cette dernière transformation de manière que la ligne droite à l'infini se corresponde à elle-même. Pour cela, on n'aura qu'à prendre la ligne droite à l'infini pour la transversale L de l'article 7, et la conique Σ elle même pour la conique S. Lorsque Σ est une conique centrale, en transformera les quatre points d'intersection de S et Σ en quatre points d'une même circonférence, en transformant Σ en elle-même, de manière que les points extrêmes d'un certain couple de diamètres conjugués deviennent les sommets de la courbe. Lorsque Σ est une parabole, on aura une transformation encore plus simple. [Soit A le sommet de Σ, a le point de contact de la tangente à Σ menée parallèlement à la polaire par rapport à S du point à l'infini sur l'axe de Σ; on n'aura qu'a transformer Σ homologiquement en elle-même, en prenant pour centre d'homologie le point à l'infini sur Aa.]

Troisième Partie.

1. Problème.—*Abaisser d'un point donné P des normales sur une conique complètement décrite.*

Pour abréger le discours nous supposerons que la conique soit centrale. On sait que ce problème a été résolu par Apollonius de Perge, qui a démontré que les pieds des normales se trouvent sur une hyperbole équilatère, passant par P et par le centre de la conique, et ayant ses asymptotes parallèles aux axes principaux de cette courbe. L'hyperbole est le lieu des points d'intersection des diamètres de la conique par des perpendiculaires abaissées de P sur les diamètres conjugués.

En employant les diverses méthodes précédentes, on parviendrait à résoudre ce problème d'un grand nombre de manières différentes. Mais c'est surtout la belle solution de Joachimsthal qui peut intéresser les géomètres; c'est pourquoi nous la reproduirons ici avec une démonstration fondée sur les principes précédents.

Soit Σ la conique tracée; Γ l'hyperbole équilatère d'Apollonius: S, C les centres respectifs de ces deux courbes. Soit encore A_1A_2 l'axe réel, ou l'un des deux axes réels, de Σ; α le point à l'infini sur cet axe, β le point à l'infini sur l'axe conjugué. Qu'on désigne par γ un point quelconque de Γ, et qu'on abaisse du sommet A_1 une perpendiculaire sur la droite $P\gamma$. Cette perpendiculaire, qui sera en même temps parallèle au diamètre de Σ conjugué à $S\gamma$, coupera Σ en un second point que nous désignerons par σ. Il est évident qu'on pourra considérer les deux courbes Γ et Σ comme homographiques par rapport aux deux séries de

points γ et σ. Qu'on transforme Γ en Σ, et que, dans la seconde figure, Σ' soit la courbe correspondant à Σ dans la première. Nous allons voir que les points d'intersection de Σ et Σ' appartiendront à une même circonférence. Aux points α et β de Γ correspondront évidemment les points A_1 et A_2 de Σ; d'où l'on conclut qu'au point C de la première figure correspond le point β de la seconde. Soit L la droite de la première figure qui correspond à la droite à l'infini de la seconde : il faut que cette droite passe par C, puisque le point correspondant de C est à l'infini. En se rappelant la propriété caractéristique des points de Γ, on verra que, si Σ est une ellipse, la droite L est l'axe d'homologie de la dyade asymptotique de Σ et de la dyade des perpendiculaires abaissées de P sur la dyade asymptotique; et que les points imaginaires, où L est coupée par l'une ou l'autre de ces dyades, appartiennent à Γ. Soit D le point à l'infini de la droite L; les deux points CD seront évidemment des points conjugués par rapport à la conique Γ; donc, au point D de la première figure il correspondra le point α de la seconde. Mais les points CD sont aussi des points conjugués par rapport à Σ, puisqu'ils sont harmoniquement conjugués aux points où la droite L rencontre la dyade asymptotique de Σ; d'où il s'ensuit que L est parallèle à la polaire de C par rapport à Σ. Par conséquent, les points α, β seront des points réciproques par rapport aux deux courbes Σ et Σ'; donc il passe une circonférence de cercle par les points d'intersection de Σ et Σ'. La démonstration serait tout aussi simple si l'on supposait que Σ fût une hyperbole. Soit Ω le cercle appartenant au faisceau (Σ, Σ'); pour le déterminer complètement, il faudra trouver trois couples de points réciproques par rapport à ce faisceau. Pour cela, soit s le point de Σ qui correspond au point S de Γ, et qui se trouve sur la perpendiculaire abaissée de A_1 sur PS; les droites sA_1, sA_2 de la seconde figure correspondront aux droites $S\alpha$, $S\beta$ de la première, c'est-à-dire, aux deux axes de Σ. Qu'on prenne les points réciproques de α et β par rapport au système des deux courbes Σ et Γ; les points de la seconde figure qui correspondront à ces deux points réciproques seront évidemment les points d'intersection de la tangente $A_1\beta$ par sA_2, et de la tangente $A_2\beta$ par sA_1; points qu'on désignera par a_1 et a_2. Donc les deux points A_1a_1, et les deux points A_2a_2 seront réciproques par rapport au système (Σ, Σ'). Le cercle Ω coupera orthogonalement les cercles (A_1a_1) et (A_2a_2) décrits sur A_1a_1 et A_2a_2 comme diamètres; mais il est évident que ces cercles coupent orthogonalement le cercle (A_1A_2); de plus la droite qui joint leurs centres est précisément la tangente à Σ au point s. On aura donc le beau résultat démontré par Joachimsthal que les cercles (A_1A_2) et (Ω) ont pour axe radical la tangente à Σ au point s.

Pour achever la détermination du cercle Ω, il reste à trouver un troisième couple de points réciproques par rapport à Σ et Σ'. C'est ce qu'on peut faire d'une infinité de manières différentes. Soit, par exemple, F le pied de la perpendiculaire abaissée du point P sur la polaire de ce point relativement à Σ; P, F seront des points réciproques par rapport au système (Σ, Γ), puisque PF est la tangente à Γ au point P. On trouvera le point p de Σ qui correspond au point P de Γ en menant par le point A une parallèle à la polaire de P relativement à Σ. Pour trouver le point f de la seconde figure qui correspond au point F de la première, on mènera dans la conique Σ la corde $A_1 f'$ parallèle à la polaire de F; l'intersection de la droite sf' par la tangente à Σ au point p sera le point f cherché. Le cercle Ω coupera orthogonalement le cercle (pf), et se trouvera par cela complètement déterminé.

On pourrait aussi se servir de la méthode suivante. Qu'on fasse varier le point P sur la droite fixe SP, et qu'on prenne sur cette droite $SP' = -SP$. Les points s et p resteront fixes; les cercles Ω passeront toujours par les points d'intersection de la tangente au point s et du cercle $(A_1 A_2)$. Aux deux points P et P' correspondra le même cercle; de plus les cercles Ω varieront anharmoniquement avec les segments de l'involution $P_1 P'_1$, $P_2 P'_2$, etc. On en tirera la construction suivante. Soit π le centre du cercle Ω qui passe par le point p; et qu'on désigne par ρ l'intersection de SP par la polaire de P. Le centre du cercle Ω cherché sera le point d'intersection de la ligne des centres par une droite menée de P parallèlement à $\rho\pi$ *.

2. On sait qu'un grand nombre de problèmes cubiques et biquadratiques conduisent à l'étude des *correspondances* déterminées par les équations

$$x_1(A_1 y_1^2 + 2B_1 y_1 y_2 + C_1 y_2^2) + x_2(A_2 y_1^2 + 2B_2 y_1 y_2 + C_2 y_2^2) = 0, \quad . \ . \ . \quad (1)$$

$$x_1^2(A_1 y_1^2 + 2B_1 y_1 y_2 + C_1 y_2^2) + 2x_1 x_2(A_2 y_1^2 + 2B_2 y_1 y_2 + C_2 y_2^2) + x_2^2(A_3 y_1^2 + 2B_3 y_1 y_2 + C_3 y_2^2) = 0, \quad . \ . \ . \quad (2)$$

$$x_1(A_1 y_1^3 + 3B_1 y_1^2 y_2 + 3C_1 y_1 y_2^2 + D_1 y_2^3) + x_2(A_2 y_1^3 + 3B_2 y_1^2 y_2 + 3C_2 y_1 y_2^2 + D_2 y_2^3) = 0, \quad . \ . \ . \quad (3)$$

dans lesquelles on peut supposer que les quotients $\frac{x_1}{x_2}$, $\frac{y_1}{y_2}$ sont des rapports anharmoniques qui déterminent la position des points variables x et y, soit sur une droite, soit sur une conique. Une théorie complète de ces correspondances

* Note VIII (p. 55).

dépasserait de beaucoup les limites de ce mémoire. Il nous suffira de placer ici quelques observations qui sont d'une grande importance dans cette théorie.

(1.) Soient a, b, c trois points d'une conique déterminés par l'équation cubique

$$F = Ax_1^3 + 3Bx_1^2x_2 + 3Cx_1x_2^2 + Dx_2^3 = 0.$$

Soient aussi ABC le triangle circonscrit, L l'axe d'homologie des deux triangles ABC, abc; a', b', c' les points d'intersection des droites Aa, Bb, Cc par la conique. Alors les trois points $a'b'c'$ sont les points déterminés par le covariant cubique de F; et les deux points d'intersection de L par la conique sont déterminés par le covariant quadratique.

Supposons qu'on donne sur une droite trois points P, Q, R, et trois autres points p, q, r harmoniquement dérivés des premiers par rapport à un système F de trois points inconnus. Proposons-nous de déterminer (1) le point s harmoniquement dérivé d'un point donné S par rapport au système F, (2) les deux points covariants de ce système, (3) le système de deux points S_1S_2 harmoniquement dérivés d'un point donné s, (4) les trois points inconnus eux-mêmes. De ces problèmes le premier n'est que linéaire, le second et le troisième sont quadratiques, le quatrième est cubique. On projètera les points PQR, pqr sur une conique, en prenant pour centre de projection un point de la conique; nous désignerons les points projétés par les mêmes lettres. Soient $P'Q'R'$ les pôles des droites Pp, Qq, Rr relativement à cette conique. Le point λ, pôle de la droite L, satisfait à l'équation

$$\lambda . [P', P, Q', Q, R', R] = [P, p, Q, q, R, r];$$

donc ce point pourra être déterminé linéairement, puisqu'on pourra trouver deux sections coniques dont il sera le quatrième point d'intersection, les trois autres étant connus. Le point s sera déterminé linéairement par l'équation

$$\lambda . [P, Q, R, S] = [p, q, r, s];$$

de même, en supposant que s soit donné, la droite λS, déterminée par cette équation anharmonique coupera la conique aux deux points S_1S_2. Enfin, en prenant un point quelconque ω de la conique, et en faisant correspondre anharmoniquement les faisceaux $\lambda . [P, Q, R, \ldots]$ et $\omega . [p, q, r, \ldots]$, on aura une section conique S qui coupera Σ au point connu ω, et aux trois points inconnus. On pourra se passer, comme on voit, du tracé de la conique Σ, si l'on ne veut déterminer que les points s et λ, et la droite L.

Dans la solution du problème cubique on remarquera que le point ω peut

être pris à volonté sur la conique Σ; on pourra même déterminer d'avance la position de ce point de sorte qu'on puisse faire passer une circonférence de cercle par les quatre points ω, α, β, γ. Pour cela, on observera que les points ω correspondent anharmoniquement aux coniques du faisceau $(\alpha, \beta, \gamma, \lambda)$. On prendra trois positions du point ω, et l'on déterminera les points rectangulaires à l'infini appartenant aux axes principaux des coniques correspondantes $(\alpha, \beta, \gamma, \lambda, \omega)$; ou, plutôt, les trois points ρ harmoniquement conjugués à un point fixe par rapport à ces trois systèmes de points rectangulaires. Il est évident que les points ρ correspondront anharmoniquement aux points ω; donc, en désignant par σ le conjugué harmonique du point fixe par rapport aux points à l'infini appartenant aux axes principaux de Σ, le point ω sera déterminé linéairement par l'équation

$$[\rho_1, \rho_2, \rho_3, \sigma] = [\omega_1, \omega_2, \omega_3, \omega].$$

(2.) Dans l'équation (2), qu'on considérera relativement à une conique Σ, on fera varier le point y, ou ce qui revient au même, le rapport $y_1 : y_2$. Les cercles qui joignent les deux points déterminés par les valeurs correspondantes de $x_1 : x_2$ envelopperont une section conique X. Pareillement, on aura une section conique Y, enveloppe des droites joignant les points y correspondant à une même position de x. Soit θ un point quelconque de Σ; soient $\eta_1 \eta_2$ les deux points qui correspondent à θ, considéré comme appartenant à la série des x; $\xi_1 \xi_2$ les deux points correspondant au même point θ, considéré comme appartenant à la série des y. Qu'on prenne la corde $\theta_1 \theta_2$, conjuguée harmonique de $\sigma\theta$ par rapport aux deux droites $\eta_1 \eta_2$, $\xi_1 \xi_2$ se coupant au point σ. Cette corde sera la polaire de θ par rapport à une troisième section conique, que nous désignerons par Θ. Pour les deux coniques X et Y, les droites enveloppantes, ou, si l'on veut, les points de contact sur ces droites, correspondront anharmoniquement aux points de la conique Σ.

Il y a trois problèmes biquadratiques qui se présentent naturellement, quand on considère la correspondance doublement quadratique (2).

Trouver les quatre points (x) *pour lesquels les deux points* (y) *correspondants deviennent coincidents.*

Trouver les quatre points dont chacun représente deux points (y), *qui sont devenus coincidents.*

Trouver les points où le point (x) *coincide avec l'un des points* (y) *correspondants.*

Qu'on prenne sur la conique Σ les quatre points de contact des tangentes communes à Σ et X; chacun de ces points représentera deux points y devenus

coincidents. On en déduira les quatres points x, auxquels correspondent ces quatre points doubles, en se servant de la relation anharmonique que nous avons indiquée. Enfin le point x coincide avec l'un des points y correspondants aux quatres points de rencontre de Σ et Θ.

La théorie de l'équation (2) se simplifie, si elle est symétrique relativement aux deux séries de points x et y. En ce cas, les deux coniques X et Y coincident l'une avec l'autre, et avec la conique, polaire réciproque de Σ par rapport à Θ. Les points d'intersection des deux coniques X et Σ sont précisément les points x pour lesquels les points y correspondants deviennent coincidents ; et les points de contact sur Σ des tangentes communes de Σ et X sont ces doubles points. En supposant toujours que l'équation (2) soit symétrique, prenons les points $y_1 y_2$ correspondant à un point quelconque x : soit x' l'un de ces deux points ; l'un des deux points correspondant à x' sera x, l'autre sera un nouveau point x''. Déterminons successivement de la même manière les points x''', x'''', ... ; il peut arriver, comme on sait, qu'après un nombre fini d'opérations on retombe à la fin sur le point de départ x. Ces cas particuliers ont été beaucoup étudiés par les géomètres ; mais c'est surtout le cas où l'on aurait $x''' = x$, qui est important (ainsi que nous allons voir) pour la théorie des problèmes du troisième et du quatrième ordre.

Les problèmes linéaires et quadratiques qui se rattachent à l'équation (2) peuvent se résoudre en beaucoup de cas par les méthodes connues. Par exemple, l'on voit qu'étant donnés huit points x, et un point y correspondant à chacun de ces points, les deux points y correspondant à un point x quelconque doivent s'obtenir par une construction quadratique. Et, en effet, on peut opérer cette construction, en se servant des propriétés des courbes du quatrième ordre, ayant deux points doubles.

(3.) Considérons la correspondance définie par l'équation (3) ; et supposons que cette équation soit relative à une conique Σ. A chaque point x correspondront trois points de la série y : et l'on aura ainsi une série de triangles inscrits à Σ. Mais cette série de triangles sera en même temps circonscrite à une seconde conique ; elle sera de plus une série de triangles harmoniques par rapport à une troisième conique. Ainsi, cette série de triangles pourra être définie par une correspondance quadratique double, qui sera symétrique, et dans laquelle le troisième élément dérivé coincidera avec l'élément d'où l'on est parti. Il s'ensuit que la plupart des questions relatives à l'involution cubique définie par l'équation (3) pourront être réduites aux recherches analogues relatives à cette espèce particulière de correspondances quadratiques doubles *.

* Note IX (p. 59).

3. Nous désignerons par $[a, b, c, d]$ le quotient $\frac{ac}{ad} : \frac{bc}{bd}$, qui est un des rapports anharmoniques des quatre points a, b, c, d en ligne droite ; de même, nous représenterons par $P.[a, b, c, d]$ le rapport anharmonique $\frac{\sin aPc}{\sin aPd} : \frac{\sin bPc}{\sin bPd}$ des quatre droites Pa, Pb, Pc, Pd se coupant au même point P*. Cela posé, nous aurons les deux lemmes suivants.

Lemme I. Soient

$$Pp,\ Pp_1,\ Pp_2,\quad Qq,\ Qq_1,\ Qq_2,\quad Rr,\ Rr_1,\ Rr_2,$$

neuf droites données ; le lieu d'un point x, qui satisfait à l'équation

$$P.[x,\ p,\ p_1,\ p_2] \times Q.[x,\ q,\ q_1,\ q_2] \times R.[x,\ r,\ r_1,\ r_2] = a, \quad \ldots \quad (1)$$

a étant une constante, est une courbe cubique passant par les neuf points d'intersection des droites Pp_1, Qq_1, Rr_1 avec les droites Pp_2, Qq_2, Rr_2.

Démonstration. (1°). Le lieu du point x qui satisfait à l'équation

$$Q.[x,\ q,\ q_1,\ q_2] \times R.[x,\ r,\ r_1,\ r_2] = \mu, \quad \ldots \ldots \quad (2)$$

μ étant une constante, est une conique (μ) passant par les quatre points d'intersection des droites Qq_1, Rr_1 avec les droites Qq_2, Rr_2. Soit $R\rho$ le rayon du faisceau (R) qui satisfait à l'équation

$$R.[\rho,\ r,\ r_1,\ r_2] = \mu \ldots \quad \ldots \ldots \quad (3)$$

En divisant membre à membre l'équation (2) par l'équation (3), on aura

$$Q.[x,\ q,\ q_1,\ q_2] \times R.[x,\ \rho,\ r_1,\ r_2] = 1,$$

ou, ce qui revient au même,

$$R.[x,\ q,\ q_1,\ q_2] = R.[x,\ \rho,\ r_2,\ r_1],$$

équation qui démontre ce qui a été avancé.

---

* En général, si a, b, c, d sont quatre éléments quelconques, dont on peut définir le rapport anharmonique (par exemple, quatre points d'un même conique, ou quatre courbes d'un même faisceau) nous exprimerons ce rapport pas la formule $[a, b, c, d]$; et, toutes les fois qu'il sera nécessaire de distinguer entr'eux les divers rapports anharmoniques du même système de quatre éléments, la formule $[a, b, c, d]$ désignera pour nous le rapport anharmonique analogue au rapport $\frac{ac}{ad} : \frac{bc}{bd}$ de quatre points en ligne droite. Nous nous servirons, avec quelques géomètres, des parenthèses pous exprimer des courbes passant par des points donnés ; ainsi (a, b) sera la droite qui joint les points a, b ; (a, b, c, d, e) sera la conique des cinq points a, b, c, d, e. Si P est la base d'un faisceau de courbes d'ordre quelconque, (P, a) sera la courbe de ce faisceau qui passe par le point a ; et $P.[a, b, c, d]$ sera le rapport anharmonique des quatre courbes (P, a), (P, b), (P, c), (P, d).

(2°.) En donnant à μ des valeurs successives différentes, les coniques (μ) correspondantes, lieux des points x, seront toutes circonscrites au même quadrilatère. Or, ces coniques correspondront anharmoniquement aux valeurs de μ. En effet, supposons que Qx soit une direction fixe ; on sait que les coniques correspondront anharmoniquement aux points où elles coupent cette droite. En désignant par ρ, r_1, r_2 les points d'intersection de Qx par $R\rho$, Rr_1, Rr_2, la position du point x, appartenant à la conique (μ), sera déterminée par l'équation

$$[x, \rho, r_2, r_1] = Q \,.\, [x, q, q_1, q_2],$$

dont le second membre est une constante C. En multipliant les deux membres de cette équation par les deux membres de l'équation

$$\mu = [\rho, r, r_1, r_2],$$

on aura

$$[x, r, r_1, r_2] = \frac{\mu}{C},$$

d'où il s'ensuit que le rapport anharmonique de quatre points x est égal au rapport anharmonique des valeurs correspondantes de μ.

(3.) La position de la droite Px, qui satisfait à l'équation

$$P \,.\, [x, p, p_1, p_2] = \frac{a}{\mu},$$

variera anharmoniquement avec les valeurs de μ. Donc le lieu des points d'intersection d'une conique (μ) par la droite correspondant à la même valeur de μ sera une courbe cubique ; mais il est évident que ce sont précisément ces points d'intersection qui satisfont à l'équation (1). On voit d'ailleurs que la courbe cubique passera par cinq des neuf points d'intersection des droites données ; elle passera aussi par les quatre autres, puisque dans la démonstration on peut échanger entre eux les faisceaux (P), (Q), (R).

Lemme II. Soient S_1, S_2, S_3 trois courbes du même ordre ; A_1, A_2, A_3, B_1, B_2, B_3 les courbes des faisceaux (S_2, S_3), (S_3, S_1), (S_1, S_2) qui passent par les points α et β respectivement ; on aura l'équation

$$[S_2, S_3, A_1, B_1] \times [S_3, S_1, A_2, B_2] \times [S_1, S_2, A_3, B_3] = +1. \quad . \quad . \quad . \quad (3)$$

Démonstration. Puisqu'il y a toujours une courbe du faisceau (S_2, S_3) qui appartient en même temps au faisceau déterminé par deux courbes quelconques des faisceaux (S_3, S_1), (S_1, S_2), il y a une courbe commune aux faisceaux (S_2, S_3), (A_2, A_3). Mais cette courbe commune ne peut être autre que A_1, puisque A_1 est une courbe du faisceau (S_2, S_3), et qu'elle passe par α, un des points d'intersection

de A_2, A_3. Donc les trois courbes A_1, A_2, A_3 appartiennent au même faisceau ; le même raisonnement s'applique aux trois courbes B_1, B_2, B_3. Soient α et β les points où vont concourir les droites polaires d'un même point O par rapport à A_1, A_2, A_3, et à B_1, B_2, B_3 respectivement ; soient aussi σ_1, σ_2, σ_3 les points d'intersection de (α, β) par les polaires de O relativement à S_1, S_2, S_3 ; on aura évidemment

$$[S_2, S_3, A_1, B_1] = [\sigma_2, \sigma_3, \alpha, \beta]$$
$$[S_3, S_1, A_2, B_2] = [\sigma_3, \sigma_1, \alpha, \beta]$$
$$[S_1, S_2, A_3, B_3] = [\sigma_1, \sigma_2, \alpha, \beta],$$

valeurs qui satisfont identiquement à l'équation (3).

Avant de terminer ces préliminaires nous rappellerons qu'étant donné trois points sur chacune de deux cubiques, et en outre six des neuf points d'intersection des deux courbes, on trouve aisément les trois autres points d'intersection par une construction cubique qu'on doit à M. Chasles. Soient 1, 2, ..., 6 les six points d'intersection donnés, 7, 8, 9 les trois points cherchés ; on détermine linéairement deux coniques telles que (5, 6, 7, 8, 9), (4, 6, 7, 8, 9), dont les quatre points d'intersection sont le point connu 6, et les trois points cherchés. Pareillement, étant donné trois des points d'intersection d'une conique et d'une cubique, qu'on suppose déterminées toutes les deux par un nombre suffisant de points, on trouvera les trois autres points d'intersection par une construction cubique facile. Soient a, b, c les points donnés, d, δ, des points donnés de la cubique et de la conique respectivement, ω le point opposé au système a, b, c, d, relativement à la cubique. En désignant par x un point quelconque de la cubique, les coniques (a, b, c, d, x) et les droites (ω, x) se correspondront anharmoniquement. Soit ξ le quatrième point d'intersection de la conique donnée par (a, b, c, d, x) ; les deux faisceaux (δ, ξ), (ω, x) seront homographiques, et la conique, lieu des points d'intersections des rayons correspondants, coupera la conique donnée au point connu δ et, en outre, aux trois points cherchés.

4. Problème. *Étant donné treize des points d'intersection de deux courbes du quatrième ordre, trouver les trois autres.*

Nous supposerons que les treize points soient tels qu'on peut faire passer actuellement par ces points une vraie courbe du quatrième ordre. Nous exclurons donc absolument les cas où l'on aurait, soit cinq points en ligne droite, soit neuf points sur une même conique, soit treize points sur une même cubique. Mais, afin de simplifier la discussion générale, nous en exclurons aussi, pour le moment, les cas où l'on aurait

(1) Neuf points formant la base d'un faisceau de cubiques.

(2) Huit points sur une même conique.

(3) Onze points sur une même cubique.

(4) Quatre points en ligne droite.

Pour tous ces cas la solution générale se simplifie plus ou moins ; nous les considérerons séparément plus tard. Enfin, nous supposerons que les treize points soient tous réels, et tous différents ; nous reviendrons ci-après sur les cas où l'on aurait des points imaginaires.

Soient 1, 2, 3, ..., 13 les treize points donnés, 14, 15, 16 les points qu'il s'agit de trouver.

Prenons six points quelconques des treize points, par exemple les points 8, 9, 10, 11, 12, 13 ; nous allons montrer comment on peut déterminer la cubique qui passe par ces six points, et par les trois points inconnus.

Pour cela, nous prenons un quelconque des six points que nous avons choisis, par exemple le point 8 ; nous le joignons aux sept points 1, ..., 7, et nous considérons le système des huit points 1, ..., 8 comme formant la base P_8 d'un faisceau de courbes cubiques. Déterminons le point p_8, de sorte que les cinq droites

$$(p_8, 9),\ (p_8, 10),\ (p_8, 11),\ (p_8, 12),\ (p_8, 13)$$

correspondent anharmoniquement aux courbes cubiques

$$(P_8, 9),\ (P_8, 10),\ (P_8, 11),\ (P_8, 12),\ (P_8, 13).$$

La détermination du point p_8 se fera linéairement par une construction sur laquelle nous reviendrons plus tard ; nous dirons que ce point est *biquadratiquement opposé* aux points 1, ..., 8 de la base P_8. La courbe du quatrième ordre, qu'on peut faire passer par les treize points et par le point p_8, aura pour faisceaux générateurs le faisceau de droites (p_8), et le faisceau de courbes cubiques (P_8) ; de plus, les droites

$$(p_8, 14),\ (p_8, 15),\ (p_8, 16)$$

du premier faisceau correspondront aux courbes cubiques

$$(P_8, 14),\ (P_8, 15),\ (P_8, 16)$$

du second faisceau, puisque 14, 15, 16 sont des points de la courbe du quatrième ordre. Substituons successivement au point 8 deux autres points du système de six points, par exemple les points 9 et 10 ; soient P_9, P_{10} les bases cubiques qu'on aura ainsi, p_9, p_{10} les points biquadratiquement opposés à ces bases ; nous allons

voir que les trois points opposés p_8, p_9, p_{10}, et les trois points d'intersection des trois couples de droites

$$p_9(10),\ p_{10}(9);\quad p_{10}(8),\ p_8(10);\quad p_8(9),\ p_9(8),$$

sont des points de la cubique cherchée, qui sera dès lors complètement déterminée puisqu'on en connaîtra douze points. Considérons la courbe cubique, lieu des points x qui satisfont à l'équation

$$p_8 . [x, 11, 9, 10] \times p_9 . [x, 11, 10, 8] \times p_{10} . [x, 11, 8, 9] = +1.$$

D'après le lemme I, cette courbe passe par les points p_8, p_9, p_{10}, 8, 9, 10, et par les trois points d'intersection des trois couples de droites

$$(p_9, 10),\ (p_{10}, 9);\quad (p_{10}, 8),\ (p_8, 10);\quad (p_8, 9),\ (p_9, 8);$$

elle passe en outre par le point 11, puisque chacun des trois rapports anharmoniques devient égal à l'unité positive si l'on fait coïncider x avec ce point. Mais les points 12, 13, 14, 15, 16 appartiennent aussi à la même courbe. En effet, soit ξ un quelconque de ces points ; d'après la relation anharmonique qui subsiste entre les faisceaux (p_8), (P_8) ; (p_9), (P_9) ; (p_{10}), (P_{10}), on aura

$$\begin{aligned} p_8 . [\xi, 11, 9, 10] &= P_8 . [\xi, 11, 9, 10], \\ p_9 . [\xi, 11, 10, 8] &= P_9 . [\xi, 11, 10, 8], \\ p_{10} . [\xi, 11, 8, 9] &= P_{10} . [\xi, 11, 8, 9]. \end{aligned}$$

Mais le produit des seconds membres de ces équations est l'unité positive ; comme il résulte du lemme II, en y écrivant

$$\begin{aligned} S_1 &= (P_9, 10) = (P_{10}, 9), \\ S_2 &= (P_{10}, 8) = (P_8, 10), \\ S_3 &= (P_8, 9) = (P_9, 8). \end{aligned}$$

Il résulte de ce qui précède que la cubique, qui passe par les neuf points 8, ..., 16, passe aussi par les neuf points biquadratiquement opposés aux systèmes de huit points, qu'on obtient en joignant successivement aux sept points 1, ..., 7 chacun des huit points 8, ..., 16 ; et par les trente-six points d'intersection des couples de droites (p_α, β), (p_β, α), en désignant par α, β deux nombres inégaux de la série 9, ..., 16. De ces cinquante-quatre points, on en connaîtra ving-sept, qui ne dépendent pas des points inconnus 14, 15, 16.

Pour avoir une autre cubique passant par ces trois points, nous remarquons que, par hypothèse, la cubique C_1 des neuf points 8, ..., 16, ne peut pas passer

par tous les sept points 1, ..., 7. Soit donc 7 un de ces points qui n'appartient pas à C_1; on échangera entr'eux dans la construction précédente le point 7 et un point quelconque 8 des six points 8, ..., 13; et l'on déterminera ainsi la cubique C_2 des neuf points 7, 9, ..., 16. Les cinq points 9, ..., 13 seront des points communs aux deux courbes C_1 et C_2; le point biquadratiquement opposé aux points 1, ..., 8 sera un sixième point commun; enfin, les points qu'il s'agit de trouver seront précisément les trois autres points communs. Puisque par hypothèse les huit points 1, ..., 8 n'appartiennent pas tous à une même conique, on pourra dire autant des huit points 9, ..., 16. Soient 11, 12, 13 trois des points 9, ..., 13 qui n'appartiennent pas à une même conique avec les points inconnus. Qu'on détermine les deux coniques (12, 13, ..., 16) et (11, 13, ..., 16), se coupant au point connu 13; les trois autres points d'intersection de ces courbes seront finalement les points cherchés.

On remarquera que les deux courbes du quatrième ordre dont nous nous sommes servis dans la démonstration précédente, ne sont pas deux courbes quelconques du faisceau déterminé par les treize points donnés. Chacune des deux courbes est assujettie à passer par le neuvième point appartenant à la base cubique formée par un système de huit points choisis parmi les treize points donnés *.

5. La solution précédente dépend essentiellement de la détermination des points biquadratiquement opposés aux divers systèmes de huit points qu'on peut former avec les treize points donnés. Soit $P=[1, 2, 3, ..., 8]$ l'un quelconque de ces systèmes; pour avoir le point opposé biquadratiquement à P, il nous faudra avant tout un système de cinq points, ou de cinq droites, qui correspondent anharmoniquement aux cubiques $(P, 9)$, $(P, 10)$, $(P, 11)$, $(P, 12)$, $(P, 13)$. À cet effet, on pourrait se servir, comme on sait, soit des tangentes à ces courbes en un point quelconque de la base P, soit de leurs points d'intersection par une des droites qui joignent deux des points P, soit enfin de cinq droites, polaires d'un même point par rapport aux cinq courbes. Mais nous préférons la méthode suivante, qui conduit à des opérations moins pénibles. On choisira parmi les points P un système Q de quatre points quelconques 1, 2, 3, 4, et l'on déterminera pour chacune des cinq cubiques le point opposé au système Q. Pour cela on déterminera la conique Γ qui passe par les quatre points 5, 6, 7, 8, et qui admet le rapport anharmonique $Q.[5, 6, 7, 8]$; puis, on prendra sur cette conique des points 9′, 10′, 11′, 12′, 13′, tels qu'on ait

$$[5, 6, 7, 8, 9', 10', 11', 12', 13'] = Q.[5, 6, 7, 8, 9, 10, 11, 12, 13];$$

* Note X (p. 63).

les intersections de Γ par les droites

$$(9, 9'), (10, 10'), (11, 11'), (12, 12'), (13, 13')$$

seront les points opposés à Q, appartenant respectivement aux cubiques

$$(P, 9), (P, 10), (P, 11), (P, 12), (P, 13).$$

Nous les désignerons par $\omega_9, \omega_{10}, \omega_{11}, \omega_{12}, \omega_{13}$, et nous nous en servirons pour un système de points correspondants anharmoniquement aux cinq cubiques. Ensuite, on déterminera la conique Σ qui passe par quatre points quelconques 9, 10, 11, 12 des cinq points 9, 10, 11, 12, 13, et qui admet le rapport anharmonique $[\omega_9, \omega_{10}, \omega_{11}, \omega_{12}]$. Soit $13''$ le point de cette conique qui satisfait à l'équation

$$[9, 10, 11, 12, 13''] = [\omega_9, \omega_{10}, \omega_{11}, \omega_{12}, \omega_{13}];$$

le point d'intersection de Σ par la droite $(13, 13'')$ est le point opposé biquadratiquement au système P. On voit que la construction revient au fond à la construction si connue du point opposé à un système de quatre points appartenant à une courbe cubique.

La construction ne réussit pas si trois des points Q sont en ligne droite, mais elle ne devient que plus facile si trois des points 5, 6, 7, 8, ou bien trois des points 9, 10, 11, 12, 13 sont en ligne droite, puisqu'alors l'une ou l'autre des coniques Γ, Σ est remplacée par un système de deux droites. On peut donc toujours faire en sorte que le système Q ne contienne pas trois points en ligne droite. Cependant, s'il y avait quatre points en ligne droite parmi les points P, on ferait bien de les prendre pour le système Q, puisque en ce cas la détermination du point biquadratiquement opposé au système P se réduirait tout simplement à la détermination du point opposé au système des quatre points 5, 6, 7, 8, relativement à la courbe cubique $(5, 6, \ldots, 13)$. Encore, s'il y avait quatre des cinq points 9, 10, 11, 12, 13 en ligne droite, la construction précédente ne serait plus applicable. En effet, dans ce cas il n'y a aucun point opposé biquadratiquement au système P, à moins que la condition

$$P.[9, 10, 11, 12] = [9, 10, 11, 12]$$

ne soit satisfaite par les quatre points en ligne droite. En supposant que cette condition eût lieu, on déterminerait sur la droite $(9, \ldots, 12)$ le point $13'$ qui satisfait à l'équation

$$P.[9, 10, 11, 12, 13] = [9, 10, 11, 12, 13'];$$

et on trouverait que tout point de la droite $(13, 13')$ aurait la propriété caractéristique d'un point opposé biquadratiquement au système P.

La construction cesse encore d'être applicable si les points P et trois des points 9, ..., 13 appartiennent à une même cubique. Elle deviendrait indéterminée si l'un des cinq points 9, ..., 13 était le neuvième point appartenant à la base cubique P. Ainsi en supposant que 13 appartînt à cette base, tout point de la conique Σ serait opposé biquadratiquement au système P. C'est ce qui arriverait aussi si l'on avait la relation

$$P.[9, \ldots, 13]=[9, \ldots, 13],$$

ce dernier symbole se rapportant à la conique qu'on peut mener par les cinq points.

6. Nous allons maintenant revenir sur les cas particuliers que nous avons exclus de la discussion générale (Art. 4). Dans tous ces cas, comme on a pu voir par ce qui précéde, on pourra simplifier la détermination des points opposés biquadratiquement à certains systèmes de huit points, si toutefois ces points ne cessent pas d'exister.

(1.) Supposons que les neuf points 1, 2, ..., 9 forment la base P d'un faisceau de courbes cubiques. La conique Σ, qui passe par les points 10, ..., 13, et admet le rapport anharmonique $P.[10, 11, 12, 13]$ passera aussi par les points 14, 15, 16. Car on pourra faire passer par les treize points et par un point quelconque σ de cette conique, une courbe S du quatrième ordre qui aura pour faisceaux générateurs le faisceau de cubiques (P), et le faisceau de droites (σ). Les courbes Σ et S se couperont en huit points, dont σ, 10, 11, 12, 13 seront cinq ; nous allons voir que les trois autres seront précisément les points inconnus 14, 15, 16. Soit, en effet, x un des trois points d'intersection de Σ et S, autres que σ, 10, 11, 12, 13 ; en prenant x pour centre du faisceau générateur, on aura une courbe X du quatrième ordre, qui passera par x et par les treize points, mais qui ne pourra pas coïncider avec S, puisque, en désignant par a un point quelconque qui n'appartient pas à la conique Σ, les deux rapports anharmoniques

$$x.[10, 11, 12, a], \quad \sigma.[10, 11, 12, a]$$

ne sauraient être égaux. Donc le point x est bien un des trois points 14, 15, 16, puisq'il appartient en même temps aux deux courbes S et X. On arriverait au même résultat en s'appuyant sur la proposition générale de l'article 4, d'où l'on conclurait qu'un point quelconque de Σ appartient à la cubique déterminée par les sept points 10, ..., 16 et deux quelconques des points P ; c'est-à-dire que cette cubique est composée de Σ et de la droite qui joint les deux points. Pour compléter la solution du problème, on déterminera par la méthode générale une cubique qui passera par les trois points inconnus, par trois des points 10, ..., 13,

et par trois des points P. Trois des points d'intersection de cette cubique par Σ seront connus ; donc on trouvera les trois autres par la construction cubique que nous avons déjà indiquée.

Nous ferons remarquer que si les points 10, ..., 13 appartiennent, deux à deux, à deux courbes cubiques du faisceau (P), la solution du problème sera linéaire. En effet, soient $(P, 10, 11)$, $(P, 12, 13)$ les deux cubiques ; on pourra considérer la droite (12, 13) et la cubique $(P, 10, 11)$, prises ensemble, comme une courbe du quatrième ordre passant par les treize points ; pareillement la droite (10, 11) et la cubique $(P, 12, 13)$ composeront une autre courbe du quatrième ordre passant par les mêmes points ; donc le point d'intersection de (10, 11) et (12, 13) sera un des points cherchés ; et l'un des deux autres sera le troisième point d'intersection de la cubique $(P, 10, 11)$ par (10, 11), point qu'on sait déterminer linéairement.

La construction ne sera que quadratique, si trois des points qui n'appartiennent pas à P sont en ligne droite. Soient 10, 11, 12 trois points d'une même droite λ ; soit aussi α le point de λ qui satisfait à l'équation

$$[10, 11, 12, \alpha] = P.[10, 11, 12, 13];$$

la conique Σ sera remplacée par l'ensemble des deux droites λ, et $(13, \alpha)$. Qu'on détermine par la méthode générale ci-dessus une cubique qui passe par les trois cherchés, par trois des points P, par deux des points 10, 11, 12, et enfin par 13 ; un des points cherchés sera le troisième point d'intersection de la cubique par λ ; celui-ci se trouvera linéairement ; les deux autres seront les deux points d'intersection, autres que 13, de la droite $(13, \alpha)$ par la cubique ; on les aura par une construction quadratique.

Enfin, si tous les quatre points 10, 11, 12, 13 appartiennent à la même droite, il n'y aura aucune vraie courbe du quatrième ordre qui pourra passer par les treize points donnés, à moins que la condition

$$P.[10, 11, 12, 13] = [10, 11, 12, 13]$$

ne soit vérifiée ; donc tous les fois que cette relation ne subsistera pas, ce cas sera un de ceux que nous devons rejeter. De plus, quand même la condition se trouverait réalisée, on pourra faire abstraction du point 13, puisque toute courbe du quatrième ordre qui passe par les points 1, ..., 12, passera nécessairement par ce point. Il faudra donc qu'un quatorzième point soit donné pour que les seize points soient complètement déterminés ; mais ce point étant donné, les deux autres se trouveront par la construction quadratique précédente.

Nous avons déjà remarqué qu'en désignant par P le système 1, ..., 8, tout

point appartenant à la conique (9, ..., 13) est opposé biquadratiquement à P, si l'on a la relation

$$P.[9, 10, 11, 12, 13]=[9, 10, 11, 12, 13].$$

Ce cas se réduit immédiatement à celui que nous venons de traiter. En effet, le neuvième point appartenant à la base cubique P, est évidemment un des trois points cherchés, puisqu'il est un point commun à deux courbes du quatrième ordre, passant toutes les deux par les treize points, et ayant le même faisceau générateur de courbes cubiques (P), mais ayant des points différents biquadratiquement opposés à ces faisceaux. Donc on trouvera linéairement un des trois points 14, 15, 16; on obtiendra les autres par une construction quadratique, puisqu'on connaîtra quatre des points d'intersection de la conique (9, ..., 13) par une cubique qu'on fera passer par les deux points inconnus, par quatre des points 9, ..., 13, et par trois des points de la base cubique.

(2.) Supposons que les huit points 1, ..., 8 appartiennent tous à une même conique (P). La conique $\sigma=(9, 10, 11, 12, 13)$ sera une première conique passant par les points inconnus 14, 15, 16. Soit C une cubique déterminée par les trois points inconnus, par trois des points 9, ..., 12, et par trois des points P. On pourra déterminer cette cubique par la méthode générale, et puisqu'on connaîtra trois des points d'intersection de C par σ, on pourra trouver une seconde conique σ', qui coupera σ en quatre points, dont un sera connu d'avance, tandis que les autres seront les points cherchés.

(3.) Supposons que onze des treize points appartiennent à une même cubique (P); soient 1, 2, ..., 11 ces onze points, que nous désignerons par P. On sait que toute courbe du quatrième ordre, qui passe par onze points d'une cubique, rencontre la cubique en un douzième point fixe. Ce point fixe sera un des trois points cherchés; nous le désignerons par 14; il pourra se déterminer linéairement; de plus, cette détermination dépendra uniquement des onze points P, et nullement des points 12, 13. On pourra donc substituer à ces deux points deux autres points quelconques choisis à volonté, pourvu qu'ils ne soient pas situés sur la cubique (P). Nous prendrons actuellement, au lieu de 12 et 13, deux points 12′ et 13′ qui forment avec sept des onze points les neuf points basiques Q d'un faisceau de courbes cubiques, auquel la cubique (P) n'appartient pas. Soient 1, 2, ..., 7 les sept points; on prendra pour 12′ un point quelconque qui n'appartient pas à (P); 13′ sera le neuvième point appartenant à la base cubique 1, ..., 7, 12′, mais il ne sera pas nécessaire de le trouver, quisqu'on n'en fait aucun usage dans la construction. On déterminera la conique qui passe par les quatre points 8, 9, 10, 11, et qui admet le rapport anharmonique $Q.[8, 9, 10, 11]$: cette co-

nique passera par le point 14 cherché, puisqu'elle doit passer (Art. 6, 1) par les trois points qui complètent la base biquadratique 1, ..., 11, 12′, 13′. En échangeant entre eux un des quatre points et un des sept points, on aura une seconde conique, qui coupera la première en trois points connus d'avance; le quatrième point d'intersection sera le point 14. Pour trouver les points 15 et 16, on considérera un système de huit points Ω, composé des deux points 12 et 13, et de six points quelconques des points P. Soient 1, ..., 6 ces six points, et désignons par ω le point biquadratiquement opposé a Ω. La cubique (P) et la droite (12, 13) composent une courbe du quatrième ordre passant par les treize points donnés. Donc les points 15, 16 sont les deux points d'intersection (autres que 12, 13) de la droite (12, 13) par la courbe du quatrième ordre qui a pour faisceaux générateurs le faisceau de cubiques (Ω) et le faisceau de droites (ω). Donc enfin les points 15, 16 sont les points doubles des deux divisions homographiques déterminées par les deux faisceaux sur la droite (12, 13).

Si douze des points donnés se trouvaient sur une même cubique, il faudrait qu'un quelconque de ces douze points fût déterminé par les onze autres de la manière que nous avons indiquée; autrement on ne pourrait faire passer aucune vraie courbe du quatrième ordre par ces points. Mais en ce cas il faudrait aussi qu'un quatorzième point fût donné, afin de déterminer complètement le système des seize points; alors la construction des points 15 et 16 serait la même que ci-dessus.

(4.) Lorsque quatre des treize points sont en ligne droite, on aura tout d'abord une cubique passant par les trois points inconnus, puisque ces points appartiendront évidemment à la cubique des neuf autres points. Mais on peut aussi opérer de la manière suivante, sans faire usage de cette cubique. Soient 10, 11, 12, 13 les quatre points en ligne droite; les points p_{10}, p_{11}, p_{12}, p_{13} pourront se déterminer par la méthode générale de l'article 5. Ces points, ainsi que les points d'intersection des droites $(p_{10}, 11)$, $(p_{11}, 10)$, etc., appartiendront à la conique (8, 9, 14, 15, 16), qui se trouvera ainsi complètement déterminée. De même on pourra déterminer la conique (7, 9, 14, 15, 16), dont les intersections avec la première conique feront connaître la solution du problème.

7. Il y a encore quelques cas particuliers du problème qui ne sont pas dépourvus d'intérêt, mais dont la discussion, d'ailleurs très facile, dépasserait les limites que nous nous sommes prescrites. Mais nous ne saurions nous dispenser de placer ici les observations suivantes qui serviront à éclaircir la solution générale.

(1.) La détermination de la cubique (8, ..., 16) exige la connaissance de deux

seulement des points $p_8, p_9, \ldots, p_{13}$. En effet, quand on aura trouvé les deux points p_8 et p_9 on aura neuf points de la cubique, puisque le point d'intersection des droites $(p_8, 9)$, $(p_9, 8)$ appartient aussi à cette courbe. De plus, tant que le problème reste cubique, il ne peut arriver que les neuf points forment la base d'un faisceau de courbes cubiques. Pour que le point d'intersection de $(p_8, 9)$, $(p_9, 8)$ fût le neuvième point appartenant à la base cubique 8, ..., 13, p_8, p_9, il faudrait que les six points p_8, 9, ..., 13 appartinssent à une même conique. Or la conique $(p_8, 10, \ldots, 13)$ n'est autre que la conique qui satisfait à la relation

$$P_8.[10, 11, 12, 13]=[10, 11, 12, 13];$$

en outre, x étant le point de cette conique qui vérifie l'équation

$$P_8.[9, 10, 11, 12, 13]=[x, 10, 11, 12, 13],$$

p_8 sera le second point d'intersection de la droite $(x, 9)$ avec la conique. Donc, si le point 9 appartient lui-même à la conique, p_8 viendra se confondre avec 9, mais la position limite de la droite qui joindra ces deux points coincidents sera toujours la droite $(x, 9)$; d'où il s'ensuit que les six points p_8, 9, ..., 13 (dont les deux premiers sont coincidents) ne peuvent pas être censés appartenir à une même conique, à moins que le point x ne coincide avec 9. Mais si cela arrivait, l'équation

$$P_8.[9, 10, 11, 12, 13]=[9, 10, 11, 12, 13]$$

serait satisfaite ; c'est-à-dire, le neuvième point appartenant à la base P_8 serait un des trois points cherchés, et la détermination des deux autres ne serait que quadratique.

On conclura aussi, de ce qui vient d'être dit, que si p_8, l'un des deux points opposés qu'on aura à déterminer, venait à coincider avec l'un des points 9, ..., 10, la cubique (8, ..., 16) n'en serait pas moins complètement déterminée, puisqu'on aurait remplacé deux points par une tangente et son point de contact.

Nous ajouterons que si les deux points p_8 et p_9, tout en restant déterminés l'un et l'autre, venaient à se confondre en un seul point p, ce point serait un des trois points cherchés ; puisqu'en considérant successivement les deux bases P_8 et P_9 (qui auraient le même point opposé p, mais qui ne pourraient pas être identiques, parceque nous avons supposé que la position du point opposé p n'est pas indéterminée), on aurait deux courbes du quatrième ordre, passant par les treize points, et se coupant en outre au point p.

Donc on conclura généralement qu'afin d'avoir les deux cubiques, dont on a besoin pour déterminer les trois points cherchés, il suffira de trouver trois points

opposés biquadratiquement à trois systèmes de huit points, choisis convenablement parmi les treize points donnés.

(2.) Puisque les deux courbes cubiques (8, ..., 16) et (7, 9, ..., 16) se coupent au point p_8, il est évident que p_8 est le neuvième point qui appartient à la base cubique 9, ..., 16. Nous aurons donc le théorème que voici :—

'Si l'on partage les seize points d'une base biquadratique en deux systèmes de huit points, le point biquadratiquement opposé à l'un de ces systèmes appartient en même temps à la base cubique déterminée par l'autre système. La courbe du faisceau qui passe par le neuvième point appartenant à l'un des deux systèmes, passe aussi par le neuvième point appartenant à l'autre système.'

8. Nous allons maintenant supposer que quelques uns des treize points deviennent imaginaires. Il suffira de considérer les deux cas (1) où l'on n'aurait que trois points réels, (2) où l'on n'aurait qu'un seul point réel. Pour abréger, nous supposerons que la position des treize points soit tout-à-fait générale, et nous ne nous occuperons pas encore des circonstances spéciales que nous avons considérées dans l'article 6.

(1.) Supposons que 1, 2 ; 3, 4 ; 5, 6 ; 10, 11 ; 12, 13 soient des dyades de points imaginaires, mais 7, 8, 9 soient des points réels. La détermination des points a, b, c biquadratiquement opposés aux bases cubiques (1, ..., 6, 8, 9), (1, ..., 6, 9, 7), (1, ..., 6, 7, 8), que nous désignerons par A, B, C, se fera à peu près comme si les treize points étaient tous réels. En effet, d'après ce que nous avons dit dans la première partie de ce mémoire, on saura déterminer (1) la conique Γ de l'article 5 ; (2) le point réel ω_9 et les dyades $\omega_{10}\omega_{11}$, $\omega_{12}\omega_{13}$ appartenant à cette conique et correspondant anharmoniquement à la cubique réelle $(C, 9)$ et aux cubiques imaginaires conjuguées $(C, 10)$, $(C, 11)$ et $(C, 12)$, $(C, 13)$; (3) la conique Σ qui passe par les points 10, ..., 13, et qui satisfait à l'équation

$$[10, 11, 12, 13] = [\omega_{10}, \omega_{11}, \omega_{12}, \omega_{13}] = C.[10, 11, 12, 13];$$

(4) le point 9′ de cette conique qui vérifie la relation

$$[9', 10, 11, 12, 13] = [\omega_9, \omega_{10}, \omega_{11}, \omega_{12}, \omega_{13}] = C.[9, 10, 11, 12, 13];$$

(5) enfin, le point c cherché, où la droite (9, 9′) rencontre pour la seconde fois la conique Σ. Les points a, b, c une fois trouvés, les cubiques (8, 9, 10, .., 16), (9, 7, 10, ..., 16), (7, 8, 10, ..., 16), dont deux suffisent pour notre but, seront complètement déterminées. En effet, on connaîtra neuf points, dont trois sont réels, de chacune de ces courbes ; et nous avons vu que ces neuf points ne peuvent pas appartenir à une même base cubique.

(2.) On étendra la solution au cas où l'on n'aurait qu'un seul point réel 7, au

moyen des principes généraux que nous avons établis dans la première partie. La cubique (8, ..., 16) sera réelle ; la détermination de cette courbe se réduira à celle des points b, c qui seront des points imaginaires conjugués. Pour les trouver il faudra substituer dans la solution précédente la dyade 8, 9 aux deux points réels que nous avons désignés par les mêmes nombres. Quoiqu'en cette solution il soit question de points imaginaires, elle ne consiste actuellement que d'une certaine suite d'opérations linéaires, portant sur des points et des droites réelles. Donc, en substituant aux points réels 8, 9 la dyade de points imaginaires, on parviendra à déterminer linéairement les deux points imaginaires c, b biquadratiquement opposés aux systèmes 1, ..., 7, 8 et 1, ..., 7, 9 ; il est d'ailleurs évident que ces deux points appartiendront à la même dyade, dont on connaîtra l'homologie avec la dyade 8, 9. Le centre d'homologie des deux dyades appartiendra lui-même à la cubique cherchée ; il sera le seul point réel qu'on en connaîtra. Pour éviter la considération de courbes cubiques imaginaires, on pourra substituer successivement à la dyade 8, 9 les dyades 1, 2 et 3, 4 ; on aura ainsi trois courbes cubiques réelles se coupant en quatre points connus. On déterminera, par la méthode de M. Chasles, les trois coniques dont chacune passe par les cinq points d'intersection inconnus de deux de ces trois courbes. Ces coniques se couperont aux trois points cherchés, qu'on pourra dès lors déterminer par une construction cubique.

APPENDICE.

Note I (p. 3).

Descartes, *Géométrie*, livre troisième (*Œuvres* de Descartes, éd. Cousin, vol v. p. 409).

'Or, quand on est assuré que le problème proposé est solide, soit que l'équation par laquelle on le cherche monte au carré de carré, soit qu'elle ne monte que jusques au cube, on peut toujours en trouver la racine par l'une des trois sections coniques, laquelle que ce soit, ou même par quelque partie de l'une d'elles, tant petite qu'elle puisse être, en ne se servant au reste que de lignes droites et de cercles. Mais je me contenterai de donner une règle générale pour les trouver toutes par le moyen d'une parabole, à cause qu'elle est en quelque façon la plus simple.'

Note II (p. 3).

De la Hire, *La construction des équations analytiques*, opuscule qui fait partie (pp. 297–452) des *Nouveaux éléments des sections coniques*, Paris, 1679. Le problème des normales d'une section conique sert à l'auteur pour exemple de la méthode algébrique générale.

Maclaurin, *Treatise of Algebra*, London, 1748, p. 352.

Joachimsthal, *Journal de Crelle*, vol. xxvi, p. 172, vol. xlviii, p. 377. Voici la critique que fait Joachimsthal de la solution qui lui est propre, et de celle qu'il attribue à De la Hire.

'On sait que chaque problème qui dépend d'une équation du quatrième degré, se résout par la règle et le compas, en supposant une seule section conique complètement décrite. C'est sur ce principe que repose la solution du problème en question, due à De la Hire. Tandis que le géomètre grec qui a traité le premier cette question, Apollonius de Perge, se sert, outre la conique donnée, d'une hyperbole equilatère, De la Hire ne fait usage que d'une circonférence de cercle, dont les intersections avec la courbe ont, à un facteur près, les mêmes abscisses que les pieds des normales. Mais cette méthode offre plusieurs inconvénients qu'il est bon de signaler. En premier lieu, telle que De la Hire l'a représentée, sa

méthode n'est qu'un résultat de calcul, et ne se rattache à aucune proposition de géométrie; ensuite le choix de l'inconnue comporte nécessairement une ambiguité, et, en dernier lieu, les formules qui déterminent la position et le rayon de la circonférence sont trop compliquées pour se prêter aisément aux constructions graphiques. On adressera peut-être, et à juste titre, ce dernier reproche également à la nouvelle solution qu'on va lire; et néanmoins je n'hésite pas à la publier; car, quoiqu'elle ne soit pas encore une solution définitive, les propositions si simples sur lesquelles elle repose, pourront servir de point de départ pour arriver à une solution purement géométrique du problème dont il s'agit.'

La solution, dont parle ici Joachimsthal, est due non pas à De la Hire, mais à M. E. Catalan. Il est vrai que M. Catalan ne parle que d'une reproduction avec quelques simplifications de la solution ancienne (*Nouvelles Annales de Mathématiques*, per MM. Terquem et Gerono, vol. vii, p. 332); mais il est pervenu à une solution qui est entièrement différente de celle de De la Hire, et qui nous a paru plus élégante, et surtout plus naturelle. Nous n'avons pas cherché l'interprétation géométrique des formules analytiques un peu compliquées dont s'est servi De la Hire; mais nous sommes parvenus à traiter par les méthodes de la géométrie pure la solution de M. Catalan, aussi bien que celle de Joachimsthal. (Voir la note VIII.)

Note III (p. 7).

Soient $p_1 p_2$, $q_1 q_2$ deux couples de l'involution qui détermine une dyade donnée $\lambda_1 \lambda_2$; nous dirons que cette dyade est représentée par $[p_1 p_2, q_1 q_2]$. Soit encore $[x_1 x_2, y_1 y_2]$ une représentation d'une seconde dyade $\mu_1 \mu_2$; et supposons que l'équation anharmonique

$$[p_1, p_2, q_1, q_2] = [x_1, x_2, y_1, y_2]$$

soit satisfaite. Cette équation entraine nécessairement l'une ou l'autre des deux équations suivantes

$$[p_1, p_2, q_1, q_2, \lambda_1, \lambda_2] = [x_1, x_2, y_1, y_2, \mu_1, \mu_2],$$
$$[p_1, p_2, q_1, q_2, \lambda_2, \lambda_1] = [x_1, x_2, y_1, y_2, \mu_1, \mu_2];$$

mais il importe d'observer que ces équations ne sauraient avoir lieu toutes les deux ensemble. Selon que la première ou la seconde équation est satisfaite, nous dirons que les éléments imaginaires $\lambda_1 \lambda_2$, $\mu_1 \mu_2$, ou bien les éléments imaginaires $\lambda_2 \lambda_1$, $\mu_1 \mu_2$, sont représentés homographiquement par les systèmes $[p_1 p_2, q_1 q_2]$, $[x_1 x_2, y_1 y_2]$. D'après la définition que nous avons donnée de l'homologie des dyades, il est évident que lorsqu'on connaît l'homologie de deux dyades, on connaît en même temps deux représentations homographiques des éléments homologues de ces dyades. Et, réciproquement, il résulte de la solution du problème fondamental de l'article II, que l'on peut trouver linéairement le centre ou l'axe d'homologie de deux dyades de même espèce, dont les axes ou les centres ne sont pas coincidents, lorsqu'on connaît deux représentations homographiques des éléments imaginaires qu'on veut regarder comme homologues dans ces deux dyades. De plus, on verra ci-après qu'en désignant par $a_1 a_2$, $b_1 b_2$, $c_1 c_2$, trois dyades quelconques, on peut déterminer linéairement deux représentations homographiques de $b_1 b_2$, $c_1 c_2$, lorsqu'on connaît deux représentations homographiques de $a_1 a_2$, $b_1 b_2$, et deux représentations homographiques de $a_1 a_2$, $c_1 c_2$. D'après cela, on peut se servir des représentations homographiques pour définir l'homologie des dyades, en disant que l'homologie de deux dyades est donnée, lorsqu'on connaît une

représentation homographique de ces dyades. Cette manière de définir l'homologie des dyades a le double avantage de ne pas exiger l'emploi de dyades auxiliaires, ni dans le cas de deux dyades dont les axes ou les centres sont coincidents, ni dans le cas de deux dyades d'espèce différente, et de se prêter facilement à la théorie des dyades de droites qui ne se rencontrent pas dans l'espace.

Ce que nous venons de dire se rattache immédiatement à la belle théorie des imaginaires, qui a formé l'objet principal des savantes recherches de l'auteur de la *Géométrie de Situation* *. Nous croyons faire plaisir aux lecteurs de l'ouvrage de cet excellent géomètre, en ajoutant les observations suivantes, qui feront voir combien nous nous sommes peu éloignés de la route qu'il a tracée.

(1.) Étant donné deux représentations homographiques

$$[p_1 p_2, q_1 q_2], \quad [x_1 x_2, y_1 y_2]$$

des éléments imaginaires $\lambda_1 \lambda_2$, $\mu_1 \mu_2$, et une autre représentation quelconque $[r_1 r_2, s_1 s_2]$ de la première de ces deux dyades ; on pourra déterminer linéairement les éléments $u_1 u_2$, $v_1 v_2$, qui satisfont à l'équation anharmonique

$$[x_1 x_2, y_1 y_2, u_1 u_2, v_1 v_2] = [p_1 p_2, q_1 q_2, r_1 r_2, s_1 s_2],$$

et on aura ainsi une nouvelle représentation $[u_1 u_2, v_1 v_2]$ des éléments $\mu_1 \mu_2$, qui sera homographique à la représentation $[r_1 r_2, s_1 s_2]$ des éléments $\lambda_1 \lambda_2$. Soient donc $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ trois dyades quelconques données ; $[A]$, $[B]$ deux représentations homographiques des éléments $a_1 a_2$, $b_1 b_2$; $[A_1]$, $[C_1]$ deux représentations homographiques des éléments $a_1 a_2$, $c_1 c_2$. On déterminera une représentation $[C]$ des éléments $c_1 c_2$ qui soit homographique à la représentation $[A]$ des éléments $a_1 a_2$; et on aura ainsi deux représentations homographiques $[B]$, $[C]$, des éléments $b_1 b_2$, $c_1 c_2$.

(2.) Étant donné une représentation $[p_1 p_2, q_1 q_2]$ d'une dyade $\lambda_1 \lambda_2$, et un troisième couple quelconque $r_1 r_2$ de l'involution qui détermine cette dyade, il existe toujours un quatrième couple de cette même involution, qui satisfait à l'équation anharmonique

$$[p_1, p_2, q_1, q_2, \lambda_1, \lambda_2] = [r_1, r_2, s_1, s_2, \lambda_1, \lambda_2].$$

On trouve linéairement ce quatrième couple en prenant pour $s_1 s_2$ les éléments conjugués à $p_2 p_1$ dans l'involution déterminée par $q_1 r_2$, $q_2 r_1$. On peut donc trouver une infinité de représentations des éléments imaginaires $\lambda_1 \lambda_2$, homographiques à une représentation donnée.

(3.) Il convient d'attribuer un signe algébrique déterminé à chaque représentation d'une dyade ; ainsi nous dirons que la représentation $[p_1 p_2, q_1 q_2]$ de la dyade $\lambda_1 \lambda_2$ est positive ou négative, selon que le sens du mouvement continu indiqué par $p_1 q_1 p_2$ est positif ou négatif. Il s'ensuit de là que deux représentations homographiques d'une même dyade ont le même signe, ou des signes contraires, selon que les éléments imaginaires se correspondent directement ou réciproquement dans les deux représentations. En effet, en désignant par $[p_1 p_2, q_1 q_2]$, $[r_1 r_2, s_1 s_2]$ deux représentations homographiques de la dyade $\lambda_1 \lambda_2$, on a, dans le premier de ces deux cas, l'équation anharmonique

$$[p_1, p_2, q_1, q_2, \lambda_1, \lambda_2] = [r_1, r_2, s_1, s_2, \lambda_1, \lambda_2] ;$$

d'où il s'ensuit que le sens indiqué par $p_1 q_1 p_2$ est le même que le sens indiqué par $r_1 s_1 r_2$,

* Staudt, *Beiträge zur Geometrie der Lage* (Nürnberg, 1856).

puisque deux éléments correspondants, qui parcourent deux divisions homographiques, dont les éléments doubles sont imaginaires, se mouvent toujours dans le même sens. Donc, en ce premier cas, les deux représentations sont de même signe ; tandis que dans le second cas, on aura l'équation

$$[p_1, p_2, q_1, q_2, \lambda_1, \lambda_2] = [r_1, r_2, s_1, s_2, \lambda_2, \lambda_1],$$

ou, ce qui est la même chose, l'équation

$$[p_1, p_2, q_1, q_2, \lambda_1, \lambda_2] = [s_1, s_2, r_1, r_2, \lambda_1, \lambda_2].$$

Cette équation implique que le sens indiqué par $p_1 q_1 p_2$ coincide avec le sens indiqué par $s_1 r_1 s_2$; d'où l'on conclura que les deux sens indiqués par $p_1 q_1 p_2$ et $r_1 s_1 r_2$ sont opposés, c'est à dire, que les deux représentations $[p_1 p_2, q_1 q_2]$, $[r_1 r_2, s_1 s_2]$ sont de signes contraires.

(4.) Étant donné deux systèmes géométriques, dont chacun consiste d'une série d'éléments simplement infinie, et dont l'un dépend linéairement de l'autre, à chacun des deux sens qu'on peut attribuer à la succession des éléments dans le premier, il correspondra un sens déterminé dans l'autre. C'est ainsi qu'en rattachant chacun des deux éléments d'une même dyade à l'un des deux sens opposés que l'on peut concevoir dans le système géométrique auquel cette dyade appartient, de manière à établir une distinction fictive entre les deux éléments de la dyade, on établit en même temps une distinction correspondante entre les deux éléments d'une dyade quelconque linéairement dérivée de la première ; puisque les éléments homologues des deux dyades peuvent être censés appartenir aux sens correspondants dans les deux systèmes dont elles font partie.

Comme exemple très particulier de cette remarque générale, considérons deux dyades de points $\lambda_1 \lambda_2$, $\mu_1 \mu_2$ ayant le même axe LM. Soient $[p_1 p_2, q_1 q_2]$, $[x_1 x_2, y_1 y_2]$ deux représentations homographiques des éléments imaginaires $\lambda_1 \lambda_2$, $\mu_1 \mu_2$. Puisque ces deux représentations peuvent être de même signe ou de signes contraires, et puisque la correspondance des éléments imaginaires est différente dans les deux cas, on voit très clairement dans ce cas particulier, qu'en considérant les deux sens opposés dans lesquels un point peut décrire une droite, on arrive à distinguer *à priori*, non pas entre les deux éléments d'une même dyade (ce qui serait absurde), mais entre les deux manières d'établir l'homologie de deux dyades. D'ailleurs on peut observer que, dans ce même cas particulier, il y a une autre distinction très marquée entre les deux manières d'établir l'homologie des dyades données. En effet, si les deux représentations homographiques données des éléments $\lambda_1 \lambda_2$, $\mu_1 \mu_2$ sont de même signe, les points doubles de l'involution $(\lambda_1 \mu_1, \lambda_2 \mu_2)$ seront imaginaires, et les points doubles de l'involution $(\lambda_1 \mu_2, \lambda_2 \mu_1)$ seront réels, tandis que, dans le cas contraire, les points doubles de la première involution seront réels, et ceux de la seconde seront imaginaires. Pour démontrer cette assertion, prenons un point quelconque sur la circonférence d'une conique, et de ce point, comme centre de projection, projetons les points de la droite LM sur la courbe. Pour plus de simplicité, nous désignerons les projections de ces points par les mêmes lettres que les points eux-mêmes. Les dyades $\lambda_1 \lambda_2$, $\mu_1 \mu_2$ considérées sur la conique, donnent lieu à deux centres d'homologie, dont l'un est intérieur, et l'autre extérieur à la courbe. Soit Ω le centre intérieur, et menons les cordes $x_1 \Omega \xi_1$, $y_2 \Omega \xi_2$, $y_1 \Omega \eta_1$, $y_2 \Omega \eta_2$; $[\xi_1 \xi_2, \eta_1 \eta_2]$ sera une représentation de la dyade $\lambda_1 \lambda_2$, homographique à la représentation $[p_1 p_2, q_1 q_2]$ de cette même dyade, et aussi à la représentation $[x_1 x_2, y_1 y_2]$ de la dyade $\mu_1 \mu_2$; de plus, le sens de $\xi_1 \eta_1 \xi_2$ est le même que le sens de $x_1 y_1 x_2$, puisque Ω est un point intérieur. Donc la représentation

$[\xi_1 \xi_2, \eta_1 \eta_2]$ comporte le même signe que la représentation $[p_1 p_2, q_1 q_2]$, ou le signe contraire, selon que les deux représentations $[x_1 x_2, y_1 y_2]$, $[p_1 p_2, q_1 q_2]$ sont de même signe, ou de signes contraires. Par conséquent $[\xi_1 \xi_2, \eta_1 \eta_2]$ et $[x_1 x_2, y_1 y_2]$ sont deux représentations homographiques des éléments $\lambda_1 \lambda_2$, $\mu_1 \mu_2$ dans le premier cas, et des éléments $\lambda_2 \lambda_1$, $\mu_1 \mu_2$ dans le second. Mais cela revient à dire que Ω est le point d'intersection des cordes imaginaires $\lambda_1 \mu_1$, $\lambda_2 \mu_2$ dans le premier cas, et des cordes $\lambda_1 \mu_2$, $\lambda_2 \mu_1$ dans le second. Et de là, en revenant des points de la conique à ceux de la droite, on conclut immédiatement la vérité de la proposition qu'il fallait démontrer.

Note IV (p. 7).

Les deux solutions que nous avons données de ce problème ne diffèrent pas essentiellement, puisque la droite, lieu des points d'intersection des couples de tangentes que l'on considère dans la première solution, est en même temps l'axe des deux divisions homographiques que l'on obtient en échangeant entre eux sur l'une des droites B, C, les deux points de chaque couple de l'involution qui détermine la dyade sur cette droite. La solution suivante, peu différente d'ailleurs des autres, nous paraît aussi simple qu'on peut le désirer.

En se servant des centres d'homologie donnés on obtient deux représentations homographiques $[\beta_1 \beta_2, \beta'_1 \beta'_2]$, $[\gamma_1 \gamma_2, \gamma'_1 \gamma'_2]$ des éléments imaginaires $b_1 b_2$, $c_1 c_2$; puis on détermine les axes des quatre systèmes homographiques que voici,

$$[\beta_1, \beta_2, \beta'_1, \beta'_2] = [\gamma'_1, \gamma'_2, \gamma_1, \gamma_2]$$
$$[\beta_1, \beta_2, \beta'_1, \beta'_2] = [\gamma'_2, \gamma'_1, \gamma_2, \gamma_1]$$
$$[\beta_1, \beta_2, \beta'_1, \beta'_2] = [\gamma_1, \gamma_2, \gamma'_1, \gamma'_2]$$
$$[\beta_1, \beta_2, \beta'_1, \beta'_2] = [\gamma_2, \gamma_1, \gamma'_2, \gamma'_1].$$

Les deux premières de ces droites se croisent au point P ; les deux dernières au point P'. Si le premier ou le second système devient homologique, le centre d'homologie est le point P' ; pareillement, si le troisième ou le quatrième système devient homologique, le centre d'homologie est le point P.

On peut encore remarquer que si les dyades $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ appartiennent à une même section conique, les six systèmes de trois points, $P'QR$, $PQ'R'$, $PQ'R$, $P'QR'$, PQR', $P'Q'R$, seront chacun en ligne droite.

Note V (p. 12).

Supposons seulement que l'axe d'homologie A de $a_1 a_2$, $\alpha_1 \alpha_2$ soit donné. Faisons passer une conique réelle Σ par PΠ, et par la dyade de points déterminée par $a_1 a_2$, ou $\alpha_1 \alpha_2$, sur l'axe A. Soient $k_1 k_2$, $\kappa_1 \kappa_2$ les dyades de points déterminées sur Σ par les dyades données $b_1 b_2$, $\beta_1 \beta_2$. L'axe A passera par l'intersection des droites imaginaires conjuguées $k_1 \kappa_2$, $k_2 \kappa_1$; ce qui suffit pour faire voir qu'on peut distinguer linéairement entre les deux centres d'homologie de $k_1 k_2$, $\kappa_1 \kappa_2$, et pourtant entre les deux axes d'homologie de $b_1 b_2$, $\beta_1 \beta_2$. La construction est entièrement linéaire, puisque pour déterminer les dyades $k_1 k_2$, $\kappa_1 \kappa_2$ il n'est pas nécessaire de tracer la conique Σ.

Note VI (p. 16).

La détermination de la dyade $\pi_1\pi_2$ ne présente aucune difficulté théorique. Soit $[x_1x_2,\ y_1y_2]$ une représentation donnée de la dyade p_1p_2; σ une conique qui passe par cette dyade. D'un point quelconque réel de σ projetons sur cette conique l'involution déterminée par les coniques du faisceau $[c_1, c_2, d_1, d_2]$ sur l'axe de p_1p_2. Soit π le pôle de l'involution qu'on aura ainsi sur la conique σ. Aux quatre rayons $\pi.[x_1, x_2, y_1, y_2]$ il correspondra anharmoniquement quatre coniques du faisceau. Soient $\xi_1, \xi_2, \eta_1, \eta_2$ les points de la conique auxiliaire C, qui correspondent anharmoniquement à ces quatre coniques; $[\xi_1\xi_2, \eta_1\eta_2]$ sera une représentation de la dyade $\pi_1\pi_2$, et cette représentation sera homographique à la représentation donnée $[x_1x_2,\ y_1y_2]$ de la dyade p_1p_2.

Note VII (p. 22).

Toute conique du réseau circonscrit qui passe par Γ rencontre la droite Δ en deux points harmoniquement conjugués par rapport à la conique cherchée. De même, les deux tangentes menées du point Γ à une conique du réseau inscrit, qui est tangente à la droite Δ, sont deux droites conjuguées par rapport à cette même conique du réseau harmonique. On peut donc trouver très simplement l'involution que détermine cette conique, soit au point Γ, soit sur la droite Δ; et, d'après ce que nous avons dit dans l'article précédent, cela suffit pour déterminer le système polaire de la courbe.

Les expressions 'ellipse *minima* circonscrite,' 'ellipse *maxima* inscrite,' dont nous nous sommes servis dans cet article, sont relatives au cas d'un triangle harmonique réel. Lorsque ce triangle est imaginaire, les coniques polaires du centre des distances moyennes, et de la droite à l'infini, sont toutes les deux des hyperboles.

Note VIII (p. 32).

Soient X, Y les projections orthogonales de P sur les axes A_1A_2, B_1B_2 respectivement; ST la tangente à Γ au point S. Menons la corde ss' perpendiculaire à A_1A_2. D'après la définition de Γ, ST est le diamètre de Σ conjugué aux cordes perpendiculaires à SP. Donc ST est parallèle à A_2s; de plus, puisque l'hyperbole Γ est équilatère, les droites bissectrices de l'angle CST sont parallèles aux asymptotes de la courbe; donc SC est parallèle à A_2s'. On voit en même temps que SC est le diamètre de Σ conjugué aux cordes perpendiculaires à XY, puisque XY et SP font des angles égaux avec les axes de Σ. Mais la droite XY passe elle-même par C, puisqu'elle est une des diagonales du quadrilatère $SP\alpha\beta$, inscrit à Γ. Donc C est le point d'intersection de XY avec le diamètre de Σ conjugué aux cordes perpendiculaires à XY, ou, si l'on veut, avec le diamètre parallèle à A_2s'; et la polaire de C par rapport à Σ est la perpendiculaire abaissée du pôle de XY sur cette droite elle-même. Ces déterminations nous seront utiles plus tard.

Puisque nous tenons à faire voir que la solution de Joachimsthal ne conduit pas à des opérations impraticables à cause de leur longueur, nous indiquerons ici la manière de les effectuer, qui nous paraît la plus simple. On supposera connus les axes et l'un des foyers H de la conique centrale Σ; on prendra pour A_1A_2 l'axe focal. En se servant d'un équerre, on

mènera la corde $A_1 s$ perpendiculaire à PS, la corde $A_2 p$ perpendiculaire à $A_1 s$, la corde ss' perpendiculaire à $A_1 A_2$; enfin, la normale et la tangente à Σ au point s (on sait que cela peut se faire très simplement avec l'équerre). Soit μ le point d'intersection de l'axe focal avec la normale ; σ, η les projections orthogonales de S, H sur la tangente, Y la projection orthogonale de P sur l'axe conjugué, Y_1 le point de ce même axe pour lequel l'angle YHY_1 est droit ; enfin, $A_1 p_1$ étant menée perpendiculaire à PY_1, soit y le point d'intersection des cordes $p_1 p_2$, sB, et θ la projection orthogonale de y sur ss'. Le centre ω du cercle de Joachimsthal sera le point d'intersection des droites $\theta\mu$, $S\sigma$; le rayon de ce cercle sera $\omega\eta$. On décrira le cercle, et on joindra les points d'intersection des deux courbes au point A_1 par des cordes $A_1 x$: les normales issues du point P seront perpendiculaires à ces cordes, et leurs pieds se trouveront sur les diamètres de Σ parallèles aux cordes $A_2 x$. D'après ce que nous avons dit, on vérifiera facilement les détails de cette construction, dans laquelle on n'aura besoin du compas qu'au moment où l'on veut tracer le cercle Ω. En effet, puisque D se transforme en α, et C en β, les axes principaux de la conique transformée Σ' sont les droites qui correspondent au diamètre de Σ qui passe par C, et à la polaire de C. L'un de ces deux axes est ss' ; c'est celui qui correspond à SC ; l'autre est $y\theta$, puisque y correspond au point Y_1, qui se trouve sur la polaire de C. Donc θ est le centre de Σ' ; et, puisque la perpendiculaire abaissée de θ sur sa polaire par rapport à Σ doit passer par ω, et aussi par μ, la construction du point ω se trouve justifiée.

On remarquera qu'en général, étant donné un point quelconque Z, pour en trouver le point correspondant dans la figure transformée, on mènera les cordes $A_1 p_1$, $A_2 s_1$, dont la première est perpendiculaire à PZ, et la seconde est parallèle à SZ ; les cordes ss_1, pp_1 se croiseront au point z. Et, en effet, c'est ainsi que nous avons déterminé les points f et y, correspondant aux points F et Y_1, dans les constructions précédentes.

La solution analytique du problème qui nous occupe, donnée il y a presque deux cents ans par De la Hire, a été l'objet de recherches intéressantes par M. E. Catalan (*Nouvelles Annales de Mathématiques* par MM. Terquem et Gerono, Vol. vii. p. 332 et 396, année 1848). Nous allons voir que la solution de M. Catalan, aussi bien que celle de Joachimsthal, se déduit naturellement de la méthode dont nous nous sommes servis.

Soient n_1, n_2, n_3, n_4 les pieds des normales abaissées de P sur Σ ; à ces quatre points M. Catalan substitue quatre autres n'_1, n'_2, n'_3, n'_4, qui sont les points d'intersection de Σ par une circonférence de cercle, et dont les abscisses, mesurées du centre de Σ sur l'un des deux axes, sont proportionnelles aux abscisses de n_1, n_2, n_3, n_4 *. Soient toujours $A_1 A_2$, $B_1 B_2$ les axes de Σ ; α, β les points à l'infini sur ces axes respectivement ; en supposant que $A_1 A_2$ soit l'axe des abscisses, on aura l'équation anharmonique

$$\beta . [\alpha, S, n_1, n_2, n_3, n_4] = \beta . [\alpha, S, n'_1, n'_2, n'_3, n'_4]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

Qu'on transforme la figure de manière qu'aux points n de la figure donnée correspondent les points n' de la nouvelle figure. Puisque les points $\beta, n_1, n_2, n_3, n_4$ appartiennent à la

* Dans la solution plus compliquée de De la Hire ce sont les ordonnées des points n' qu'on fait proportionnelles aux abscisses des points n. De plus, au lieu du centre de la conique, on prend pour origine un point tel que les sommes des ordonnées des points n', et des abscisses des points n, s'évanouissent séparément. On voit qu'il en doit résulter une construction géométrique entièrement différente de celle que nous allons déduire de l'analyse de M. Catalan. D'ailleurs, nous avons reconnu que la solution de De la Hire ne s'applique pas au cas où il y aurait quatre normales réelles : mais, malgré cet inconvénient, cette solution nous paraît mériter une étude plus approfondie.

conique Γ, on conclut de l'équation (A), que β appartiendra à Γ′, la transformée de Γ. Or, Γ′ ne peut être qu'une parabole, dont l'axe est parallèle à $B_1 B_2$. Car le faisceau (Γ′, Σ), dont les points n' forment la base, contient par hypothèse un cercle ; donc $a\beta$ sont les points doubles de l'involution que ce faisceau détermine sur la ligne droite à l'infini, et la conique du faisceau, qui passe par β, y touche cette droite. Cela posé, il s'ensuit de l'équation (A) que le point β de Γ′ correspond au point a de Γ ; donc a se transforme en β, et l'asymptote Ca devient la droite à l'infini $a\beta$. Soient a, b les projections de C sur les axes $A_1 A_2$, $B_1 B_2$; a se transformant en β, b se transforme en a, puisque ab, βa sont des points réciproques par rapport aux faisceaux (Σ, Γ), (Σ, Γ′), respectivement. De plus, d'après l'équation (A), le point S doit se transformer en un point S' situé sur βS ; donc $A_1 A_2$, ou Sa, se transforme en $S'\beta$, ou $B_1 B_2$. Soit Σ_1 la conique du faisceau (Σ, Γ) qui se transforme en Σ ; il faut que pour cette conique les droites aS, aC soient des droites conjuguées. Cette condition détermine Σ_1 sans ambiguité, puisque, des deux coniques du faisceau (Σ, Γ) qui y satisfont, l'une est Γ, qui ne se transforme pas en Σ. En désignant par $X_1 Y_1$ les points d'intersection des axes de Σ avec la polaire de C, soient c, x les points placés symétriquement à C, X_1, par rapport à l'axe B_1, B_2, y le point placé symétriquement à Y_1 par rapport à l'axe $A_1 A_2$. La conique du faisceau (Σ, Γ) pour laquelle C est le pôle de xy ne peut être autre que Σ_1. Car la polaire de c par rapport à Γ est l'axe $A_1 A_2$, puisque Sc est tangent à Γ au point S ; et la polaire de c par rapport à Σ doit passer par x, à cause de la situation symétrique des points CX_1, cx ; donc c, x sont des points réciproques par rapport au faisceau (Σ, Γ), et la conique de ce faisceau par rapport à laquelle C est pôle de xy, aura Cc, ou Ca, pour polaire de x ; c'est à dire que aS, aC seront des droites conjuguées par rapport à cette conique. Il résulte de là que abx sera un triangle harmonique par rapport à Σ_1, et que pour satisfaire à toutes les conditions du problème, il suffira de transformer Σ_1 en Σ, de manière que le triangle abx devienne le triangle βaS. Soit (1°) Σ une ellipse ; les involutions déterminées par le faisceau (Σ, Γ) sur les axes de Σ ont évidemment des points doubles imaginaires ; donc toute conique du faisceau rencontre ces droites en deux points réels. Soient $\lambda_1 \lambda_2$ les points d'intersection de Σ_1 avec $A_1 A_2$; x sera le point milieu du segment $\lambda_1 \lambda_2$; mais S, qui est le centre de l'involution, se trouvera aussi sur ce même segment ; donc les droites bS, bx seront situées dans le même angle formé par les droites $b\lambda_1$, $b\lambda_2$, tangentes Σ_1 en λ_1, λ_2. Mais l'axe bS rencontre Σ_1 en deux points réels ; donc aussi bx rencontre Σ_1 en deux points réels $\nu_1 \nu_2$. D'ailleurs, les points $\lambda_1 \lambda_2$, $\nu_1 \nu_2$ sont quatre points harmoniques de Σ_1 ; donc en transformant homographiquement ν_1, ν_2 en A_1, A_2, et λ_1, λ_2 en B_1, B_2 (ce qui peut se faire par une quelconque de quatre transformations différentes), on transformera Σ_1 en Σ, abx en βaS, Γ en une parabole ayant son axe parallèle à $B_1 B_2$, et, enfin, les quatre points n en quatre points n' situés sur la circonférence d'un cercle et satisfaisant à l'équation (A). Quelque soit l'axe qu'on a choisi pour $A_1 A_2$, ce sera toujours la même conique Σ_1 qui se transformera en Σ ; puisque la définition que nous avons trouvé pour Σ_1 est symétrique par rapport aux deux axes. Et l'on peut ajouter que dans les quatre transformations, relatives à un même axe, ce sera toujours la même conique qui se transformera en un cercle, et que les quatre cercles résultants, ainsi que les quatre paraboles Γ′, seront placés symétriquement par rapport aux axes principaux de Σ. Si (2°) Σ est une hyperbole, on voit d'abord qu'il faudra prendre pour $A_1 A_2$ l'axe qui ne rencontre pas la courbe. Car si les sommets $A_1 A_2$ étaient réels, les points $\lambda_1 \lambda_2$ seraient réels, comme dans le cas de l'ellipse, et il faudrait transformer deux points réels $\lambda_1 \lambda_2$ en deux points imaginaires $B_1 B_2$. On doit donc supposer que les sommets

$A_1 A_2$ sont imaginaires ; en ce cas, on aura à transformer les points $\lambda_1 \lambda_2$ en deux points réels $B_1 B_2$, mais, pour que les points $\lambda_1 \lambda_2$ restent eux-mêmes réels, il faudra que Sx^2 soit plus grand que $-SA^2$. Qu'on mène des perpendiculaires aux asymptotes de Σ par les points où ces droites rencontrent la tangente à l'un des sommets B_1, B_2 ; et qu'ensuite par les points d'intersection de ces perpendiculaires avec l'axe conjugué on mène des parallèles à l'axe focal. La condition ci-dessus revient à dire que le point P doit être compris entre les deux parallèles qu'on a tracées. D'ailleurs, cette limitation de la méthode de M. Catalan résulte clairement des formules analytiques dont il l'a fait dépendre. Lorsque la condition de possibilité est satisfaite, les angles $\lambda_1 b \lambda_2$, Sbx empiètent l'un sur l'autre ; mais l'axe bS rencontre la conique Σ_1 en deux points réels ; donc bx ne la rencontre pas, et ba la coupe en deux points réels $\mu_1 \mu_2$. En faisant correspondre $\lambda_1 \lambda_2$ à $B_1 B_2$, et $\mu_1 \mu_2$ aux deux points à l'infini sur les asymptotes de Σ, on aura quatre transformations différentes, dont chacune pourra servir pour la solution du problème.

Lorsqu'on veut faire usage de cette méthode, on commencera par la détermination de C, et de la polaire de ce point par rapport à Σ ; on aura ainsi les points a, b, c, x. Pour avoir les points $\lambda_1 \lambda_2$, on projettera du point B_1 sur l'axe $A_1 A_2$ les deux extrémités du diamètre de Σ conjugué à la corde $B_1 x$. De même, si Σ est une ellipse, on trouvera $\phi_1 \phi_2$, les deux points d'intersection de $B_1 B_2$ avec Σ_1, en projetant du point A_1 sur $B_1 B_2$ les extrémités du diamètre parallèle à $A_1 b$. En faisant correspondre λ_1 à B_1, λ_2 à B_2, on prendra sur $B_1 B_2$ les points a', S', qui satisfont à l'équation anharmonique

$$[a, \lambda_1, \lambda_2, a, S] = [\beta, B_1, B_2, a', S'],$$

et l'on mènera par S' une parallèle à $A_1 A_2$. Cette parallèle correspond à $B_1 B_2$; elle rencontre Σ en deux points réels $\phi'_1 \phi'_2$, correspondant à $\phi_1 \phi_2$. On fixera à volonté la correspondance de $\phi'_1 \phi'_2$, $\phi_1 \phi_2$, et on prendra le point β' qui satisfait à l'équation

$$[S, \phi_1, \phi_2, \beta] = [S', \phi'_1, \phi'_2, \beta'],$$

et qui, par conséquent, correspond à β. Le point C', correspondant à C, est évidemment le point à l'infini sur la droite $a'\beta'$; donc D', qui correspond à D, et qui est le point réciproque de C' par rapport au faisceau (Σ, Γ'), se trouve à l'intersection de $\beta'\beta$ avec le diamètre de Σ conjugué à $a'\beta'$. Du point D' abaissons sur $a'\beta'$ une perpendiculaire ; soit ω le point d'intersection de cette perpendiculaire avec une droite $S\omega$, qui fait avec l'un des axes principaux le même angle que la perpendiculaire, mais de l'autre côté de cet axe. Le cercle, dont ω est le centre, et qui coupe orthogonalement le cercle dont $a'\beta'$ est le diamètre, appartient au faisceau transformé. C'est ce qu'on vérifie en observant que cx, $a\beta$, CD sont des couples de points réciproques par rapport au faisceau (Σ, Γ), et qu'en désignant par c' le point à l'infini harmoniquement conjugué à C' par rapport à $a\beta$, les points correspondants $c'S$, $a'\beta'$, $C'D'$, sont également des couples de points réciproques par rapport au faisceau (Σ, Γ').

Pour avoir le rapport des abscisses des points n' aux abscisses des points correspondants n, projetons P sur les deux axes. En désignant, comme nous avons fait plus haut, ces projections par X, Y, prenons Y' le point correspondant à Y ; le rapport cherché sera celui de $S'Y'$ à SX.

Avant de terminer cette longue note, nous indiquerons une troisième méthode, qui ne s'applique pas à l'hyperbole, mais qui conduit à une solution assez simple pour le cas de l'ellipse. Soient $a_1 a_2$, $b_1 b_2$ les diamètres conjugués égaux de l'ellipse Σ. Qu'on transforme

la courbe en elle-même, de manière que, la ligne à l'infini restant la même, les axes $A_1 A_2$, $B_1 B_2$ deviennent les diamètres égaux $a_1 a_2$, $b_1 b_2$. C'est ce qu'on peut faire par quatre transformations différentes, en supposant, pour abréger, qu'on échange entre eux les points à l'infini de la courbe. Mais quelle soit la transformation qu'on choisit, le faisceau, qui correspond au faisceau (Σ, Γ), contiendra un cercle, puisque les points doubles de l'involution que ce faisceau déterminera sur la ligne droite à l'infini, seront les points rectangulaires $\alpha\beta$. Soient donc $a_1 a_2$, $b_1 b_2$ les points qui correspondent à $A_1 A_2$, $B_1 B_2$ respectivement; et soit X un point donné de la courbe. On trouvera le point correspondant x en menant la corde Xx parallèle à $A_1 a_1$; on aura ainsi le diamètre Sx correspondant au diamètre SX; de plus, la droite qui joint deux points correspondants de ces diamètres sera parallèle à Xx; donc on pourra trouver très facilement, dans l'une des deux figures, le point correspondant à un point donné de l'autre. Soient toujours a, b les projections orthogonales de C sur les axes $A_1 A_2$, $B_1 B_2$ respectivement, a', b' les points correspondants dans la nouvelle figure. Qu'on abaisse de a', b' des perpendiculaires sur $b_1 b_2$, $a_1 a_2$ respectivement; le point de concours de ces droites sera le centre ω du cercle Ω qu'on cherche. Avec le rayon Sa_1, décrivons un cercle concentrique à Σ; soit $d_1 d_2$ le diamètre de ce cercle qui fait un angle droit avec $S\omega$; les points d_1, d_2 appartiendront à la circonférence de Ω. On tire cette dernière conclusion d'un théorème qui n'est qu'un cas particulier d'un autre plus général, mais qui vaut la peine d'être énoncé:

'Toute hyperbole, ayant ses asymptotes parallèles à Sa_1, Sb_1, et passant par S, détermine par ses intersections avec Σ une circonférence de cercle, qui coupe orthogonalement le cercle imaginaire, dont S est le centre, et $-Sa_1^2$ le carré du rayon.'

On peut encore remarquer que la somme des angles excentriques de deux points correspondants x et X est égale à un multiple impair de $\frac{1}{4}\pi$. Cela vérifie que les points correspondants aux pieds des normales appartiennent à une même circonférence.

Note IX (p. 35).

La théorie des correspondances (1), (2), (3), a été étudiée par M. Battaglini, dans un excellent Mémoire ('*Sulle forme binarie dei primi quattro gradi, appartenenti ad una forma ternaria quadratica*,' Giornale di Matematiche, vol. v. p. 39), qui, malheureusement, nous était encore inconnu, lorsque nous écrivions l'article précédent. Cependant, nous placerons ici quelques remarques additionnelles, qui ne sont pas sans intérêt pour les constructions graphiques.

La théorie géométrique de la correspondance (2) peut être présentée de la manière suivante. Soient (A) et (B) deux systèmes corrélatifs dans le plan d'une conique donnée Σ; et soient X et Y les coniques qui correspondent dans les systèmes A et B à la conique Σ, considérée comme appartenant aux systèmes B et A respectivement. Soit Θ la conique des pôles des deux systèmes corrélatifs, c'est à dire, la conique lieu des points qui se trouvent sur leurs droites corrélatives; de même soit Θ' la conique des polaires, ou la conique enveloppe des droites qui passent par leurs points corrélatifs. En considérant un point quelconque y de Σ comme appartenant au second système, la droite corrélative rencontrera Σ en deux points x, qui seront liés au point y, par une équation de la forme (2). D'après cela, on aura les théorèmes suivants qui donnent immédiatement la solution des problèmes biquadratiques dépendant de la correspondance (2).

(1.) Les points d'intersection de X et Σ sont les quatre points x pour lesquels il y a coincidence des points y correspondants : et les quatre points de contact avec Σ des tangentes communes à Y et Σ sont les points y, qui sont devenus coincidents. La corrélation des deux systèmes (A) et (B) fera connaître l'un des deux systèmes de quatre points lorsqu'on aura trouvé l'autre. Pareillement, les points d'intersection de Y et Σ, et les points de contact avec Σ des tangentes communes à X et Σ, sont respectivement les points y dont les correspondants coincident, et les points coincidents eux-mêmes.

(2.) Les quatre points d'intersection de Θ et Σ sont des points de coincidence d'un point x avec l'un des points correspondants y, et les tangentes menées à Θ' de l'un quelconque θ de ces quatre points (tangentes dont l'une est aussi tangente à X, et l'autre à Y), rencontrent la conique Σ en deux points, qui sont les points correspondants à θ, autres que θ lui-même.

On remarquera qu'une seule construction biquadratique suffit pour trouver, soit les points $x_1 x_2$, soit les points $y_1 y_2$, qui deviennent coincidents, soit enfin les points qui correspondent à ces points coincidents dans chacun des deux systèmes ; puisque, ayant trouvé les points d'intersection de deux coniques, on n'a besoin que d'une construction quadratique pour trouver leurs tangentes communes. Mais, pour trouver les points x, qui coincident avec un de leurs points correspondants y, il faut une construction biquadratique nouvelle.

Le problème linéaire ' Étant donné huit points x, et un point y correspondant à chacun de ces points, trouver la droite $y_1 y_2$ correspondante à un point quelconque x ' peut s'enoncer plus généralement de la manière suivante, ' Étant donné huit points dans l'une de deux figures corrélatives, et huit autres points situés respectivement sur les droites corrélatives des premiers points, déterminer la corrélation des deux figures.' Or, c'est de ce problème que dépend (ainsi que l'ont fait voir MM. Seydewitz et Schröter) la construction de la surface du second ordre qui passe par neuf points donnés. On en trouvera la solution complète dans le Mémoire de M. Schröter (*Journal de Crelle-Borchardt*, vol. lxii. p. 215).

Si l'équation (2) est symétrique, les coniques X et Y coincident, de même que les coniques Θ et Θ', et les deux systèmes (A) et (B) deviennent polaires réciproques par rapport à Θ et Θ'. En ce cas particulier la solution du problème linéaire est tout-à-fait élémentaire.

En passant maintenant à la correspondance cubique, définie par l'équation (3), supposons que $yy'y''$ soient les trois points correspondants à un même point x, $YY'Y''$ le triangle des tangentes à Σ en ces trois points. L'axe d'homologie des triangles $yy'y''$, $YY'Y''$ enveloppera une section conique C_1 ; pareillement, le lieu du centre d'homologie de ces deux triangles sera une seconde conique C_2, polaire réciproque de C_1 par rapport à Σ. Soit, de plus, σ la conique inscrite aux triangles $yy'y''$, Ω la conique par rapport à laquelle ces mêmes triangles sont harmoniques, de sorte que Ω est une des coniques réciproquantes de Σ et σ. Les trois coniques Ω, C_2, Σ ont les mêmes points d'intersection ; ces points sont en même temps les points de contact avec Σ des tangentes communes aux trois courbes σ, C_1, Σ ; de plus, la tangente à Ω en un quelconque de ces points rencontre Σ, pour la seconde fois, en un des points d'intersection de Σ avec σ, et y est tangente à cette dernière conique. Les points de coincidence de deux des points y, qui correspondent à un même point x, sont évidemment les points de contact avec Σ des tangentes communes à Σ et σ ; d'où l'on voit que, pour trouver ces points, il suffit de connaître l'une quelconque des quatre coniques auxiliaires C_1, C_2, σ, Ω. Les axes d'homologie des triangles $yy'y''$ et $YY'Y''$, considérés

comme des tangentes à C_1, et les centres d'homologie de ces mêmes triangles, considérés comme des points de C_2, correspondent anharmoniquement soit aux triangles $yy'y''$, soit aux points x; cette observation servira pour déterminer les points x qui correspondent aux quatre triangles évanescents que nous venons de trouver. La détermination des points x, qui coincident avec un des points correspondants y, se fait un peu différemment. Soient P, Q deux points quelconques de Σ, P' un point qui n'appartient pas à cette conique. Les coniques du faisceau (P, P', y, y', y'') correspondront anharmoniquement aux points x; par conséquent, le lieu géométrique des intersections des droites Qx et des coniques correspondantes (P, P', y, y', y'') sera une courbe cubique, qui passera par les points P et Q, et qui, en outre, rencontrera Σ en quatre points, qui sont ceux qu'on cherche. On les déterminera en se servant de la construction biquadratique indiquée par M. Chasles.

Comme vérification des résultats précédents, nous ajouterons quelques unes des principales formules analytiques qui se rattachent à la théorie des correspondances (2) et (3). Soient p, q, r les coordonnées homogènes d'un point quelconque du plan que l'on considère; on prendra l'équation $pr-q^2=0$ pour l'équation de la conique Σ, et on représentera par θ_1^2, $\theta_1\theta_2$, θ_2^2 les coordonnées d'un point quelconque θ de cette conique. Les deux systèmes corrélatifs, dont dépend la correspondance (2), seront définis par l'équation

$$p_1(A_1p_2+2B_1q_2+C_1r_2)+2q_1(A_2p_2+2B_2q_2+C_2r_2)+r_1(A_3p_2+2B_3q_2+C_3r_2)=0;$$

et en mettant dans cette équation les coordonnées du point θ, soit pour $p_1q_1r_1$, soit pour $p_2q_2r_2$, on aura l'équation, soit de la droite $\eta_1\eta_2$, soit de la droite $\xi_1\xi_2$. Donc l'équation de la conique X, enveloppe de $\xi_1\xi_2$, sera

$$(A_1p+2A_2q+A_3r)(C_1p+2C_2q+C_3r)=(B_1p+2B_2q+B_3r)^2;$$

et, pareillement, l'équation de Y, enveloppe de $\eta_1\eta_2$, sera

$$(A_1p+2B_1q+C_1r)(A_3p+2B_3q+C_3r)=(A_2p+2B_2q+C_2r)^2.$$

Enfin, on aura pour la conique des pôles l'équation

$$\Theta=A_1p^2-4B_2q^2+C_3r^2+2(C_2+B_3)qr+(A_3+C_1)pr+2(B_1+A_2)pq=0,$$

et pour la conique des polaires l'équation

$$\Theta'=4\Delta\Theta-\Phi^2=0,$$

en désignant par Δ le déterminant

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

et par Φ la fonction linéaire

$$p[A_2C_1-A_1C_2-A_3B_1+A_1B_3]+2q[A_2B_3-A_3B_2-B_1C_2+B_2C_1]+r[B_3C_1-B_1C_3-A_3C_2+A_2C_3].$$

Passons à la correspondance (3). Soit toujours $\Sigma=pr-q^2=0$ l'équation de la conique Σ, et désignons par P, Q, R, S, T, U les déterminants du système

$$\begin{matrix} A_1, & B_1, & C_1, & D_1, \\ A_2, & B_2, & C_2, & D_2, \end{matrix}$$

pris dans leur ordre naturel, et par

$$\begin{matrix} a_1\,, & b_1\,, & c_1\,, \\ a_{12}, & b_{12}, & c_{12}, \\ a_2\,, & b_2\,, & c_2\,, \end{matrix}$$

les neuf quantités

$$\begin{matrix} A_1C_1-B_1^2, & A_1D_1-B_1C_1, & B_1D_1-C_1^2, \\ A_1C_2-2B_1B_2+A_2C_1, & A_1D_2+A_2D_1-B_1C_2-B_2C_1, & B_1D_2-2C_1C_2+B_2D_1, \\ A_2C_2-B_2^2, & A_2D_2-B_2C_2, & B_2D_2-C_2^2. \end{matrix}$$

Soient, de plus,

$$\begin{matrix} \lambda_1^2, & \lambda_1\lambda_2, & \lambda_2^2, \\ \mu_1^2, & \mu_1\mu_2, & \mu_2^2, \end{matrix}$$

les coordonnées de deux des trois points y qui correspondent à un même point x. En éliminant x_1 et x_2, et divisant par $3(\lambda_1\mu_2-\lambda_2\mu_1)$, on aura l'équation suivante :

$$\begin{aligned} &P\lambda_1^2\mu_1^2+Q(\lambda_1\mu_2+\lambda_2\mu_1)\lambda_1\mu_1+\tfrac{1}{3}R(\lambda_1^2\mu_2^2+\lambda_2^2\mu_1^2) \\ &+(3S+\tfrac{1}{3}R)\lambda_1\lambda_2\mu_1\mu_2+T(\lambda_1\mu_2+\lambda_2\mu_1)\lambda_2\mu_2 \\ &+U\lambda_2^2\mu_2^2=0, \end{aligned}$$

qui est celle d'une correspondance quadratique double. Cette correspondance est évidemment symétrique ; elle est aussi triangulaire, puisque, la fonction

$$K=PU+RS-QT$$

étant identiquement zéro, la conique enveloppe de la droite qui joint les deux points λ correspondants à un point μ donné, c'est à dire la conique

$$\sigma=4(\tfrac{1}{3}Rp+Tq+Ur)(Pp+Qq+\tfrac{1}{3}Rr)-\{Qp+(3S+\tfrac{1}{3}R)q+Tr\}^2=0,$$

ne diffère pas de la conique enveloppe de la droite qui joint le point μ à l'un ou l'autre des deux points correspondants λ ; en effet, on trouve pour l'équation de cette dernière conique $\sigma-4K\Sigma=0$.

Cela posé, la conique σ est la conique inscrite à tous les triangles du système ; et

$$\Omega=Pp^2+(3S+\tfrac{1}{3}R)q^2+Ur^2+2Tqr+\tfrac{2}{3}Rpr+2Qpq=0$$

est l'équation de la conique par rapport à laquelle ces mêmes triangles sont des triangles harmoniques. L'équation

$$\begin{aligned} x_1(a_1y_1^2+b_1y_1y_2+c_1y_2^2)+x_1x_2(a_{12}y_1^2+b_{12}y_1y_2+c_{12}y_2^2) \\ +x_2^2(a_2y_1^2+b_2y_1y_2+c_2y_2^2)=0 \end{aligned}$$

exprime la correspondance doublement quadratique, mais non symétrique, qui a lieu entre un point donné x et les deux points covariants du triangle correspondant. Donc l'équation de la conique C_1, enveloppe de la corde qui joint ces deux points, sera

$$C_1=4(a_1p+b_1q+c_1r)(a_2p+b_2q+c_2r)-(a_{12}p+b_{12}q+c_{12}r)^2=0\,;$$

et on aura l'expression suivante pour l'équation de la conique C_2, polaire réciproque de C_1 par rapport à Σ,

$$\begin{vmatrix} r, & a_{12}, & a_2 \\ -2q, & b_{12}, & b_2 \\ p, & c_{12}, & c_2 \end{vmatrix} \times \begin{vmatrix} r, & a_{12}, & a_1 \\ -2q, & b_{12}, & b_1 \\ p, & c_{12}, & c_1 \end{vmatrix} + \begin{vmatrix} r, & a_1, & a_2 \\ -2q, & b_1, & b_2 \\ p, & c_1, & c_2 \end{vmatrix}^2 = 0.$$

Chaque terme de cette dernière équation est divisible par le déterminant

$$D = \begin{vmatrix} a_1, & b_1, & c_1 \\ a_{12}, & b_{12}, & c_{12} \\ a_2, & b_2, & c_2 \end{vmatrix};$$

en supprimant ce facteur constant, on a l'expression plus simple:

$$C_2 = 3\Omega + (R - 3S)\Sigma = 0.$$

Les fonctions D et $R-3S$ sont très connues; l'évanescence de la première implique que les axes d'homologie des triangles $yy'y''$, $YY'Y''$ passent tous par un même point, et que le système contient un triangle dont les trois sommets se confondent en un seul; l'évanescence de la seconde exprime que deux triangles quelconques du système sont harmoniquement conjugués l'un à l'autre, et que, par conséquent, le système donné coincide avec le système harmonique correspondant. (Voir le Mémoire de M. Battaglini, pp. 44–49.)

Note X (p. 41).

Si les sept points 1, ..., 7, appartiennent tous à une même conique, la détermination de la cubique (9, ..., 16) ne réussit pas. En effet, dans ce cas, le point biquadratiquement opposé au système (1, ..., 7, a) est le point a lui-même; ou, plus exactement, il n'y a aucun point biquadratiquement opposé à ce système, puisqu'il n'y a aucune courbe biquadratique passant par les treize points, et ayant (1, ..., 7, a) pour base de courbes cubiques génératrices. Il faudra donc, dans cette hypothèse particulière, éviter de faire usage de la cubique (9, ..., 16), ce qui sera toujours possible.

Si les dix points 1, ..., 10 appartiennent à une même cubique, la détermination de la cubique (9, ..., 16) devient illusoire, et doit être remplacée par une autre, puisque, dans ce cas, les trois systèmes de points (p_8, 9, 10), (p_9, 10, 8), (p_{10}, 8, 9), sont respectivement en ligne droite; d'où il résulte que l'équation anharmonique

$$p_8 . [x, 11, 9, 10] \times p_9 . [x, 11, 10, 8] \times p_{10} . [x, 11, 8, 9] = +1$$

devient identique quel que soit le point x, et ne peut servir à définir aucun lieu géométrique. La même chose arriverait si les droites $p_9 p_{10}$, $p_{10} p_7$, $p_8 p_9$ passaient par les points 8, 9, 10 respectivement. Pour éviter cet inconvénient, on prendra arbitrairement les points 8, 9, et on déterminera les points p_8, p_9 avant de choisir le point 10. La droite p_8, p_9 peut bien passer par un des points 10, ..., 13, mais elle ne peut pas passer par deux de ces points α, β, puisqu'il s'ensuivrait de cette supposition que les neuf points 1, ..., 7, α, β forment la base d'un faisceau de courbes cubiques, ou bien que les onze points 1, ..., 9, α, β appartiennent à une même cubique. Il y aura donc au moins trois des points 10, ..., 13, qui ne se trouvent pas sur la droite $p_8 p_9$; de plus, de ces trois points il y aura au moins deux qui ne peuvent pas appartenir à la cubique (1, ..., 9); on prendra à volonté l'un ou l'autre pour le point 10.

Il correspond une analyse très simple à la démonstration géométrique du théorème

de cet article. Soient (a, b), (a, b, c, d, e), ..., les fonctions algébriques qui, égalées à zéro, donnent les équations de la droite (a, b), de la conique (a, b, c, d, e), ..., et ainsi de suite. En désignant par λ une constante indéterminée, les courbes cubiques du faisceau

$$(1, \ldots, 8, 9) = \lambda (1, \ldots, 8, 10);$$

correspondent anharmoniquement aux droites

$$(p_8, 9) = \lambda (p_8, 10);$$

et puisque les fonctions $(P_8, 9)$, $(p_8, 9)$, ..., qui entrent dans ces équations, peuvent être multipliées par des constantes quelconques, on peut faire en sorte que la droite $(p_9, 11)$ corresponde à la cubique $(P_8, 11)$. Cela posé,

$$(1, \ldots, 7, 8, 9)(p_8, 10) = (1, \ldots, 7, 8, 10)(p_8, 9) \quad \ldots\ldots (a)$$

sera l'équation d'une courbe biquadratique qui passe par le point p_8 et par les treize points donnés. Pareillement, les courbes biquadratiques

$$(1, \ldots, 7, 9, 10)(p_9, 8) = (1, \ldots, 7, 9, 8)(p_9, 10), \quad \ldots\ldots (b)$$

$$(1, \ldots, 7, 10, 8)(p_{10}, 9) = (1, \ldots, 7, 10, 9)(p_{10}, 8) \quad \ldots\ldots (c)$$

passeront par les mêmes treize points, et par les points p_9, p_{10}, respectivement. En multipliant ces trois équations, membre à membre, et divisant par le produit

$$(1, \ldots, 7, 8, 9)(1, \ldots, 7, 9, 10)(1, \ldots, 7, 10, 8)$$

on aura l'équation cubique

$$(p_8, 10)(p_9, 8)(p_{10}, 9) = (p_8, 9)(p_9, 10)(p_{10}, 8). \quad \ldots\ldots (d)$$

Il résulte du choix que nous avons fait du point 10, que les deux membres de cette équation ne peuvent pas être identiques, à moins que le triangle p_8, p_9, p_{10} ne coincide avec le triangle 8, 9, 10. Mais, en supposant toujours que les sept points 1, ..., 7 n'appartiennent pas tous à une même conique, on remarquera que, si cette coincidence a lieu, les courbes biquadratiques (a), (b), (c) doivent avoir des points doubles aux points 8, 9, 10 respectivement; c'est à dire, que les trois points cherchés coincident avec ces mêmes points, et que toutes les courbes biquadratiques qui passent par les treize points ont en ces trois points des tangentes communes, dont on trouve facilement la direction, en se servant des courbes (a), (b), (c). En revenant donc au cas général, l'équation (d) sera l'équation d'une courbe cubique, qui passe évidemment par les six points 8, 9, 10, p_8, p_9, p_{10} et par les trois points d'intersection de $(p_8, 10)$, $(p_{10}, 8)$, $(p_8, 9)$; $(p_{10}, 9)$, $(p_9, 10)$. Mais cette courbe passe aussi par les points 11, ..., 16. Soit ξ un de ces points; la chose est évidente, si ξ n'appartient à aucune des courbes cubiques

$$(1, \ldots, 7, 8, 9), \quad (1, \ldots, 7, 9, 10), \quad (1, \ldots, 7, 10, 8);$$

ensuite, si ξ appartient à une seulement de ces courbes, par exemple à la première, ξ sera le point d'intersection de $(p_9, 8)$ et $(p_8, 9)$, et ne cessera point d'appartenir à (d); enfin, si ξ appartient à la fois à deux des mêmes courbes cubiques (ce qui par hypothèse ne peut pas arriver, à moins que ξ ne soit un des trois points inconnus), ξ sera le neuvième point appartenant à une base cubique donnée, et pourra être déterminé linéairement, sans qu'il soit nécessaire de chercher la cubique (d).

Il est assez remarquable que la solution que nous avons donnée du problème de cet article, s'applique aussi à cet autre problème plus général :

'Étant donné $4n-3$ des $4n$ points d'intersection d'une courbe d'ordre n avec une courbe biquadratique, trouver les trois autres points.' On suppose $n>4$.

Prenons $4n-8$ des $4n-3$ points donnés ; ajoutons-y

$$\frac{(n-1)(n+2)}{1.2}-1-(4n-8)$$

points choisis arbitrairement, et considérons l'ensemble des $\frac{(n-1)(n+2)}{1.2}-1$ points comme déterminant la base P_{4n-8} d'un faisceau de courbes d'ordre $n-1$. En considérant cette base par rapport aux cinq points $4n-7$, $4n-6$, $4n-5$, $4n-4$, $4n-3$, on aura un point *opposé* p_{4n-8}, qui sera le centre d'un faisceau de droites correspondant anharmoniquement aux courbes du faisceau P_{4n-9}. Le lieu des intersections des lignes correspondantes des deux faisceaux sera une courbe de l'ordre n, qui passera par les $4n-3$ points donnés, et, par conséquent, par les trois points cherchés. En permutant cycliquement les points $4n-8$, $4n-7$, $4n-6$ (sans changer autrement la base des courbes de l'ordre $n-1$), on aura deux autres courbes de l'ordre n passant, comme la première, par les $4n$ points. De là en suivant, soit notre démonstration géométrique, soit l'analyse précédente, on conclura que la courbe cubique des neuf points $4n-8, \ldots, 4n$ passe aussi par les points opposés p_{4n-8}, p_{4n-7}, etc ..., et par les points d'intersection tels que celui des droites $(p_{4n-8}, 4n-7)$ et $(p_{4n-7}, 4n-8)$. On voit donc que tout se réduit à la détermination des points opposés, détermination qui sera facile, lorsqu'on connaîtra les rapports anharmoniques des faisceaux

$$P_{4n-8}.[4n-7,\ 4n-6,\ 4n-5,\ 4n-4,\ 4n-3].$$

Or, il n'est pas douteux, qu'étant donné des points en nombre suffisant pour déterminer une courbe géométrique d'ordre n, on ne puisse trouver linéairement, soit la tangente à cette courbe en un point donné, soit la droite polaire d'un point donné par rapport à la courbe ; mais, puisqu'il paraît qu'on n'a pas encore cherché la solution de ce problème général de géométrie linéaire, nous ferons voir qu'on peut s'en passer ici, en se servant d'une méthode particulière qui se présente naturellement. Considérons les points $1, \ldots, 8$, comme déterminant la base A d'un faisceau de courbes cubiques. Soit (a) la conique qui satisfait à l'équation

$$(9, 10, 11, 12) = A.[9, 10, 11, 12].$$

En prenant successivement différents points a de cette conique pour des points opposés à la base A, on aura un faisceau de courbes biquadratiques, correspondant anharmoniquement aux points a. Soit B la base du faisceau biquadratique ; il est évident, qu'étant donné un point quelconque x, on pourra trouver linéairement le point a correspondant à la courbe (B, x). Déterminons la conique (b) qui satisfait à l'équation

$$(13, 14, 15, 16) = B.[13, 14, 15, 16].$$

Prenons un point quelconque b de (b) ; nous le considérons comme un point opposé à la base biquadratique B ; et nous aurons de la sorte un faisceau de courbes du cinquième ordre qui correspondront anharmoniquement aux points b. Soit C la base de ce nouveau faisceau ; pour trouver linéairement le point b qui correspond à une courbe quelconque (C, x) du faisceau, on détermine sur la conique (b) le point x' qui satisfait à l'équation anharmonique

$$[13, 14, 15, 16, x'] = B.[13, 14, 15, 16, x'];$$

le point b est le second point d'intersection de la conique par la droite xx'. Après avoir déterminé une troisième conique (c), qui satisfait à l'équation

$$(17, 18, 19, 20) = C.[17, 18, 19, 20],$$

on sera conduit à considérer une base D de courbes du sixième ordre, et, en continuant de la sorte, on arrivera enfin à une base M de courbes de l'ordre $n-1$; cette base comprendra tous les points 1, 2, 3, ..., $4n-8$, puisque chaque base de la série ascendante comprend évidemment tous les points de la base précédente ; de plus, les courbes du faisceau M correspondront anharmoniquement aux points d'une certaine conique (l) passant par les points $4n-11$, $4n-10$, $4n-9$, $4n-8$; de sorte qu'étant donné un point quelconque x, on pourra trouver linéairement le point l de cette conique qui correspond à la courbe (M, x). Enfin, on déterminera la conique (m) qui satisfait à l'équation

$$(4n-7,\ 4n-6,\ 4n-5,\ 4n-4) = M.(4n-7,\ 4n-6,\ 4n-5,\ 4n-4),$$

et le point m'_{4n-3} de cette dernière conique qui correspond à la courbe $(M, 4n-3)$; le second point d'intersection de la conique (m) avec la droite $(4n-3, m'_{4n-3})$ sera le point p_{4n-8}, qu'il s'agissait de trouver.

On peut dire que cette construction est composée de transformations homographiques successives. On commence par déterminer la conique Γ qui satisfait a l'équation

$$[5, 6, 7, 8] = (1, 2, 3, 4).[5, 6, 7, 8].$$

Soit $\nu'[\nu = 9, ..., 4n-3]$ le point de cette conique qui corresponde anharmoniquement à la conique $(1, 2, 3, 4, \nu)$: le second point d'intersection de la droite (ν, ν') avec Γ est le point ω_ν de Γ qui correspond anharmoniquement à la cubique (A, ν). On transforme homographiquement la figure de manière que les points ω_9, ω_{10}, ω_{11}, ω_{12} deviennent les points 9, 10, 11, 12 ; la transformée de Γ sera (a) ; soit $a'_\nu[\nu = 13, ..., 4n-3]$ le point de cette dernière conique qui correspond au point ω_ν de la conique Γ ; le second point d'intersection de la droite (ν, a'_ν) avec (a) sera le point a_ν, qui correspond anharmoniquement à la courbe biquadratique (B, ν). On transforme encore la figure de manière que les points a_{13}, a_{14}, a_{15}, a_{16} deviennent les points 13, 14, 15, 16 ; on détermine la conique (b) correspondant à (a), et le point $b'_\nu[\nu = 17, ..., 4n-3]$, correspondant à a_ν ; on mène la droite (ν, b'_ν), qui détermine le point b_ν, correspondant anharmoniquement à la courbe (C, ν) ; et en continuant cette série uniforme d'opérations linéaires on parvient enfin à déterminer le point m_{4n-3} de la conique (m), qui n'est autre que le point opposé p_{4n-8}.

XXIV.

ARITHMETICAL NOTES.

[Proceedings of the London Mathematical Society, vol. iv. pp. 236–253. The three papers which form these Notes were read on January 9 and February 13, 1873.]

I.—*On the Arithmetical Invariants of a Rectangular Matrix, of which the Constituents are Integral Numbers.*

1. LET $\|a_{ij}\|$ represent a rectangular matrix of the type $n \times (n+m)$, and let ∇_s be the greatest common divisor of the $\dfrac{\Pi n}{\Pi s \,.\, \Pi (n-s)} \times \dfrac{\Pi (n+m)}{\Pi s \,.\, \Pi (n+m-s)}$ minor determinants of order s which appertain to it; so that, in particular, ∇_n is the greatest common divisor of the $\dfrac{\Pi (m+n)}{\Pi n \,.\, \Pi m}$ determinants of the matrix; the numbers $\nabla_n, \nabla_{n-1}, \ldots, \nabla_1$ are the *arithmetical invariants* of the matrix. We suppose that the matrix is asyzygetic; *i.e.* that the $\dfrac{\Pi (m+n)}{\Pi n \,.\, \Pi m}$ determinants are not all equal to zero. We then have the theorem (Memoir, p. 388 *sqq.* *):

Theorem (α). 'The quotient $\dfrac{\nabla_s}{\nabla_{s-1}}$ is the greatest common divisor of the quotients obtained by dividing each minor determinant of order s by the greatest common divisor of its own first minors.'

Let p be a prime dividing ∇_i, and let I_i be the exponent of the highest power of p which divides ∇_i; any minor of order i which is divisible by $p^{I_i + \sigma}$

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* The references in this and the two following notes are to a Memoir 'On Systems of Linear Indeterminate Equations and Congruences' (Phil. Trans. vol. cli. pp. 293–326.) [This Memoir is No. XII of vol. i, pp. 367–406. The references in the text are to the pages of the Memoir as printed in vol. i.]

may be said to be divisible *in excess* by p^σ. Using this abbreviated mode of expression, we may enunciate the two following corollaries:

Corollary (*b*). 'Any minor determinant, which is not divisible in excess by p, contains at least one first minor which is not divisible in excess by p.'

Corollary (*c*). 'Any minor determinant, which is not divisible in excess by p^σ, contains at least one first minor which is not divisible in excess by p^σ.'

Or, which is the same thing,

'If all the first minors of a given minor are divisible in excess by p^σ, the given minor is itself divisible in excess by p^σ.'

Of these corollaries, the first is a particular case of the second; and the second is only a re-statement, in other words, of the theorem (*a*).

2. It is the object of this note to establish a theorem, which may be regarded as reciprocal to the theorem (*a*).

Theorem (A). 'The fraction $\frac{\nabla_{s-1}}{\nabla_s}$ is the greatest common divisor of the fractions obtained by dividing each minor of order $s-1$ by the greatest common divisor of its first majors; or, which is the same thing, the integral number $\frac{\nabla_s}{\nabla_{s-1}}$ is the least common denominator of these fractions.'

Any square matrix which contains a given square matrix is here called, for brevity, a major of that matrix; if the given square matrix be of order $s-1$, its *first* majors are the square matrices of order s which contain it.

The theorem (A) admits of two corollaries, corresponding to the corollaries (*b*) and (*c*):

Corollary (B). 'Any minor determinant, which is not divisible in excess by p, is contained in at least one first major which is not divisible in excess by p.'

Corollary (C). 'Any minor determinant, which is not divisible in excess by p^σ, is contained in at least one first major which is not divisible in excess by p^σ.'

Or, which is the same thing,

'If all the first majors of a given minor are divisible in excess by p^σ, the given number is itself divisible in excess by p^σ.

3. To prove the theorem (A), we consider, in the first place, a *square* matrix $\| a_{ij} \|$ of the type $n \times n$, so that $\nabla_n = \Sigma \pm a_{11} a_{22} \dots a_{nn}$; and we represent the minor determinant $\frac{d \nabla_n}{d a_{ij}}$ by A_{ij}, so that the reciprocal matrix is $\| A_{ij} \|$. The determinant of this matrix is $\overset{n-1}{\nabla}_n$, and the greatest common divisor of its minor determinants of order i is $\overset{i-1}{\nabla}_n \times \nabla_{n-i}$.

Let $M = \dfrac{\Pi n}{\Pi(s-1).\Pi(n-s+1)}$, $N = n-s+1$; let k_μ $[\mu = 1, 2, 3, \ldots, M^2]$ represent any one of the M^2 minors of order $s-1$ which appertain to the matrix $\|a_{ij}\|$; let $k_{\mu,\nu}$ $[\nu = 1, 2, 3, \ldots, N^2]$ represent any one of the first majors of k_μ; let d_μ be the greatest common divisor of these N^2 numbers; lastly, let K_μ and $K_{\mu,\nu}$ represent the minors which, in the reciprocal matrix $\|A_{ij}\|$, are reciprocal to the minors k_μ and $k_{\mu,\nu}$ in the matrix $\|a_{ij}\|$; so that, for example, if

$$k_\mu = \Sigma \pm a_{11} a_{22} \ldots a_{s-1,\,s-1}, \qquad K_\mu = \Sigma \pm A_{s,s} A_{s+1,\,s+1} \ldots A_{n,n};$$

and if $$k_{\mu,\nu} = \Sigma \pm a_{11} a_{22} \ldots a_{s,s}, \qquad K_{\mu,\nu} = \Sigma \pm A_{s+1,\,s+1} \ldots A_{n,n}.$$

Thus the N^2 first minors of K_μ are precisely the determinants $K_{\mu,\nu}$; we also have the equations $K_\mu = \underset{n}{\overset{n-s}{\nabla}} \times k_\mu$, $K_{\mu,\nu} = \underset{n}{\overset{n-s-1}{\nabla}} \times k_{\mu,\nu}$; so that, applying the theorem (a) to the matrix $\|A_{ij}\|$, we find that the greatest common divisor of the M^2 integral numbers,

$$\left[\underset{n}{\overset{n-s}{\nabla}} \times k_\mu\right] \div \left[\underset{n}{\overset{n-s-1}{\nabla}} \times d_\mu\right] = \nabla_n \times \frac{k_\mu}{d_\mu},$$

is the quotient

$$\left[\underset{n}{\overset{n-s}{\nabla}} \times \nabla_{s-1}\right] \div \left[\underset{n}{\overset{n-s+1}{\nabla}} \times \nabla_s\right] = \nabla_n \times \frac{\nabla_{s-1}}{\nabla_s};$$

that is to say, the greatest common divisor of the M^2 fractions $\dfrac{k_\mu}{d_\mu}$ is the fraction $\dfrac{\nabla_{s-1}}{\nabla_s}$, in accordance with the theorem (A). The corollaries (B) and (C) may be immediately verified by observing that, if either of them were supposed untrue for any given minor k_μ, the corresponding fraction $\dfrac{k_\mu}{d_\mu}$ would acquire a denominator which could not be a sub-multiple of $\dfrac{\nabla_s}{\nabla_{s-1}}$.

4. To extend the demonstration to the case of an oblong matrix of the type $n \times (m+n)$, we retain the preceding notations, so far as they are applicable; and we put $M_1 = \dfrac{\Pi(n+m)}{\Pi(s-1).\Pi(n+m-s+1)}$, $N_1 = n+m-s+1$; so that the values of μ and ν are now $[1, 2, \ldots, M \times M_1]$ and $[1, 2, \ldots, N \times N_1]$ respectively. Let Q be a common multiple of the $M \times M_1$ numbers d_μ, and let us complete the given rectangular matrix into a square one by adding rows of constituents each of which is divisible by Q. In the resulting square matrix, any given minor k_μ (appertaining to the given rectangular matrix) will have N_1^2 first majors; but it

is evident that the greatest common divisor of these N_1^2 first majors will be the same as that of the $N \times N_1$ first majors which appertain to the given rectangular matrix. Hence the $M \times M_1$ fractions $\frac{k_\mu}{d_\mu}$, obtained by dividing each minor of order $s-1$ in the given matrix by the greatest common divisor of its first majors, occurs among the M_1^2 fractions similarly derived from the completed square matrix; also the numbers ∇_s and ∇_{s-1} are evidently the same for both matrices. Applying, therefore, the theorem (A) to the square matrix, we see that $\frac{\nabla_{s-1}}{\nabla_s}$ is a common divisor of the $M \times M_1$ fractions $\frac{k_\mu}{d_\mu}$; that it is the greatest common divisor of these fractions, may be proved by considering a minor determinant of order s in the given matrix, which is not divisible in excess by p; this minor determinant contains [by the corollary (b)] a first minor which is itself not divisible in excess by p; if k_μ be the first minor, the denominator of the corresponding fraction $\frac{k_\mu}{d_\mu}$ necessarily contains the prime p raised to the power $I_s - I_{s-1}$.

It will be observed that, whether we consider a rectangular or a square matrix, the corollary (C) is an immediate consequence from the theorem (A); but the theorem (A) does not follow conversely from the corollary (C). For the absence of factors prime to $\frac{\nabla_s}{\nabla_{s-1}}$ from the least common denominator of the fractions $\frac{k_\mu}{d_\mu}$ is asserted in the theorem, but is not asserted in the corollary.

5. Every common divisor of the first minors of a given minor is evidently a common divisor of the first majors of that minor; and, if the matrix be square, a consideration of the reciprocal matrix shows that, conversely, every common divisor, prime to the determinant of the matrix, of the first majors of any given minor, divides the first minors of that minor. We thus obtain the self-reciprocal theorem:

Theorem (dD). 'In any square matrix, the greatest common divisor of the first minors of any given minor is identical with the greatest common divisor of its first majors, so far as factors prime to the determinant of the matrix are concerned.'

This theorem is not universally true in the case of oblong matrices, at least if we suppose that, in its enunciation, the words 'determinant of the matrix' are replaced by the words 'greatest common divisor of the determinants of the matrix.' If, for example, $\|a\|$ is a square matrix of order $n-1$, and $\|b\|$ is a

matrix of the type $n \times (n+m)$, of which the determinants are relatively prime, the symbol $\left\| \begin{matrix} 0 \\ a \end{matrix} \; b \right\|$ may serve to represent a matrix of the type $n \times (2n+m-1)$, in which the first $n-1$ constituents of the uppermost row are zeros. In this matrix the greatest common divisor of the first majors of $|a|$ is evidently $|a|$ itself; and this greatest common divisor is prime to the greatest common divisor of the determinants of the matrix, because these determinants are relatively prime; but unless $\|a\|$ is an unit matrix, its determinant cannot be equal to the greatest common divisor of its own first minors.

6. It may be added, that the properties to which this note refers admit of being stated in a generalized form. Thus, we may replace the enunciations of the theorems (a), (A), and (dD) by the following:

Theorem (a'). 'The quotient $\frac{\nabla_s}{\nabla_{s-i}}$ is the greatest common divisor of the quotients obtained by dividing each minor determinant of order s by the greatest common divisor of its own first minors of order $s-i$.'

Theorem (A'). 'The fraction $\frac{\nabla_{s-i}}{\nabla_s}$ is the greatest common divisor of the fractions obtained by dividing each minor of order $s-i$ by the greatest common divisor of its own majors of order s.'

Theorem (dD'). 'In any square matrix, the greatest common divisor of the minors of order $s-i$ appertaining to a given minor of order s is identical with the greatest common divisor of the majors of order $s+i$ appertaining to the given minor, so far as factors prime to the determinant of the matrix are concerned.'

A very slight modification of the proof of the theorem (a), (Memoir, pp. 397–399,) supplies a proof of the theorem (a'), and from it the theorems (A') and (dD') may be inferred by means of the methods employed in this note to establish the theorems (A) and (dD).

II.—*On Systems of Linear Congruences.*

1. Let $$A_{i,1}\,x_1 + A_{i,2}\,x_2 + \ldots + A_{i,n}\,x_n \equiv A_{i\,n+1} \pmod{M}, \quad \ldots\ldots \quad (1)$$
$$i = 1, 2, 3, \ldots. n,$$

represent a system of n linear congruences; let

$$D_n,\ D_{n-1},\ \ldots\ldots,\ D_1,$$
$$\nabla_n,\ \nabla_{n-1},\ \ldots\ldots,\ \nabla_1,$$

respectively denote the arithmetical invariants of the augmented and unaugmented matrices of the system; *i.e.* of the matrices

$$\left.\begin{matrix} \|A_{ij}\| & i=1, 2, \ldots, n, \\ & j=1, 2, \ldots, n+1, \end{matrix}\right\} \quad \ldots\ldots\ldots \quad (2)$$

and

$$\left.\begin{matrix} \|A_{ij}\| & i=1, 2, \ldots, n, \\ & j=1, 2, \ldots, n; \end{matrix}\right\} \quad \ldots\ldots\ldots \quad (3)$$

let d_s and δ_s be the greatest common divisors of M with $\frac{D_s}{D_{s-1}}$, and of M with $\frac{\nabla_s}{\nabla_{s-1}}$ respectively; lastly, let $d = d_1 \times d_2 \times \ldots \times d_n$, $\delta = \delta_1 \times \delta_2 \times \ldots \times \delta_n$. We then have the two theorems (Memoir, p. 399):

'The necessary and sufficient condition for the resolubility of the system (1) is $d = \delta$.'

'When this condition is satisfied, the number of its incongruous solutions is $d = \delta$.'

There are similar theorems (Memoir, p. 402) relating to defective and redundant systems of congruences. Some observations which may serve in certain cases to facilitate the applications of the theory are contained in the present note.

2. Consider separately the powers of the different prime numbers dividing the modulus M. Let p be one of these primes; and let μ, a_s, α_s be the exponents of the highest powers of p which divide M, D_s, ∇_s respectively. Then, because $\frac{D_{s+1}}{D} \div \frac{D_s}{D_{s-1}}$, $\frac{\nabla_{s+1}}{\nabla_s} \div \frac{\nabla_s}{\nabla_{s-1}}$, $\nabla_s \div D_s$, and $\frac{\nabla_s}{\nabla_{s-1}} \div \frac{D_s}{D_{s-1}}$, are all integral (Memoir, pp. 396–398), we have the inequalities

$$a_n - a_{n-1} \geqq a_{n-1} - a_{n-2} \geqq \ldots \geqq a_1 - a_0; \quad \ldots\ldots\ldots \quad (4)$$

$$\alpha_n - \alpha_{n-1} \geqq \alpha_{n-1} - \alpha_{n-2} \geqq \ldots \geqq \alpha_1 - \alpha_0; \quad \ldots\ldots\ldots \quad (5)$$

$$\alpha_s \geqq a_s; \quad \ldots\ldots\ldots \quad (6)$$

$$\alpha_s - \alpha_{s-1} \geqq a_s - a_{s-1}. \quad \ldots\ldots\ldots \quad (7)$$

Let $a_\sigma - a_{\sigma-1}$ be the first term in the series (4) which is less than μ; so that, if $a_n - a_{n-1} < \mu$, we have $\sigma = n$, and in every other case

$$a_{\sigma+1} - a_\sigma \geqq \mu > a_\sigma - a_{\sigma-1}. \quad \ldots\ldots\ldots \quad (8)$$

We may then replace the two theorems of the Memoir by the two following:—

'The necessary and sufficient condition for the resolubility of the congruences (1), considered with regard to the modulus p^μ, is

$$\alpha_\sigma = \alpha_\sigma.' \quad \ldots\ldots\ldots\ldots \quad (9)$$

'When this condition is satisfied, the number of incongruous solutions is $p^{\alpha_\sigma + (n-\sigma)\mu}$.'

For the condition of resolubility is (Memoir, p. 400) that the greatest common divisor of the numbers

$$p^{\alpha_n},\ p^{\alpha_{n-1}+\mu},\ p^{\alpha_{n-2}+2\mu},\ \ldots,\ p^{n\mu}, \quad \ldots\ldots\ldots \quad (10)$$

should be the same as the greatest common divisor of the numbers

$$p^{\alpha_n},\ p^{\alpha_{n-1}+\mu},\ p^{\alpha_{n-2}+2\mu},\ \ldots,\ p^{n\mu}. \quad \ldots\ldots\ldots \quad (11)$$

The greatest common divisor of the numbers (10) is $p^{\alpha_\sigma + (n-\sigma)\mu}$; for it follows from the inequalities (8) and (4) that, in the series (10), the exponent $\alpha_\sigma + (n-\sigma)\mu$ is less than any exponent which follows it, and not greater than any which precedes it. Again, if $\alpha_\sigma = \alpha_\sigma$, $p^{\alpha_\sigma + (n-\sigma)\mu}$ is also the greatest common divisor of the numbers (11); for it is equal to one of them, and it certainly divides all of them, because it divides all the numbers (10), each of which, by virtue of the inequalities (6), divides one of the numbers (11). But if $\alpha_\sigma > \alpha_\sigma$, the greatest common divisor of the numbers (11) is a power of p having an exponent higher than $\alpha_\sigma + (n-\sigma)\mu$. For the inequality $\alpha_\sigma > \alpha_\sigma$, combined with the inequality (7), implies the successive inequalities $\alpha_{\sigma+1} > \alpha_{\sigma+1}$, $\alpha_{\sigma+2} > \alpha_{\sigma+2}$, ..., $\alpha_n > \alpha_n$; so that every exponent in the series (11), which precedes $\alpha_\sigma + (n-\sigma)\mu$, surpasses the corresponding exponent in (10), and therefore certainly surpasses $\alpha_\sigma + (n-\sigma)\mu$. And again, the exponents following $\alpha_\sigma + (n-\sigma)\mu$ in the series (11) also surpass $\alpha_\sigma + (n-\sigma)\mu$, for these exponents are not less than the corresponding exponents in (10), each of which is greater than $\alpha_\sigma + (n-\sigma)\mu$. Thus the condition for the equality of the two greatest common divisors, *i.e.* for the resolubility of the congruences (1), is $\alpha_\sigma = \alpha_\sigma$. And, by transforming the proposed system (Memoir, p. 401) into an equivalent system of the type

$$\frac{\nabla_{n-s+1}}{\nabla_{n-s}} b_s \equiv c_s \pmod{p^\mu}, \quad \ldots\ldots\ldots \quad (12)$$

it is immediately ascertained that, when this condition is satisfied, the number of incongruous solutions is equal to $p^{\alpha_\sigma + (n-\sigma)\mu}$, *i.e.* to the common value of the two greatest common divisors.

3. The criterion of resolubility just established differs in form only from that

given in the Memoir; and the coincidence of the two may be shown by observing that the equation $\alpha_\sigma = a_\sigma$, combined with the inequalities (7), implies the equations $\alpha_{\sigma-1} = a_{\sigma-1}$, $\alpha_{\sigma-2} = a_{\sigma-2}$, ..., $\alpha_1 = a_1$. As here stated, the criterion of resolubility, and the expression for the number of solutions, depend only on the numbers σ, ∇_σ, D_σ, and do not involve explicitly the consideration of the two complete series of invariants. When μ surpasses $a_n - a_{n-1}$, the condition of resolubility is $\alpha_n = a_n$, and the number of solutions is p^{a_n}. In this case we only need to determine the invariants ∇_n and D_n, or rather the exponents of the highest powers of p dividing those numbers. But when $\mu \leqq a_n - a_{n-1}$, it is necessary to calculate successively the invariants D_n, D_{n-1}, D_{n-2}, ... in order to ascertain the index σ for which the inequalities (8) are satisfied. It will be observed, however, that since every minor of order s, which is not divisible in excess by p, contains a minor of order $s-1$ which is itself not divisible in excess by p, it is always possible to restrict the examination of the minors of order $s-1$ (which is required in order to obtain the value of D_{s-1}) to those which are contained in a single minor of order s.

We may also notice that if, in the series of differences

$$a_n - a_n,\ a_{n-1} - a_{n-1},\ \ldots,\ a_1 - a_1,$$

$a_s - a_s$ is the first which is equal to zero, $p^{a_{s+1} - a_s}$ is the highest power of p for which the proposed congruences are resoluble.

4. In the case of a redundant or defective system, as well as in the case of a system such as (1), which is neither defective nor redundant, it will be found that the equation of condition $d = \delta$ may be replaced by the equation $\alpha_\sigma = a_\sigma$. And in all these cases alike, whenever the system admits of any solution at all, the number of incongruous solutions is $p^{a_\sigma + (n-\sigma)\mu}$, n denoting the number of the indeterminates.

5. In the case of a redundant system, the condition $d = \delta$, or $\alpha_\sigma = a_\sigma$, although necessary is not sufficient (Memoir, p. 404). The complete condition of resolubility in this case is that the greatest common divisor of the numbers

$$p^{a_{n+1}},\ p^{a_n + \mu},\ p^{a_{n-1} + 2\mu},\ \ldots,\ p^{(n+1)\mu}, \quad \ldots\ldots\ldots \quad (13)$$

should be the same as the greatest common divisor of the numbers

$$p^{a_n + \mu},\ p^{a_{n-1} + 2\mu},\ \ldots,\ p^{(n+1)\mu}. \quad \ldots\ldots\ldots \quad (14)$$

And the identity of these two greatest common divisors implies not only (as in Art. 2) the equation $\alpha_\sigma = a_\sigma$, but also the inequality

$$\mu \leqq a_{n+1} - a_n; \quad \ldots\ldots\ldots\ldots \quad (15)$$

since, if this condition be not satisfied, a_{n+1} will be less than any of the exponents following it in the series (13), and therefore less than any exponent in the series (14).

The two conditions expressed by the equation $a_\sigma = a_\sigma$, and the inequality (15), are sufficient as well as necessary. But it is remarkable that the exponent μ not only satisfies the inequality (15), but also the inequality

$$\mu \leqq a_{n+1} - a_n, \quad \ldots \ldots \ldots \ldots \quad (16)$$

which is, in general, closer than (15), because $a_n \geqq a_n$. To prove this, consider, in the given redundant system, any set of $n+1$ congruences such that the determinant of their augmented matrix is not divisible in excess by p; and multiply these $n+1$ congruences, taken in order, by the determinants of their unaugmented matrix. The sum of the products is the determinant of the augmented matrix; we infer that this determinant is divisible by $p^{a_n+\mu}$, *i.e.* that $a_{n+1} \geqq a_n + \mu$.

Since (as has been said) the two conditions of resolubility, taken together, are sufficient, they must involve the inequality (16). To deduce it from them, we may employ the following lemma:

'Let $\|Q_{ij}\|$ $\begin{matrix} i = 1, 2, \ldots, n, \\ j = 1, 2, \ldots, n+m, \end{matrix}$

represent any square or oblong matrix of integral numbers, and let $\|q\|$ be a partial matrix consisting of any s of the n horizontal rows of $\|Q\|$; also let Q_n, $Q_{n-1}, \ldots$; $q_s, q_{s-1}, \ldots$ be the arithmetical invariants of $\|Q\|$ and $\|q\|$ respectively; then, if r is any number not surpassing s, $\dfrac{Q_n}{Q_{n-r}}$ is divisible by $\dfrac{q_s}{q_{s-r}}$.'

To establish this lemma, let us suppose that $\|q\|$ consists of the last s rows of $\|Q\|$, and let us replace $\|Q\|$ by a compound matrix of the form $\|Q'\| \times \|Q''\|$, in which $\|Q''\|$ is a prime matrix of the same type as $\|Q\|$, and $\|Q'\|$ is a square matrix of the form

$$\left\| \begin{matrix} h_1, & k_{1,2}, & k_{1,3}, & \ldots, & k_{1,n} \\ 0, & h_2, & k_{2,3}, & \ldots, & k_{2,n} \\ 0, & 0, & h_3, & \ldots, & k_{3,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \ldots, & h_n \end{matrix} \right\|$$

Then the greatest common divisor of any horizontal row of minors in $\|Q\|$ is the same as the greatest common divisor of the corresponding row of minors in $\|Q'\|$

(Memoir, p. 389); it is sufficient, therefore, to verify the lemma in the case of this last matrix. In it we have

$$h_1 \times h_2 \times \ldots \times h_n = Q_n,$$
$$h_{n-s+1} \times h_{n-s+2} \times \ldots \times h_n = q_s;$$

and if we multiply $h_1 \times h_2 \times \ldots \times h_{n-s} = \frac{Q_n}{q_s}$ by any minor of order $s-r$ contained in the last s rows of $\|Q'\|$, we obtain a product which is either a minor of $\|Q'\|$ of the order $n-r$, or else is equal to zero; this product is therefore in every case divisible by Q_{n-r}; *i.e.* $\frac{Q_n}{q_s} \times q_{s-r}$ is divisible by Q_{n-r}, or $\frac{Q_n}{Q_{n-r}}$ is divisible by $\frac{q_s}{q_{s-r}}$.

To apply this lemma to the case which we are considering here, we change n into $n+1$, and we put $s=n$, $r=n-\sigma$. Attending only to the powers of p, we thus find

$$a_{n+1} - a_n \geqq a_{\sigma+1} - a_\sigma.$$

But $a_\sigma = a_\sigma$, and $a_{\sigma+1} - a_\sigma \geqq \mu$; therefore $a_{n+1} - a_n \geqq \mu$.

6. When the determinants of the augmented or unaugmented matrix of a proposed system of congruences vanish, we may add multiples of the modulus to the constituents of these matrices, so as to obtain determinants which do not vanish. We proceed, however, to show that this preliminary operation is unnecessary, and that the preceding theory is immediately applicable, notwithstanding the evanescence of the determinants.

If, in any matrix of which the arithmetical invariants are $\nabla_n, \nabla_{n-1}, \ldots, \nabla_1$ the minor determinants of order $s+1$ are all equal to zero, we must attribute the same value to the invariant ∇_{s+1}; and *vice versâ*, if $\nabla_{s+1} = 0$, the minor determinants of order $s+1$ must themselves be all equal to zero. Let $\nabla_{s+1} = 0$, $\nabla_s > 0$, so that $\nabla_{s+r} = 0$ $(r = 1, 2, \ldots, n-s)$, $\nabla_{s-r} > 0$ $(r = 0, 1, 2, \ldots, s-1)$. The quotients $\frac{\nabla_s}{\nabla_{s-1}}, \frac{\nabla_{s-1}}{\nabla_{s-2}}, \ldots, \frac{\nabla_1}{\nabla_0}$ are determinate positive integers, and the quotient $\frac{\nabla_{s+1}}{\nabla_s}$ is zero; but the quotients $\frac{\nabla_{s+2}}{\nabla_{s+1}}, \ldots, \frac{\nabla_n}{\nabla_{n-1}}$, assume the indeterminate form $\frac{0}{0}$. To these quotients we attribute the value zero: this may be regarded as an arbitrary definition of the value of these indeterminate numbers; it is, however, suggested by the observation that if the constituents of the matrix, instead of being integral numbers, were rational and integral functions of an indeterminate quantity, every factor of ∇_i would also be a factor of ∇_{i+1}, and would be contained in ∇_{i+1} oftener than in ∇_i; *i.e.* $\frac{\nabla_{i+1}}{\nabla_i}$ would vanish with ∇_i.

7. Let us first suppose that the proposed system of congruences,

$$A_{i,1}\,x_1 + A_{i,2}\,x_2 + \ldots + A_{i,n+m}\,x_{n+m} \equiv A_{n+m+1} \pmod{p^\mu}, \quad . \quad . \quad . \quad (17)$$
$$i = 1, 2, 3, \ldots, n, \qquad m \geqq 0,$$

is not redundant; and let the invariants ∇_n, ∇_{n-1}, ... of its unaugmented matrix vanish as far as ∇_{s+1} inclusively, but let ∇_s not vanish. The integers

$$\alpha_n, \ \alpha_{n-1}, \ \ldots, \ \alpha_{s+1},$$
$$\alpha_n - \alpha_{n-1}, \ \ldots, \ \alpha_{s+1} - \alpha_s,$$
$$\alpha_n, \ \alpha_{n-1}, \ \ldots, \ \alpha_{s+2},$$
$$\alpha_n - \alpha_{n-1}, \ \ldots, \ \alpha_{s+2} - \alpha_{s+1},$$

may then be regarded as greater than any assigned number; and if, in the inequalities (8), which determine the index σ, we adopt this interpretation of these symbols, the criterion of resolubility, and the expression for the number of solutions, will continue applicable to the system (17) without any modification whatever. For the criterion of resolubility this is evident, because the demonstration of Art. 2 subsists unchanged. To prove the same thing for the formula which expresses the number of solutions, we employ the following lemma:

'The matrix $\quad \|A_{i,j}\| \quad \begin{matrix} i = 1, 2, \ldots, n, \\ j = 1, 2, \ldots, n+m, \end{matrix}$

(in which ∇_{s+1} vanishes, but ∇_s does not vanish,) satisfies an equation of the form

$$\|\beta\| \times \|A_{i,j}\| \times \|\gamma\| = \begin{Vmatrix} q_{i,j}, & 0 \\ 0, & 0 \end{Vmatrix}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

in which $\|\beta\|$ is an unit matrix of the type $n \times n$, $\|\gamma\|$ an unit matrix of the type $(n+m) \times (n+m)$, $\begin{Vmatrix} q_{i,j}, & 0 \\ 0, & 0 \end{Vmatrix}$ a matrix of the type $n \times (n+m)$, which has all its constituents equal to zero, except those composing the partial matrix $\|q_{ij}\|$; this partial matrix being of the type $s \times s$, and having for its arithmetical invariants $\nabla_s, \nabla_{s-1}, \ldots, \nabla_0$.'

Consider the system

$$A_{i,1}\,x_1 + A_{i,2}\,x_2 + \ldots + A_{i,n+m}\,x_{n+m} = 0, \quad . \quad . \quad . \quad . \quad . \quad (18)$$
$$i = 1, 2, 3, \ldots, n,$$

which is equivalent to s independent equations; and take for $\|\gamma\|$ any unit

matrix of the type $(n+m)\times(n+m)$, in which the last $n+m-s$ columns are the $n+m-s$ rows of a fundamental set of solutions of the system (18). The compound matrix $\|A\|\times\|\gamma\|$ will be of the form $\|c_{ij}, 0\|$, where $\|c_{ij}\|$ is a partial matrix of the type $n\times s$, and $\|0\|$ is a partial matrix, composed entirely of zeros, of the type $n\times(n+m-s)$; so that $\|c_{ij}, 0\|$ is of the same type as $\|A\|$. Again, consider the system

$$c_{1,j}\,x_1+c_{2,j}\,x_2+\ldots+c_{n,j}\,x_n=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$
$$j=1,\ 2,\ 3,\ \ldots,\ s,$$

of which the equations are independent; and take for $\|\beta\|$ any unit matrix of the type $n\times n$, in which the $n-s$ lowest rows are the $n-s$ rows of a fundamental set of solutions of the system (19). The compound matrix $\|\beta\|\times\|A\|\times\|\gamma\|$ is then of the form $\left\|\begin{matrix} q_{ij}, & 0 \\ 0, & 0 \end{matrix}\right\|$.

If therefore, in the system (17), we transform the indeterminates by the substitution

$$(x_1,\ x_2,\ \ldots,\ x_{n+m})=\|\gamma\|\times(y_1,\ y_2,\ \ldots,\ y_{n+m}),$$

and at the same time replace the congruences themselves by others linearly derived from them by multiplication with the constituents of the rows of $\|\beta\|$, we obtain an equivalent system of the form

$$q_{1,j}\,y_1+q_{2,j}\,y_2+\ldots+q_{s,j}\,y_s\equiv h_j \pmod{p^\mu}, \quad . \quad . \quad . \quad . \quad . \quad (20)$$
$$j=1,\ 2,\ \ldots,\ s,$$

in which the indeterminates $y_{s+1}, \ldots, y_{n+m}$ do not appear explicitly. By the theorem of Art. 2, the number of solutions of (20) is $p^{a_\sigma+(s-\sigma)\mu}$; and to each of these solutions there correspond $p^{(n+m-s)\mu}$ solutions of (17), because each of the indeterminates $y_{s+1}, \ldots, y_{n+m}$ may have any one of p^μ values. Thus the number of solutions of (17) is $p^{a_\sigma+(n+m-\sigma)\mu}$.

8. Next, let the proposed system of congruences be redundant; and, as before, let ∇_{s+1} vanish, but not ∇_s. In this case also the demonstration relating to the criteria of resolubility subsists unchanged; to determine the number of solutions, we select from the proposed system a partial system of s equations such that the determinants of its unaugmented matrix are not divisible in excess by p. Every solution of this partial system satisfies the remaining congruences of the proposed system; *i.e.* the number of solutions of the proposed system is $p^{a_s+(n-\sigma)\mu}$, if n be the number of indeterminates.

9. It is worth while to observe that, in the equation (A), we may attribute

to the partial matrix $\|q_{ij}\|$ the simple form $\left\| \frac{\nabla_s}{\nabla_{s-1}}, \frac{\nabla_{s-1}}{\nabla_{s-2}}, \ldots, \frac{\nabla_1}{\nabla_0} \right\|$. For if $\|u\|$ and $\|v\|$ are reducing units of $\|q_{ij}\|$, so that

$$\|u\| \times \|q_{ij}\| \times \|v\| = \left\| \frac{\nabla_s}{\nabla_{s-1}}, \frac{\nabla_{s-1}}{\nabla_{s-2}}, \ldots, \frac{\nabla_1}{\nabla_0} \right\|,$$

we may replace $\|u\|$ and $\|v\|$, which are both of the type $s \times s$, by unit matrices of the types $n \times n$ and $(n+m) \times (n+m)$ respectively; and may then compound these substituted units with the unit matrices $\|\beta\|$ and $\|\gamma\|$. The unit matrices thus substituted for $\|u\|$ and $\|v\|$ contain $\|u\|$ and $\|v\|$ as partial matrices; their forms are sufficiently indicated by the symbols $\left\| \begin{matrix} u, \\ 0, \end{matrix} \begin{matrix} \\ I \end{matrix} \right\|$, $\left\| \begin{matrix} v, 0 \\ J \end{matrix} \right\|$; where I and J are matrices of the types $n \times (n-s)$ and $(n+m-s) \times (n+m)$ respectively, and are subject to no restriction other than that implied by the equations

$$\left| \begin{matrix} u, \\ 0, \end{matrix} \begin{matrix} \\ I \end{matrix} \right| = 1, \qquad \left| \begin{matrix} v, 0 \\ J \end{matrix} \right| = 1.$$

Another transformation of a matrix, in which the first $n-s$ invariants vanish, is sometimes useful; viz., we may write

$$\|A\| = \|\omega\| \times \left\| 0, 0, \ldots, 0, \frac{\nabla_s}{\nabla_{s-1}}, \frac{\nabla_{s-1}}{\nabla_{s-2}}, \ldots, \frac{\nabla_1}{\nabla_0} \right\| \times \|Q\|,$$

where $\|\omega\|$ is an unit matrix of the type $n \times n$, and $\|Q\|$ is a matrix of the same type as $\|A\|$, of which the s lowest rows form a prime matrix, while the others are arbitrary. The proof of this transformation presents no difficulty, and may be omitted here.

10. When the absolute terms in a linear system of congruences are all congruous to zero, the criteria of resolubility are always satisfied, as they ought to be, because the system is satisfied by attributing to the indeterminates values congruous to zero, and therefore certainly admits of one solution at least. In this case, however, it is usually of importance to determine, not the whole number of solutions, but the number of solutions prime to the modulus (*i.e.* the number of solutions in which one at least of the indeterminates is prime to the modulus). Reducing the proposed congruences to an equivalent system of the form (12), which can always be done by the transformations indicated in the Memoir (p. 401), and in Art. 7 of the present note, we find that if N is the whole number of solutions, $N \div p^{n-\sigma}$ is the number of solutions in which every indeterminate has a value divisible by p, so that $N\left(1 - \frac{1}{p^{n-\sigma}}\right)$ is the number of solutions prime to

the modulus. And, in general, if k_i is the number of terms in the series (5) which satisfy the inequality $a_{s+1} - a_s \geqq \mu - i$, so that, in particular, $k_0 = n - \sigma$, the number of solutions in which every indeterminate has a value divisible by p^i will be found to be $N \div p^{k_0 + k_1 + \ldots + k_{i-1}}$.

11. The transformation of a given square matrix indicated by the equation

$$\|A_{ij}\| = \|\alpha\| \times \left\| \frac{\nabla_n}{\nabla_{n-1}}, \frac{\nabla_{n-1}}{\nabla_{n-2}}, \frac{\nabla_{n-2}}{\nabla_{n-3}}, \ldots, \frac{\nabla_2}{\nabla_1}, \frac{\nabla_1}{\nabla_0} \right\| \times \|\beta\| \quad . \quad . \quad . \quad (21)$$

forms the basis of many of the demonstrations in this and the preceding note. It has been already pointed out (Memoir, p. 392) that this equation always admits of an infinite number of solutions: this observation may be verified as follows. If $\|u_{ij}\|$ is any unit matrix of which the constituents are defined by the equations

$$u_{i,j} = \lambda_{i,j}, \qquad j \leqq i,$$

$$u_{i,j} = \lambda_{i,j} \frac{\nabla_{n-i+1}}{\nabla_{n-1}} \div \frac{\nabla_{n-j+1}}{\nabla_{n-j}}, \qquad j \geqq i,$$

the matrix $\|v_{ij}\|$, of which the constituents are defined by the equations

$$v_{ij} = \lambda_{ij}, \qquad j \geqq 1,$$

$$v_{i,j} = \lambda_{i,j} \frac{\nabla_{n-j+1}}{\nabla_{n-j}} \div \frac{\nabla_{n-i+1}}{\nabla_{n-i}}, \qquad j \leqq i,$$

will also be an unit matrix; and, if $\|\alpha\|$, $\|\beta\|$ be two given unit matrices satisfying the equation (27), that equation will also be satisfied by the unit matrices comprehended in the formulae

$$\|\alpha\| \times \|u\|, \qquad \|v\|^{-1} \times \|\beta\|,$$

and by them only

III.—*On an Arithmetical Demonstration of a Theorem in the Integral Calculus.*

1. Let

$$\left.\begin{aligned} a_{i1} y_1 + a_{i2} y_2 + \ldots + a_{in} y_n &= b_i \\ i &= 1, 2, 3, \ldots, n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

represent a system of linear equations, in which the numbers a_{ij} and b_i are integral, the determinant $\nabla = |a_{ij}|$ being different from zero and positive. If, considering the numbers b_i with regard to the modulus ∇, we attribute to them in

succession the ∇^n systems of values of which they are susceptible with regard to that modulus, we have the arithmetical theorem (Memoir, p. 405):

(i.) 'The system (A) is resoluble in integral numbers y for precisely ∇^{n-1} of the ∇^n systems b.'

2. Let Σ represent a complex, or multiplicity, of order n, of which the indeterminates $y_1, y_2, \ldots, y_n$ are independent. Any system of values assigned to the indeterminates is called a point of Σ; and if the values are integral, the point is an integer point. Let

$$[x_{i1}, x_{i2}, x_{i3}, \ldots, x_{in}]$$
$$i = 1, 2, 3, \ldots, n,$$

represent n given integer points of Σ; we shall suppose these n points to be asyzygetic, so that the determinant $\nabla = |x_{ij}|$ is different from zero, and may be assumed to be positive. We then obtain the theorem:

(ii.) 'The number of integer points $[X_1, X_2, \ldots, X_n]$ which satisfy the inequalities

$$0 < X_1 \frac{d\nabla}{dx_{i1}} + X_2 \frac{d\nabla}{dx_{i2}} + \ldots + X_n \frac{d\nabla}{dx_{in}} \leqq \nabla,$$
$$i = 1, 2, 3, \ldots, n,$$

is precisely ∇.'

For, by the theorem (i.), the equations

$$\left.\begin{array}{c} X_1 \dfrac{d\nabla}{dx_{i1}} + X_2 \dfrac{d\nabla}{dx_{i2}} + \ldots + X_n \dfrac{d\nabla}{dx_{in}} = B_i \\ i = 1, 2, 3, \ldots, n \end{array}\right\}, \quad \ldots\ldots \quad \text{(B)}$$

of which the determinant is ∇^{n-1}, and in which each of the numbers B_i is supposed to receive in succession every value from 1 to ∇^{n-1} inclusive, are resoluble in integral numbers X for $\nabla^{(n-1)^2}$ of these $\nabla^{n(n-1)}$ systems of values. But the solution of the equation (B) is given by the system

$$\nabla X_j = B_1 x_{1j} + B_2 x_{2j} + \ldots + B_n x_{nj},$$
$$j = 1, 2, 3, \ldots, n.$$

If therefore the equations (B) are resoluble in integral numbers for any particular system of values of the numbers B_i, considered with regard to the modulus ∇, they are resoluble for the $\nabla^{n(n-2)}$ systems of values congruous to that particular system for the modulus ∇, but incongruous to one another for the modulus ∇^{n-1}. Hence, if k be the number of systems of values, taken with regard to the modulus ∇, for which the equations (B) are resoluble in integral

numbers, we have $k \times \nabla^{n(n-2)} = \nabla^{(n-1)^2}$, or $k = \nabla$; a result which establishes the theorem (ii.).

3. Another proof of the same theorem may be obtained as follows; The number of systems of values for which the equations (B) are resoluble in integral numbers is equal to the number of solutions of the congruences

$$B_1 x_{1j} + B_2 x_{2j} + \ldots + B_n x_{nj} \equiv 0, \pmod{\nabla},$$
$$j = 1, 2, 3, \ldots, n,$$

in which $B_1, B_2, \ldots, B_n$ are the indeterminates. And the number of solutions of this system of congruences is ∇. (Memoir, p. 399, or Arts. 1–3 of the preceding note.)

4. Let
$$[y_{i1}, y_{i2}, \ldots, y_{in}]$$
$$i = 1, 2, 3, \ldots, n,$$

represent any n asyzygetic points of Σ; and let λ_j be a positive common measure of the n quantities $y_{1j}, y_{2j}, \ldots, y_{nj}$, so that we have n^2 equations of the form $y_{ij} = x_{ij} \times \lambda_j$, in which the numbers x_{ij} are integral, and $|x_{ij}| = \nabla$ is different from zero and positive.

Let $D = |y_{ij}| = \nabla \times \prod_{j=1}^{j=n} \lambda_j$; the number of points of Σ, which are of the type $[X_1\lambda_1, X_2\lambda_2, \ldots, X_n\lambda_n]$, $X_1, X_2, \ldots$ denoting integral numbers, and which satisfy the inequalities

$$0 < (X_1\lambda_1)\frac{dD}{dy_{i1}} + (X_2\lambda_2)\frac{dD}{dy_{i2}} + \ldots + (X_n\lambda_n)\frac{dD}{dy_{in}} \leqq D, \quad . \quad . \quad . \quad . \quad \text{(C)}$$

is, by the theorem (ii.), equal to $D \div \Pi\lambda = \nabla$; that is to say, the product $\Pi\lambda$, taken as many times as there are points $(X_1\lambda_1, X_2\lambda_2, \ldots, X_n\lambda_n)$ satisfying the conditions (C), is equal to D. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be infinitesimal, and let U represent the integral $\int \ldots \int\int dY_1\, dY_2 \ldots dY_n$, extended to all values of $Y_1, Y_2, \ldots, Y_n$ which satisfy the inequalities

$$\left.\begin{array}{c} 0 < Y_1\dfrac{dD}{dy_{i1}} + Y_2\dfrac{dD}{dy_{i2}} + \ldots + Y_n\dfrac{dD}{dy_{in}} \leqq D \\ i = 1, 2, 3, \ldots, n \end{array}\right\};$$

we have, by the definition of a definite integral, $U = D$.

5. This well-known result (which may be described as a generalization of the expression for the volume of a parallelepiped in space referred to three rectangular axes) may of course be otherwise obtained by employing the formula

for the transformation of a multiple integral; viz., if we put

$$u_i = Y_1 \frac{dD}{dy_{i1}} + Y_2 \frac{dD}{dy_{i2}} + \ldots + Y_n \frac{dD}{dy_{in}},$$

the limits of u_i are 0 and D, and the functional determinant is D^{n-1}; hence

$$\int_0^D \ldots \int_0^D \int_0^D du_1\, du_2 \ldots du_n = D^{n-1} \times U, \quad \text{or} \quad D = U.$$

But as we have obtained the value of the definite integral U by a direct process, we may employ the result to demonstrate the formula for the transformation of a multiple integral; and this it may be worth while to do, because some of the demonstrations which have been given of that formula are not free from obscurity.

6. For convenience of expression, we may regard the indeterminates $X_1, X_2, X_3, \ldots, X_n$ as rectangular coordinates of a homogeneous and isotropic space of n dimensions. On this supposition we may define the volume unit of Σ to be that portion of Σ which is comprised between the limits

$$0 \leqq X_i \leqq 1,$$
$$i = 1, 2, 3, \ldots, n;$$

so that $dX_1\, dX_2 \ldots dX_n$ represents an element of Σ having the form (in n dimensions) of a rectangular parallelepiped. In any multiple integral $\int V d\Sigma$, of order n, we must regard the integration as extending over a certain 'integral space,' or portion of Σ; this integral space is to be resolved into infinitesimal elements, which may be of any form whatever, but which must exhaust the whole of it; each element is to be multiplied by the function V of its coordinates; and the definite integral is the sum obtained by adding the results. When the indeterminates are transformed by the equations

$$\left.\begin{aligned} X_i &= \phi_i(x_1, x_2, x_3, \ldots, x_n) \\ i &= 1, 2, 3, \ldots, n \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad \text{(D)}$$

we have to consider a new space σ, in which the new indeterminates $x_1, x_2, \ldots, x_n$ may be regarded as rectangular coordinates; to each element of the first space corresponds an element of the second space, and *vice versâ*; the equations (D), whether by their own nature or by restrictions imposed on their interpretation, being supposed to establish a 'one-to-one' correspondence between the two integral spaces. The problem of the transformation of a multiple integral then reduces

itself to the determination of the ratio of corresponding elementary spaces $d\Sigma$ and $d\sigma$. By the nature of infinitesimal magnitudes, this ratio is independent of the particular form assigned to either element (the form of one of course determines the form of the other). Let us take for the element $d\sigma$ the parallelepiped (in n dimensions) of which one vertex is at the point $[x_1, x_2, \ldots, x_n]$, and of which the adjacent n vertices are

$$x_1+\delta_i x_1,\ x_2+\delta_i x_2,\ \ldots,\ x_n+\delta_i x_n,$$
$$i=1, 2, 3, \ldots, n,$$

the quantities $\delta_i x_j$ being infinitesimal and asyzygetic; so that, by the theorem of Art. 4, the volume of the parallelepiped (which is an infinitesimal of order n) is $|\delta_i x_j|$. If

$$\delta_i X_j=\left(\frac{dX_j}{dx_1}\right)\delta_i x_1+\left(\frac{dX_j}{dx_2}\right)\delta_i x_2+\ldots+\left(\frac{dX_j}{dx_n}\right)\delta_i x_n,$$
$$i=1, 2, \ldots, n, \quad j=1, 2, 3, \ldots, n,$$

the element $d\Sigma$, corresponding to the element $d\sigma=|\delta_i x_j|$, is evidently $|\delta_i X_j|$. But $|\delta_i X_j| \div |\delta_i x_j| = \left|\frac{dX_i}{dx_j}\right|$; *i.e.* the ratio of corresponding spaces is the functional determinant; a conclusion which establishes the theorem of transformation.

7. If we decompose the spaces Σ and σ into elementary parallelepipeds of the type $dX_1\, dX_2 \ldots dX_n$, $dx_1\, dx_2 \ldots dx_n$, we may write the equation

$$\int U dS=\int U\times\left|\frac{dX_i}{dx_j}\right|\times d\sigma$$

in the form

$$\int\ldots\iint U\, dX_1\, dX_2 \ldots dX_n=\int\ldots\iint U\times\left|\frac{dX_i}{dx_j}\right|\times dx_1\, dx_2 \ldots dx_n.$$

In this equation $dX_1\, dX_2 \ldots dX_n$ and $dx_1\, dx_2 \ldots dx_n$ are not corresponding elements; but the substitution of elements which do not correspond for elements which do correspond is admissible, because the value of a definite integral is not altered by substituting one mode of decomposition into elements for another. It may be added that the symbol $\int U d\Sigma$, which involves no hypothesis as to the form of the element $d\Sigma$, supplies the most general and abstract expression of the definite integral; while the symbol $\int\ldots\iint U\, dX_1\, dX_2 \ldots dX_n$, which indicates a particular mode of decomposition into elements, suggests at the same time a particular method of obtaining its value by successive integration.

8. It will be seen that, when a definite order is established among the indeterminates (X) and (x), the ratio of corresponding elements in the two spaces is

determined in sign as well as in magnitude; but that if two indeterminates in either set be interchanged, the sign of this ratio is also changed. The Jacobian loci $\left|\frac{dX_i}{dx_j}\right| = 0$ and $\left|\frac{dX_i}{dx_j}\right| = \infty$ (which for our present purpose we take together) may be regarded, in general, as dividing the space Σ into two regions A and B, in the first of which $\left|\frac{dX_i}{dx_j}\right|$ is positive, and in the second negative. Similarly, σ is divided by the corresponding loci $\left|\frac{d\,x_j}{dX_i}\right| = \left|\frac{dX_i}{dx_j}\right|^{-1} = \infty,\ = 0$, into two regions a and b respectively corresponding to A and B, in such a manner that corresponding elements have the same sign in the regions A, a, and opposite signs in the regions B, b. But in using the formula for the transformation of a multiple integral, it is in general convenient to give to the functional determinant its absolute value, thus considering the elements of both spaces as positive throughout. And when the space over which the integration extends is traversed or bounded by the Jacobian loci, it is always necessary to examine the circumstances which present themselves at these loci.

XXV.

ON THE INTEGRATION OF DISCONTINUOUS FUNCTIONS.

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1. RIEMANN, in his Memoir 'Ueber die Darstellbarkeit einer Function durch eine Trigonometrische Reihe' (Abhandlungen der k. Gesellschaft der Wissenschaften zu Göttingen, vol. xiii., p. 87), has given an important theorem which serves to determine whether a function $f(x)$ which is discontinuous, but not infinite, between the finite limits a and b, does or does not admit of integration between those limits, the variable x, as well as the limits a and b, being supposed real. Some further discussion of this theorem would seem to be desirable, partly because, in one particular at least, Riemann's demonstration is wanting in formal accuracy, and partly because the theorem itself appears to have been misunderstood, and to have been made the basis of erroneous inferences.

2. Let d be any given positive quantity, and let the interval $b-a$ be divided into any *segments* whatever, $\delta_1 = x_1 - a$, $\delta_2 = x_2 - x_1$, ..., $\delta_n = b - x_{n-1}$, subject only to the condition that none of these segments surpasses d. We may term d the *norm* of the division; it is evident that there is an infinite number of different divisions having a given norm; and that a division appertaining to any given norm, appertains also to every greater norm. Let ϵ_1, ϵ_2, ..., ϵ_n be positive proper fractions; if, when the norm d is diminished indefinitely, the sum

$$S = \delta_1 f(a + \epsilon_1 \delta_1) + \delta_2 f(x_1 + \epsilon_2 \delta_2) + \ldots + \delta_n f(x_{n-1} + \epsilon_n \delta_n)$$

converges to a definite limit, whatever be the mode of division, and whatever be the fractions ϵ_2, ϵ_2, ..., ϵ_n, that limit is represented by the symbol $\int_a^b f(x)\,dx$, and

the function $f(x)$ is said to admit of integration between the limits a and b. We shall call the values of $f(x)$ corresponding to the points of any segment the *ordinates* of that segment; by the *ordinate difference* of a segment we shall understand the difference between the greatest and least ordinates of the segment. For any given division $\delta_1, \delta_2, \ldots, \delta_n$, the greatest value of S is obtained by taking the maximum ordinate of each segment, and the least value of S by taking the minimum ordinate of each segment; if D_i is the ordinate difference of the segment δ_i, the difference θ between these two values of S is

$$\theta = \delta_1 D_1 + \delta_2 D_2 + \ldots + \delta_n D_n.$$

But, for a given norm d, the greatest value of S, and the least value of S, will in general result, not from one and the same division, but from two different divisions, each of them having the given norm. Hence the difference Θ between the greatest and the least values that S can acquire for a given norm, is, in general, greater than the greatest of the differences θ. To satisfy ourselves, in any given case, that S converges to a definite limit, when d is diminished without limit, we must be sure that Θ diminishes without limit; and it is not enough to show (as the form of Riemann's proof would seem to imply) that θ diminishes without limit, even if this should be shown for every division having the norm d.

3. Let $A(d)$ be the greatest value of S appertaining to a given norm d, and let $B(d)$ be the least value of S appertaining to the same norm. If d_1 and d_2 are any two norms, of which d_1 is greater than d_2, it is evident that $A(d_1) \geqq A(d_2)$, $B(d_1) \leqq B(d_2)$, because every division appertaining to the norm d_2 also appertains to the norm d_1. And it may be proved (although, for brevity, we omit the demonstration here) that, given any norm d_1, we can always assign a norm d_2, less than d_1, which shall satisfy the inequalities $A(d_1) > A(d_2)$, $B(d_1) < B(d_2)$; except only when the function is such that the maximum (or minimum) ordinate is the same, throughout the whole interval, for all segments however small. In this excepted case, which is one by no means inconceivable, the value of $A(d)$, [or of $B(d)$,] is independent of d, and is simply $h(b-a)$, where h is the maximum (or minimum) ordinate common to all segments of the interval $b-a$. In all other cases, it is possible to assign a series of norms, decreasing without limit, and such that the corresponding maximum values of S form a decreasing series, while the corresponding minimum values of S form an increasing series.

Besides the maximum and minimum values of S corresponding to a given norm, we have also to consider the maximum and minimum values of S corresponding to a given division. Let $P(d)$ be the maximum value of S appertaining to a given division of norm d, and let $Q(d')$ be the minimum value of S

appertaining to a different division, having the same norm or a different norm. It is important to observe that we shall always have $P(d) > Q(d')$, the sign of equality being inadmissible, except when the function is such as to be represented geometrically by a single segment, or a system of segments, parallel to the axis of x. Leaving out of consideration the excepted case, we may enunciate the theorem—'The least value of S that can be obtained by taking, in any division whatever, the greatest ordinate of each segment, is greater than the greatest value that can be obtained by taking, in any division whatever, the least ordinate of each segment.' To prove this theorem, let the two divisions, which give the values $P(d)$ and $Q(d')$, be simultaneously applied to the interval $b-a$. To obtain $P(d)$, each segment in the resulting division will have to be multiplied by its greatest ordinate, or by a still greater ordinate in some adjacent segment; whereas to obtain $Q(d')$ each segment will have to be multiplied by its least ordinate, or by a still less ordinate. It follows that we have, in general, $P(d) > Q(d')$. If, however, we regard the interval $b-a$ as composed of segments $l_1, l_2, \ldots$, each of which has for its extremities points which are also extremities of segments in each of the two given divisions, we shall find that the inequality $P(d) > Q(d')$ must be replaced by the equality $P(d) = Q(d')$, if it should so happen that the maximum ordinate of each segment l is the same as its minimum ordinate; *i.e.*, if the function $f(x)$ is represented geometrically by a series of segments parallel to the axis of x, and respectively equal to the segments $l_1, l_2, \ldots$.

4. Again, let $B'(d)$ be the least value of S corresponding to the division which gives $A(d)$; and let $A'(d)$ be the greatest value of S corresponding to the division which gives $B(d)$; it is evident from what has been said that we shall have the inequalities

$$A(d) > A'(d) > B'(d) > B(d).$$

Now

$$A(d) - B(d) = [A(d) - B'(d)] + [A'(d) - B(d)] - [A'(d) - B'(d)];$$

and

$$A'(d) \geqq B'(d);$$

therefore

$$A(d) - B(d) \leqq [A(d) - B'(d)] + [A'(d) - B(d)].$$

Hence, to prove the evanescence of $A(d) - B(d)$ or Θ, it suffices to prove the evanescence of $A(d) - B'(d)$, and of $A'(d) - B(d)$, which are, in fact the two values of θ corresponding to the two divisions which give the absolutely greatest and least values of S for the norm d.

5. The theorem of Riemann may be enunciated as follows:—

'Let σ be any given quantity, however small; if, in every division of norm d,

the sum of the segments, of which the ordinate differences surpass σ, diminishes without limit, as d diminishes without limit, the function admits of integration; and, *vice versâ*, if the function admits of integration, the sum of these segments diminishes without limit with d.'

The following (with a slight modification suggested by the preceding considerations) is Riemann's demonstration of the first part of the theorem:

Let s_1 be the sum of the segments which, in the division corresponding to $A(d)$ and $B'(d)$, have ordinate differences surpassing σ; and let Ω be the greatest ordinate difference in any division appertaining to the norm d; Ω is necessarily finite, because all the ordinates are finite. The contribution of the segments s_1 to the difference $A(d) - B'(d)$ cannot surpass $s_1 \times \Omega$, and the contribution of the remaining segments cannot surpass $\sigma \times (b - a - s_1)$; *i.e.*,

$$A(d) - B'(d) \leqq s_1 \times \Omega + \sigma (b - a - s_1).$$

Similarly, if s_2 is the sum of the segments which, in the division corresponding to $A'(d)$ and $B(d)$, have ordinate differences surpassing σ,

$$A'(d) - B(d) \leqq s_2 \times \Omega + \sigma (b - a - s_2).$$

Adding these two inequalities, we find

$$A(d) - B(d) \leqq (s_1 + s_2)(\Omega - \sigma) + 2\sigma (b - a).$$

But σ may be taken as small as we please, and, by hypothesis, however small σ may be, d can always be taken so small as to render s_1 and s_2 as small as we please; *i.e.*, the difference $A(d) - B(d) = \Theta$ diminishes without limit with d, and $f(x)$ admits of integration between the limits a and b.

6. Riemann's demonstration of the second part of the theorem requires no modification. For, if S converges to a definite limit, Θ must be comminuent with d, and, *à fortiori*, each of the quantities θ must be comminuent with d. But, evidently, in any given division in which s is the sum of the segments having ordinate differences which surpass σ, $\sigma s \leqq \theta$. Hence, however small the given quantity σ may be, we can always, by taking d small enough, make $\frac{\theta}{\sigma}$ less than any assigned quantity; *i.e.*, if S converges to a definite limit, s must diminish without limit at the same time with d.

7. It will be observed that, in order to establish the convergence of S to a definite limit, it is sufficient to know that the sum of the segments, having ordinate differences surpassing σ, is comminuent with d in each of two specified divisions [viz., in the division which gives $A(d)$ the maximum value of S, and in

that which gives $B(d)$ the minimum value of S]. Hence, if these two sums are comminuent with d, the corresponding sum in any other division of norm d is also comminuent with d.

8. Let us suppose that the function $f(x)$ has any number of discontinuities between a and b; and let there be $\psi(\sigma)$ points at which there are discontinuities surpassing σ. (We say that a discontinuity surpassing σ exists at a given point, when any segment, however small, being taken which includes that point, the ordinate difference of the segment surpasses σ.) If $\psi(\sigma)$ has a finite and assignable value for every value of σ, however small, the condition of integrability is certainly satisfied, even if $\psi(\sigma)$ increase without limit, when σ diminishes without limit. For, in any division of norm d, the sum of the segments having ordinate differences which surpass σ, cannot surpass $2d \times \psi(\sigma)$; and, however small σ may be, d can be taken so small that $2d \times \psi(\sigma)$ shall be less than any quantity that can be assigned. As an example, we may take the function considered by Riemann, viz.,

$$f(x) = \frac{(x)}{1} + \frac{(2x)}{4} + \frac{(3x)}{9} + \ldots,$$

where, by (x) we are to understand the (positive or negative) excess of x above the whole number nearest to x; or, if x lies half-way between two whole numbers, the arithmetical mean between the two differences $\frac{1}{2}$ and $-\frac{1}{2}$, *i.e.*, zero. In this function, if $x = \frac{m}{2n}$, where m and $2n$ are relatively prime, we have

$$f\left(\frac{m}{2n} + 0\right) = f\left(\frac{m}{2n}\right) - \frac{\pi^2}{16n^2},$$

$$f\left(\frac{m}{2} - 0\right) = f\left(\frac{m}{2n}\right) + \frac{\pi^2}{16n^2}.$$

Thus the number of discontinuities in any given interval is infinitely great. But the number of discontinuities which in any given interval surpass a given quantity σ, is always finite. For example, the number of discontinuities between 0 and 1 which surpass σ, is equal to the number of irreducible proper fractions, having even denominators $2n$, which verify the inequality $\frac{\pi^2}{8n^2} > \sigma$; or, if $\phi(m)$ be the number of numbers not surpassing m and prime to m, and if h be the greatest integer not surpassing $\frac{\pi}{2\sqrt{2\sigma}}$, the number of discontinuities in question is

$$\phi(1) + \phi(2) + \ldots + \phi(h) = \psi(\sigma),$$

which is evidently finite for any given value of σ, although it increases without limit when σ diminishes without limit.

9. Next, let us suppose that $f(x)$ in the interval $b-a$ has an infinite number of discontinuities surpassing a given quantity σ. The points at which these discontinuities occur may either 'completely fill' one or more finite portions of the interval $b-a$, or there may be no finite portion of that interval which is 'completely filled' by them. A system of points is said to 'fill completely' a given interval when, any segment of the interval being taken, however small, one point at least of the system lies on that segment. Thus the *rational* points on any line, *i.e.*, the points of which the abscissae are rational, completely fill any segment whatever upon the line. We may observe that the assertion, that any given segment of an interval contains at least one point of a given system, is equivalent to the assertion that any given segment contains an infinite number [*i.e.*, a number greater than any that can be assigned] of the points of the system. For we may divide the given segment into as many parts as we please, and each of them must contain at least one point of the system.

10. When the points at which there occur discontinuities surpassing σ completely fill any finite portion of the interval $b-a$, the function $f(x)$ is certainly incapable of integration. For, if l be the total length of the segments which are completely filled, we have evidently $\theta > \sigma l$ for any division of any norm d; *i.e.*, it is impossible that Θ should diminish without limit with d.

But points may exist in an infinite number within a finite interval, without completely filling any portion of that interval. Whenever this happens, it must be possible in any given segment of the interval, however small, to take a finite part such that it shall contain no point of the system; otherwise, the segment in question would be completely filled. We give a few examples of such systems of points, the limits of the interval being in each case 0 and 1. We shall say, for brevity, that points are in close order on any segment when they completely fill it, and in loose order when they do not completely fill it, or any part of it however small.

11. (i.) Let the system of points be defined by the equation $x = \frac{1}{\alpha}$, α being any positive integer. It will be seen, (1) that these points are infinite in number; (2) that they are indefinitely condensed in the vicinity of the origin; (3) that they are in loose order over the whole interval, no segment, even in the immediate vicinity of the origin, being completely filled. For if d be any given quantity, however small, we can always find a finite integral number such

that $\frac{1}{m} < d$, and then the finite spaces $\left(\frac{1}{m+1}, \frac{1}{m}\right)$, $\left(\frac{1}{m+2}, \frac{1}{m+1}\right)$, &c. ... all lie on the segment $(0, d)$, and are all free from points of the system, if we leave their initial and terminal points out of account.

12. (ii.) Let the system of points be defined by the equation $x = \frac{1}{a_1} + \frac{1}{a_2}$, where a_1 and a_2 are any positive integers. Here, it is evident that the points are indefinitely condensed in the vicinity of each of the points of the system (i). But it can also be shown that they are in loose order over the whole interval from 0 to 1. Let $x = L_1$, $x = L_2$, $(L_1 < L_2,)$ be two consecutive points of the system (i); let μ be any positive quantity whatever, and consider the segment $\left(\frac{\mu L_1 + L_2}{\mu + 1}, L_2\right)$. If $x = \frac{1}{a_1} + \frac{1}{a_2}$ lies on this segment, we must have $\frac{1}{a_1} \leqq L_1$, $\frac{1}{a_2} \leqq L_1$, because no point of the system (i) lies on the interval (L_1, L_2); and also $\frac{1}{a_1} + \frac{1}{a_2} \geqq \frac{\mu L_1 + L_2}{\mu + 1}$; whence $a_1 \leqq \frac{\mu + 1}{L_2 - L_1}$, $a_2 \leqq \frac{\mu + 1}{L_2 - L_1}$. These inequalities show that, if, from the beginning of any free segment in the system (i), we cut off as small a part as we please (which we may do by taking μ great enough), the remaining portion of that segment will contain only a finite number of points belonging to the system (ii). And this suffices to prove that the points of the system are in loose order; for if d be any segment, however small, situated anywhere in the interval from 0 to 1, we can certainly find on this segment a part free from points of the system (i), and, by what has just been proved, parts of that part will be free from points of the system (ii).

13. (iii.) Let a system of points P_{s+1} be defined by the equation

$$x = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{s+1}},$$

where $a_1, a_2, \ldots, a_{s+1}$ are positive integers. Assuming (what has just been proved for $s = 2$) that the system P_s is in loose order over the whole interval from 0 to 1, we shall prove the same thing for the system P_{s+1}. Let $x = L_1$, $x = L_2$ be any two consecutive points of the system P_s; and consider as before the interval $\left(\frac{\mu L_1 + L_2}{\mu + 1}, L_2\right)$. If the point P_{s+1}, or $x = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{s+1}}$, lies on this interval, we must have, besides the inequality

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{s+1}} \geqq \frac{\mu L_1 + L_2}{\mu + 1},$$

the $s+1$ inequalities included in the formula

$$\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_{s+1}}\leqq L_1+\frac{1}{a_i},$$

because no point of the system P_s can be between L_1 and L_2. These inequalities give

$$a_i\leqq\frac{\mu+1}{L_2-L_1},\quad i=1,\ 2,\ 3,\ \ldots,\ s+1,$$

whence we may infer, precisely as in the case in which $s=2$, that the points P_{s+1} are in loose order over the whole of interval from 0 to 1.

14. Let $f(x)$ be a function, which coincides with a given continuous function $\phi(x)$ for all values of x between 0 and 1, except at the points P_{s+1}; and let the difference between $f(x)$ and $\phi(x)$ at those points not exceed the finite quantity σ. It may be shown that $f(x)$ is integrable between the limits 0 and 1, and that

$$\int_0^1 f(x)\,dx=\int_0^1\phi(x)\,dx.$$

For, take any small interval from 0 to δ; the points P_1 which lie outside it, between δ to 1, are finite in number and at finite distances from one another. Let there be Δ_1 of them; from each of them measure a space δ_1 to the right; the number of points P_2, lying outside of the measured spaces $\delta+\Delta_1\delta_1$, is necessarily finite; and these points are at finite distances from one another. Let their number be Δ_2, and measure a distance δ_3 to the right of each of them. Proceeding in this way, we shall obtain measured spaces amounting in all to

$$\delta+\Delta_1\delta_1+\Delta_2\delta_2+\ldots+\Delta_{s+1}\delta_{s+1}=H.$$

Let ϵ be any given quantity however small; and in the preceding construction let

$$\delta<\frac{\epsilon}{3(s+2)},\ \delta_1<\frac{\epsilon}{3(s+2)\Delta_1},\ \delta_2<\frac{\epsilon}{3(s+2)\Delta_2},\ \ldots,\ \delta_{s+1}<\frac{\epsilon}{3(s+2)\Delta_{s+1}};$$

we shall thus have $H<\frac{1}{3}\epsilon$. Let d be the least of the spaces $\delta,\ \delta_1,\ \delta_2,\ \ldots,\ \delta_{s+1}$; it may be shown that, in any division of norm d, the sum of the segments containing points P_{s+1} cannot exceed $3H$. For all the points P_{s+1} lie on the measured spaces; and supposing (which is the most unfavourable case) that one of those spaces begins and ends with a point P_{s+1}, we can at most triple it, by imagining a segment equal to d placed on each side of it. Thus, in every division of norm d, the sum of the segments containing the points of discontinuity is less

than ϵ; whence we infer, by Riemann's theorem, that $\int_0^1 f(x)\,dx$ has the same value as $\int_0^1 \phi(x)\,dx$.

15. (iv.) Let m be any given integral number greater than 2. Divide the interval from 0 to 1 into m equal parts; and exempt the last segment from any subsequent division. Divide each of the remaining $m-1$ segments into m equal parts; and exempt the last segment of each from any subsequent division. If this operation be continued *ad infinitum*, we shall obtain an infinite number of points of division P upon the line from 0 to 1. These points are in loose order: for if d be any segment however small, situated anywhere in the interval from 0 to 1, we may take an index k which satisfies the inequality $\frac{1}{m} < \frac{1}{2}d$; and then determine a segment of the type $\left(\frac{a}{m^k}\ \frac{a+1}{m^k}\right)$ lying entirely on the segment d. But this segment is either itself an exempted segment or its m^{th} part is so. It will be seen that, after k operations, the sum of the exempted segments amounts to $1-\left(1-\frac{1}{m}\right)^k$; so that, as k increases without limit, the points of division P occur upon segments which occupy only an infinitesimal portion of the interval from 0 to 1. And it may be inferred that a function, having any finite discontinuities at the points P, would be integrable. For, if d be any given small quantity, let the index k be determined by the inequalities $\frac{1}{m^k} > d > \frac{1}{m^{k+1}}$; the number N of excepted segments which surpass $\frac{1}{m^k}$ is

$$1+(m-1)+(m-1)^2+\ldots+(m-1)^{k-2};$$

and the sum of the remaining segments is

$$\left(1-\frac{1}{m}\right)^{k-1}.$$

It is evident that in any division of norm d, the sum of the segments containing points P cannot exceed

$$\left(1-\frac{1}{m}\right)^{k-1}+2Nd.$$

But, as d decreases, and k increases, without limit, $\left(1-\frac{1}{m}\right)^{k-1}$ and $2Nd$, which is less than $\frac{2N}{m^k}$, both decrease without limit; *i.e.*, in any division of norm d, the

sum of the segments containing points of discontinuity diminishes without limit with d; and the function is integrable.

16. (v.) Let us now, as in the last example, divide the interval from 0 to 1 into m equal parts, exempting the last segment from any further division; let us divide each of the remaining $m-1$ segments by m^2, exempting the last segment of each segment; let us again divide each of the remaining $(m-1)(m^2-1)$ segments by m^3, exempting the last segment of each segment; and so on continually. After $k-1$ operations we shall have

$$N = 1 + (m-1) + (m-1)(m^2-1) + \ldots + (m-1)(m^2-1)\ldots(m^{k-2}-1)$$

exempted segments, of which the sum will be

$$1 - \left(1-\frac{1}{m}\right)\left(1-\frac{1}{m^2}\right)\ldots\left(1-\frac{1}{m^{k-1}}\right).$$

This sum, when k is increased without limit, approximates to the finite limit $1 - E\left(\frac{1}{m}\right)$; where $E\left(\frac{1}{m}\right)$ is the Eulerian product $\prod_1^\infty \left(1-\frac{1}{m^k}\right)$, and is certainly different from zero. The points of division Q exist in loose order over the whole interval. For, if d be any small segment of that interval, and if $\frac{1}{m^{\frac{1}{2}k(k-1)}} < \frac{1}{2}d$, a segment of the type $\left(\frac{a}{m^{\frac{1}{2}k(k-1)}}, \frac{a+1}{m^{\frac{1}{2}k(k-1)}}\right)$ can be found lying entirely on the segment d, and this segment is either itself exempted, or its $\left(\frac{1}{m^k}\right)^{\text{th}}$ part is exempted. But a function having finite discontinuities at the points Q would be incapable of integration. For, if d be any norm, and $\delta < \frac{1}{m^{\frac{1}{2}k(k-1)}} < d$, in the division

$$\delta + \frac{i}{m^{\frac{1}{2}k(k-1)}}, \quad i = 0, 1, 2, 3, \ldots$$

(which is a division of norm d), the sum of the segments containing points of discontinuity is

$$\left(1-\frac{1}{m}\right)\left(1-\frac{1}{m^2}\right)\ldots\left(1-\frac{1}{m^{k-1}}\right) + \frac{N}{m^{\frac{1}{2}k(k-1)}},$$

which approximates to the finite limit $E\left(\frac{1}{m}\right)$ when d is diminished, and k is increased without limit.

17. The result obtained in the last example deserves attention, because it is opposed to a theory of discontinuous functions, which has received the sanction of an eminent geometer, Dr. Hermann Hankel, whose recent death at an early

age is a great loss to mathematical science. In an interesting memoir ('Untersuchungen ueber die unendlich oft oscillirenden und unstetigen Functionen,' Tübingen, 1870), Dr. Hankel has laid down the distinction, here adopted from him, between a system of points which completely fill a segment, and a system of points which do not completely fill any segment, but lie in loose order. [The term employed by Dr. Hankel is 'zerstreut'; the use of the equivalent English words 'dispersed' or 'scattered' has been avoided in the present note, because they might seem to exclude the sort of condensation in the vicinity of a finite or infinite number of points, which, as we have seen in the examples (i.), (ii.), (iii.), may present itself in the case of systems of points in loose order.] Dr. Hankel then asserts (see p. 26) that, when a system of points is in loose order on a line, the line may be so divided as to make the sum of the segments containing the points less than any assignable line. The proof of this assertion is, in effect, as follows:—Divide the line into segments, of which each contains a point of the system, and imagine each segment to be diminished to its n^{th} part, yet so as still to have upon it the point of the system which it contained before. The sum of the segments can thus be made less than the n^{th} part of the whole line; *i.e.*, less than any line that can be assigned, because we may suppose n as great as we please. It must be conceded that this demonstration is rigorous, if the number of points in the system is finite; but the construction indicated ceases to convey any clear image to the mind, as soon as the number of points becomes infinite. If we are allowed to divide the line from 0 to 1, in example (iii.), in such a manner as to include every point (P_{s+1}) in a segment of its own, these segments, in nu- vicinity of the points P_s, will have to be less than any line that can be d; and, if such a mode of division is admissible, it is difficult to see why not also be considered admissible so to divide the line as to include and the sional point in a segment of its own: in which case Dr. Hankel's would extend to systems of points in close order, as well as to ose order. But whether we do or do not admit the truth of roposition, the use which he makes of it (p. 31) to establish It is evident th. of Riemann's criterion to a certain class of functions would points P cannot e. eous. To prove that Riemann's condition of integrability en discontinuous function, we have to show that, given ' however small, the sum of the segments, which, *in* But, as d decreases, and *norm* d, contain the points of discontinuity, is evan- is less than $\frac{2N}{m^k}$, both decr s evident that this cannot be shown, if we confine nodes of division in which some of the segments

are from the very beginning assumed to be less than any quantity that can be assigned.

While, therefore, we may safely admit the theorem that no function can be integrable which has discontinuities, surpassing a given quantity σ, at an infinite number of points forming a system in close order; the converse assertion that, when the system of points of discontinuity is in loose order, the function is integrable, would seem to be established by no satisfactory demonstration, and to be negatived by the result obtained in example (v.).

18. Another proposition, contained in the same memoir (p. 28), appears open to a similar objection. It may be admitted that a function $f(x)$ having discontinuities, which surpass a given quantity σ however small, only at points which form a system in loose order, is necessarily continuous over finite portions of any interval however small. But it would seem to be untrue that such a function is necessarily continuous in the vicinity of any one of its points of discontinuity. If, for example, $f\left(\frac{1}{a_1}+\frac{1}{a_2}\right)=1$, and $f(x)=0$, for every other value of x, it is evident that, however small the given quantity ϵ may be, the difference

$$f\left(\frac{1}{a_1}+\epsilon\right)-f\left(\frac{1}{a_1}+\delta\right)$$

oscillates an infinite number of times between the values 0 and 1, as δ decreases from ϵ to 0; *i.e.*, the function $f(x)$ is discontinuous in the vicinity of the point $\frac{1}{a_1}$ to the right.

19. We add a few remarks which may serve still further to illustrate the meaning and use of Riemann's theorem.

(i.) The problem, 'Given a system of points upon an interval (a, b), to find, among all divisions of norm d, that in which the segments containing the points have the maximum sum,' is perfectly determinate. We may say that a point of the system is *isolated*, when it is separated from the next preceding and next following point by a distance $>2d$. Similarly a group of points may be said to be isolated, when the distance between any two consecutive points of the group is less than $2d$, but the distance between the extreme points of the group, and those which immediately precede and follow it, is greater than $2d$. It is evident that, for any given value of d, the given system of points resolves itself into a finite number of isolated groups. The first and last point of each group determine a segment; on either side of each of these segments, and on either side of each isolated point, we may place a segment

equal to d. The sum of the segments thus obtained is the maximum sum required.

It will be observed that in this solution each point of the system is regarded as double; *i.e.*, as capable of affecting two segments at once, one on each side of it. If the discontinuity of a function at any point can be removed by changing the value of the function at that point only, for example, if $f(x-0)=\alpha$, $f(x)=\alpha+\sigma$, $f(x+0)=\alpha$, the point must be regarded as single (its contribution to the difference Θ of Art. 2 would be only $\sigma \times d$). But if the values of the function preceding and following the point of discontinuity are different (*i.e.*, if $f(x-0)=\alpha$, $f(x+0)=\alpha+\sigma$), the point of discontinuity produces a double effect, its contribution to the difference Θ being $2\sigma \times d$. Similarly, in the case of functions which, like $\cos\left(\frac{\pi}{x}\right)$ in the vicinity of the origin, admit of an infinite number of maxima and minima within a finite interval, the contribution to Θ of each point at which there is a maximum, or minimum, is two-fold. For the practical application of Riemann's criterion, the distinction between points producing a one-fold effect and those producing a two-fold effect is immaterial.

20. (ii.) When a function, which is discontinuous but never infinite, does not admit of integration between the limits a and b, the symbol $\int_a^b f(x)\,dx$ becomes indeterminate. But the maximum and minimum values attributable to that symbol are perfectly determinate; and if it should become advisable to attribute a definite value to the symbol, we might select for that purpose the arithmetical mean between these two extreme values. If, for continually decreasing values of d, we calculate the corresponding maximum values of the sum S of Art. 2, these values will, as shall now be shown, converge to a determinate limit A. And similarly the successive minimum values of S will converge to a determinate limit B, different from A in the case under consideration. The difference $A-B$ is, of course, the limit of the successive differences Θ.

From the two sets of inequalities

$$A(d_1) > A(d_2) > A(d_3) > \ldots,$$
$$B(d_1) < B(d_2) < B(d_3) < \ldots,$$

combined with the inequality $A(d_n) > B(d_n)$, which holds for any value of n however great, we infer that each of the two series,

$$A(d_1)-A(d_2),\quad A(d_2)-A(d_3),\quad A(d_3)-A(d_4),\ \ldots,$$
$$B(d_2)-B(d_1),\quad B(d_3)-B(d_2),\quad B(d_4)-B(d_3),\ \ldots,$$

consists of positive terms, and that, however many terms of either series we add together, we can never surpass $A(d_1) - A(d_\infty)$ in the first, and $B(d_\infty) - B(d_1)$ in the second; *i.e.*, in neither of them can we ever surpass $A(d_1) - B(d_1)$. But if a series of positive terms be such that the sum of any number of its terms, however great, can never surpass a given finite quantity, the sum of the first n terms of the series converges to a finite and determinate limit, when n is increased without limit (see Riemann, Vorlesungen, pp. 39, 40). The sums $A(d_1) - A(d_n)$, $B(d_n) - B(d_1)$, therefore converge to finite and determinate limits; or, which is the same thing, the two series of terms

$$A(d_1), \quad A(d_2), \quad A(d_3), \ldots,$$
$$B(d_1), \quad B(d_2), \quad B(d_3), \ldots,$$

converge to finite and determinate limits.

If, for example, the function $f(x)$ have the value σ, at every point of the system considered in Art. 16, example (v.), and the value $\sigma_2 < \sigma_1$ at every other point; we shall find

$$B = \sigma_2, \quad A = \sigma_2 + (\sigma_1 - \sigma_2) \times E\left(\frac{1}{m}\right).$$

21. (iii.) Riemann's criterion of integrability is applicable to the case of any multiple integral extended over a finite space. For example, in the case of a triple integral, we must imagine the whole space of the integration divided into small spaces such that any one of them could be comprehended within a sphere of a diameter d; and any such division into spaces is a division of norm d. The criterion of integrability, then, is that, in any division whatever of norm d, the sum of the spaces in which the ordinate-differences surpass a given quantity σ, must diminish without limit with d. The ordinate-difference of any space is, of course, the difference between the greatest and least values of the function within the space.

Considering, for simplicity, the case of two dimensions only, we observe that the space of integration may not only contain points of discontinuity finite or infinite in number, but may be intersected by curves of discontinuity. The function may have values differing by a finite quantity on either side of such a curve; or its values at points along the curve may be discontinuous, or both of these kinds of discontinuity may be combined at the same curve. If $L(\sigma)$, the total length of the curves at which the discontinuities surpass σ, be finite, the function can be integrated over the given space; since, if we draw curves parallel to the curves of discontinuity and at a distance d from them on either side, the

area of the channel-like spaces thus obtained will be $2dL(\sigma)$, and will surpass the greatest sum of spaces, including the curves in any division of norm d. But the function may be integrable even if the total length of the curves of discontinuity is infinite; because an infinite number of contiguous curves may be enclosed in one and the same channel. And, provided that the curves can all be included in channels of which the length is L, and of which the breadth δ is comminuent with d, the condition that $L \times \delta$ should be comminuent with d, will suffice to ensure the integrability of the function.

XXVI.

ON THE HIGHER SINGULARITIES OF PLANE CURVES.

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THE ordinary singularities of a plane curve are its double points and double tangents, its stationary points and stationary tangents; or, as they have been also called, its nodes and links, its cusps and inflexions. The fundamental theorem, that any of the so-called higher singularities of a plane curve may be regarded as equivalent to a certain number of ordinary singularities of each of these four kinds, has been enunciated by Professor Cayley, who has also given a method for determining in every case the four indices $\delta, \tau, \kappa, \iota$, proper to any given singularity.

Several enquiries, which appear to possess some interest, are suggested by this theorem. Among them we may mention the two following—

(1) It is important to prove that the indices of singularity, as defined by Professor Cayley, satisfy the equations of Plücker; and that the 'genus' or 'deficiency' of the plane curve is correctly given by these indices.

(2). It is also of interest to examine whether any given singularity can be actually formed by the coalescence of the ordinary singularities to which it is regarded as equivalent; in other words, whether a singularity of which the indices are $\delta, \tau, \kappa, \iota$, and which is therefore to be regarded as equivalent to δ double points, τ double tangents, κ cusps, and ι inflexions, possesses a penultimate form, in which all these singularities exist, distinct from one another, but infinitely close together.

The present paper relates chiefly to the first of these enquiries; the second is reserved for a future communication.

1. Consider a plane curve C of order m and class n, defined by an equation

$F(p, q) = 0$ between the parameters of two pencils, of which the corresponding rays intersect on C, and which are represented by equations of the form $p(QP) + (QR) = 0$, $q(PQ) + (PR) = 0$; P, Q, R denoting the three vertices of a triangle, $(PQ) = 0$, $(QR) = 0$, $(PR) = 0$ the equations of its sides. It is convenient to suppose that Q and P, the centres of the two pencils, have no speciality of position with regard to C; or, more precisely, that neither Q nor P lies on the curve, nor on any singular line appertaining to the curve. Under the general name of singular lines we include (1) lines joining two singular points, (2) singular tangents, (3) tangents at singular points, (4) tangents passing through a singular point; we shall also suppose that PQ is not a tangent to C, and does not pass through any singular point. Thus to every finite value of p there will correspond m finite values of q, and *vice versâ*; and, in particular, to any singular point on the curve there will correspond a finite pair of values of p and q. To an infinite value of q there will correspond m infinite values of p, and *vice versâ*; these answer to the m intersections of PQ with the curve, no two of which, by hypothesis, are coincident. We may, if we please, project the line PQ to an infinite distance, and regard p and q as Cartesian coordinates; we prefer, however, for our present purpose, to consider them as parametric ratios; *i.e.*, as purely numerical quantities (real or complex).

2. Let $f(q)$ be the discriminant of $F(p, q) = 0$, considered as an equation of the order m in p; we may suppose the coefficient of p^m, which is certainly different from zero, to be unity. The first polar of P with regard to C is $\frac{dF}{dp} = 0$, and $f(q)$ is the resultant of the elimination of p from F and $\frac{dF}{dp}$, so that the roots of $f(q) = 0$ are the parameters of the lines drawn from P to the points of intersection of C with the first polar of P. Attending to the suppositions which have been made as to the situation of P and Q relatively to the curve C, we infer (α) that $f(q)$ has no infinite roots, and is therefore of the full order $m(m-1)$ in q; (β) that $f(q)$ has n, and only n, non-multiple roots q'; (γ) that for each of these n roots q' the equation $F(p, q') = 0$ acquires two equal roots p', its remaining roots being all different from p', and from one another; (δ) that q' is not a multiple root of the equation $F(p', q) = 0$. The n sets (p', q') give the n points of contact of tangents from P; the remaining factor of $f(q)$, viz., $f_1(q) = f(q) \div \Pi(q - q')$, consists exclusively of multiple factors, and appertains to the singular points of the curve. The index of its order, *i.e.*, $m(m-1) - n$, we may term the *total discriminantal index of the singular points of the curve.* Let q_0 be a root of $f_1(q) = 0$ of multiplicity ν; the equation $F(p, q_0) = 0$ has but

one multiple root; let this be p_0, and let its multiplicity be μ; then (p_0, q_0) is a singular point O on the curve, of which the *order* (*i.e.*, the least number of points in which it is cut by any straight line passing through it) is μ, and of which ν may be termed the *discriminantal index*. It is evident that the number of singular points is equal to the number of unequal roots of $f_1(q) = 0$, and that the total discriminantal index is equal to the sum of the discriminantal indices of the separate singular points. We shall presently (Art. 8) see that the discriminantal index of a singular point can in general be further subdivided into parts, appertaining respectively to the different branches of the curve which pass through the point, taken singly, and in pairs.

3. It is a well-known theorem of Cauchy, that so long as the analytical modulus of $q - q_0$ is less than the least of the modules of any of the quantities $q_1 - q_0$, where q_1 is any root of $f(q) = 0$ other than q_0, the m roots of the equation $F(p, q) = 0$ are developable in convergent series of the form

$$p - p_0 = A + A_0(q - q_0) + A_1(q - q_0)^{\alpha_1} + A_2(q - q_0)^{\alpha_2} + \ldots, \quad . \quad . \quad . \quad \text{(A)}$$

the exponents $\alpha_1, \alpha_2, \ldots$ being rational and positive numbers, which satisfy the inequalities $1 < \alpha_1 < \alpha_2 < \ldots$. Of the equations (A), $m - \mu$ give the values of p corresponding to the $m - \mu$ points not in the vicinity of O, in which C is cut by the line (q). The series in the right-hand members of these $m - \mu$ equations we shall designate by $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_{m-\mu}$: we observe that in them the quantities A are all different from one another and from zero; because (q_0), not being a singular line, intersects C in $m - \mu$ points, which are different from one another and from O; also, in these equations, the exponents $\alpha_1, \alpha_2, \alpha_3, \ldots$ are integral. In the remaining μ equations, which give the developments appertaining to the branches of C that pass through O, the quantities A are all equal to zero: these equations divide themselves into groups of conjugate equations, the equations of any one group being of the type

$$p - p_0 = B_0(q - q_0) + B_1\omega^{\beta_1}(q - q_0)^{\frac{\beta_1}{\Delta}} + \ldots,$$

where the numerators β are positive, integral, and increasing; Δ is less than β_1, and is the least common denominator of the fractional exponents; ω is any root of $\omega^{\Delta} = 1$: so that, if we use one and the same value of the radical in all the Δ equations of the group, they will differ from one another only by containing different values of ω; each of the μ equations defines a branch of the curve passing through O. If $\Delta = 1$, the branch is linear or of order 1; if $\Delta > 1$, the Δ conjugate equations are regarded by Professor Cayley as defining Δ partial branches

forming a single superlinear branch of order Δ; in every case the sum of the orders of the branches is equal to the order of the point, *i.e.*, $\Sigma\Delta = \mu$. The coefficients B_0 are all different from zero, and the indices $\frac{\beta_1}{\Delta}$ are all greater than unity, because neither (p_0) nor (q_0) is one of the tangents at O; but these coefficients and indices are not necessarily different in two developments belonging to two different linear or superlinear branches—indeed any two such developments may coincide for any finite number of terms; and to ascertain the true nature of any singular point it is indispensable to continue the developments until they all become different from one another. The series in the right-hand members of the μ equations we denote by $\overline{B}_1, \overline{B}_2, \overline{B}_3, \ldots, \overline{B}_\mu$.

4. The series $\overline{A}$ and $\overline{B}$ of the preceding article are *absolutely* convergent within the assigned limits; *i.e.*, any one of these series would continue to be convergent within these limits if its terms were replaced by their analytical modules. For the multiplication of two absolutely convergent series we have the theorem:—

'If the product of two given absolutely convergent series, proceeding by ascending powers of a variable, be arranged in a series proceeding in the same manner, this series is absolutely convergent for all values of the variable for which the given series are absolutely convergent, and its sum is equal to the product of the sums of the given series.' (Cauchy, 'Analyse Algébrique,' cap. vi.)

Multiplying together the m series $p - p_0 - \overline{A}$, and $p - p_0 - \overline{B}$, we obtain, by virtue of this theorem, the equation

$$F(p, q) = \Pi(p - p_0 - \overline{A}) \times \Pi(p - p_0 - \overline{B}).$$

This equation is an identity; *i.e.*, if the multiplication be actually effected on the right-hand side, all powers of $q - q_0$ above the m^{th} will disappear, and the terms that remain will be precisely the terms of $F(p_0 + \overline{p - p_0}, q_0 + \overline{q - q_0})$ or $F(p, q)$. But an arithmetical equality between the two sides of the equation subsists only so long as the analytical modulus of $q - q_0$ does not surpass the limit assigned in Cauchy's theorem (Art. 3). Subject to the same limitation, $f(q)$ is equal to the product of the squares of the differences of every two of the series $\overline{A}$ and $\overline{B}$.

5. The number of the intersections at any point O of two branches of the same curve, or of different curves, which pass through the point, and which are there represented by equations of the form

$$p^{(1)} - p_0 = B_0^{(1)}(q - q_0) + \ldots$$
$$p^{(2)} - p_0 = B_0^{(2)}(q - q_0) + \ldots$$

is defined by Professor Cayley to be the number which expresses the order of evanescence of $p^{(1)}-p^{(2)}$, *i.e.*, the integral or fractional exponent λ for which $\frac{p^{(1)}-p^{(2)}}{(q-q_0)^\lambda}\lambda$ has a finite limit, when $q-q_0$ is diminished without limit. We may justify this definition by proving that, whenever two curves C_1 and C_2 have a multiple intersection at any point, its multiplicity is correctly obtained by adding together the numbers (as thus defined) of the intersections of each branch of C_1 by each branch of C_2. If we suppose (as we may do) that the points P and Q have no speciality of position with regard to the curves C_1 and C_2 considered as one curve, the resultant $\Phi(q)$ of the equations $C_1(p, q)=0$ and $C_2(p, q)=0$ is of the order $m_1 \times m_2$; and if μ_1 branches of C_1 and μ_2 branches of C_2 pass through O, we shall have, for C_1, $m_1-\mu_1$ equations $\overline{A}^{(1)}$, and μ_1 equations $\overline{B}^{(1)}$; and similarly, for C_2, $m_2-\mu_2$ equations $\overline{A}^{(2)}$, and μ_2 equations $\overline{B}^{(2)}$. Denoting by $\Pi(\overline{B}^{(1)}-\overline{B}^{(2)})$ the product of the $\mu_1 \times \mu_2$ differences obtained by subtracting them in succession, each series $\overline{B}^{(2)}$ from each series $\overline{B}^{(1)}$, and by λ the number of intersections, as above defined, of any one branch of C_1 by any one branch of C_2, we see that the limit of $\Pi(\overline{B}^{(1)}-\overline{B}^{(2)}) \div (q-q_0)^{\Sigma\lambda}$ is finite. But $\Phi(q)=\Pi(\overline{B}^{(1)}-\overline{B}^{(2)})$, the sign of multiplication now extending to all the $m_1 \times m_2$ differences obtained by considering the m_1 series $\overline{A}^{(1)}$ and $\overline{B}^{(1)}$, and the m_2 series $\overline{A}_2$ and $\overline{B}_2$; and of these $m_1 \times m_2$ differences, none, except the $\mu_1 \times \mu_2$ differences already considered, are evanescent with $q-q_0$ (for the hypothesis that P and Q have no speciality of position with regard to the system of the two curves C_1 and C_2 implies that none of the constants $A^{(1)}$ can be equal to any of the constants $A^{(2)}$). Hence $\Sigma\lambda$ is the multiplicity of the factor $q-q_0$ in $\Phi(q)$; *i.e.*, since (p_0, q_0) is the only intersection of C_1 and C_2 which lies on (q_0), $\Sigma\lambda$ is the multiplicity of that intersection.

If we regard the equation $F(p, q)=0$ as determining a correspondence of points on a line, the coincidences of corresponding points (except indeed the coincidences $p=q=\infty$) answer in number and multiplicity to the intersections of C by the straight line $p=q$. We are thus led to a theorem given by M. Zeuthen ('Bulletin des Sciences Mathématiques,' Vol. V., p. 186).

6. As it is only the hypothesis that the points P and Q have no speciality of position with regard to C which gives us a right to assert that every one of the developments $\overline{B}$ contains a term linear in $q-q_0$, and no term in which the exponent of $q-q_0$ is less than unity, it is worth while to see how far the results of the preceding article can be depended on when this hypothesis is dispensed with. It will be found that if (p_0) is one of the tangents at O, *i.e.*, if one, or more, of the coefficients B_0 is zero, the discriminantal index of the point is still

equal to the order of evanescence of $\Pi(\overline{B}_i - \overline{B}_j)^2$. But this conclusion would no longer hold, if (q_0) were one of the tangents at O. In this case the developments appertaining to the branches to which (q_0) is a tangent would contain powers of $q - q_0$ inferior to unity; and the order of evanescence of $\Pi(\overline{B}_i - \overline{B}_j)^2$ would exceed the discriminantal index of O by τ, if $n - \tau$ is the number of tangents other than (q_0) which can be drawn to the curve from P. But the multiplicity of the intersection of two different curves, at a point which is singular for one or both of them is correctly obtained by the process of Art. 5, even when the developments contain positive powers of $(q - q_0)$ inferior to unity. Thus, in the curve $p - p_0 = (q - q_0)^{\frac{1}{a}}$, a being an integer, the order of evanescence of $\Pi[\overline{B}_i - \overline{B}_j]^2$ is $a - 1$, whereas the point is not a singular point at all, and has consequently a discriminantal index equal to zero: its tangent (q_0) however is a singular tangent, and counts as $a - 1$ tangents drawn from P. On the other hand, if we consider the two curves $(p - p_0) = (q - q_0)^{\frac{1}{a}}$, $p - p_0 = (q - q_0)^{\frac{1}{b}}$, in which a and b are both integers and $b < a$, the order of evanescence of $\Pi(\overline{B}_i - \overline{B}_j)^2$ is b; and this is the multiplicity of the intersection at (p_0, q_0).

7. We can now prove that the discriminantal index of the singular point O is equal to twice the number of the intersections of C by itself at that point; and, again, that this discriminantal index is equal to the number of the intersections at the same point of C by its first polar with regard to any point not having a special position. For (1), considering the μ equations $\overline{B}$, we see that twice the number of intersections of C by itself at the point (p_0, q_0), is the order of evanescence of $\Pi(\overline{B}_i - \overline{B}_j)^2$, the sign of multiplication extending to all the $\frac{1}{2}\mu(\mu - 1)$ differences; or, observing that $f(q) \div \Pi(\overline{B}_i - \overline{B}_j)^2$ is a product of $\frac{1}{2}m(m-1) - \frac{1}{2}\mu(\mu - 1)$ squared differences, none of which vanish with $q - q_0$, twice the number of intersections of C by itself at the point (p_0, q_0) is equal to the order of evanescence of $f(q)$ with $q - q_0$, *i.e.*, to the discriminantal index ν. And (2), since the polar of P is $\dfrac{dF}{dp} = 0$, and since the resultant of $F = 0$ and $\dfrac{dF}{dp} = 0$ is $f(q)$, we infer (Art. 5) that ν is the number of intersections at (p_0, q_0) of C by its polar with regard to P.

8. Considering a superlinear branch, of which the component branches are defined by the Δ equations

$$p - p_0 = B_0(q - q_0) + B_1 \omega^{\beta_1}(q - q_0)^{\frac{\beta_1}{\Delta}} + \ldots, \quad \ldots \quad (\Delta)$$

let Δ_1 be the greatest common divisor of Δ and β_1; and, if γ_1 is the first of the

numbers β which is not divisible by Δ_1, let Δ_2 be the greatest common divisor of Δ_1 and γ_1; if, again, γ_2 is the first of the numbers β which is not divisible by Δ_2, let Δ_3 be the greatest common divisor of Δ_2 and γ_2, and so on continually. Since the numbers β have no common divisor with Δ, we shall at last arrive in the series $\Delta_1, \Delta_2, \Delta_3, \ldots$ at a term equal to unity, when the series will terminate: and twice the number of the intersections of the superlinear branch by itself will be expressed by the formula

$$2N = \gamma(\Delta - \Delta_1) + \gamma_1(\Delta_1 - \Delta_2) + \gamma_2(\Delta_2 - \Delta_3) + \ldots,$$

in which γ is written for β_1. For if ω denote any given root of the equation $\omega^\Delta = 1$, of its remaining roots x there are $\Delta_i - 1$ which verify the equations $x^\gamma = \omega^\gamma$, $x^{\gamma_1} = \omega^{\gamma_1}$, ..., $x^{\gamma_{i-1}} = \omega^{\gamma_{i-1}}$; because Δ_i is the greatest common divisor of Δ, γ, γ_1, ..., γ_{i-1}: similarly there are $\Delta_{i+1} - 1$ roots other than ω_1 which verify the same equations, and in addition the equation $x^{\gamma_i} = \omega^{\gamma_i}$. Thus, of the $\Delta(\Delta - 1)$ differences obtained by subtracting each of the series (Δ) in turn from every other, there are $\Delta(\Delta_i - \Delta_{i+1})$ which are of the order $\frac{\gamma_i}{\Delta}$; *i.e.*, $2N = \Sigma\gamma_i(\Delta_i - \Delta_{i+1})$. The value of N depends, therefore, not on every exponent in the series (Δ), but only on certain *critical* exponents $\frac{\gamma}{\Delta}$, in the denominators of which, when reduced to their lowest terms, a new factor appears for the first time. The number $2N$, which is the 'discriminantal index' of the superlinear branch is, not itself necessarily even, but the difference $2N - (\Delta - 1)$ is always even, since we have

$$2N - (\Delta - 1) = (\gamma - 1)(\Delta - \Delta_1) + (\gamma_1 - 1)(\Delta_1 - \Delta_2) + \ldots,$$

and in this expression, if Δ is uneven, so also are $\Delta_1, \Delta_2, \ldots$; if Δ is even, let Δ_i be the first of the numbers $\Delta_1, \Delta_2, \ldots$ which is uneven; then γ_{i-1} is uneven, and so are all the subsequent numbers $\Delta_{i+1}, \Delta_{i+2}, \ldots$. In either case, therefore, every term in the expression of $2N - (\Delta - 1)$ is even.

Again, if two superlinear branches of the orders Δ and Δ' have the same tangent, let $(q - q_0)^h$ be the lowest power of $q - q_0$ which has not the same coefficient B in the two sets of series (Δ) and (Δ'): it may, of course, in one of these sets have a zero coefficient. Then the terms of lower exponent are common to the two sets; and if the exponents be reduced to their least common denominator, these initial terms will be of the form

$$B_0(q - q_0) + B_1\theta^{\alpha_1}(q - q_0)^{\frac{\alpha_1}{d}} + B_2\theta^{\alpha_2}(q - q_0)^{\frac{\alpha_2}{d}} + \ldots + B_i\theta^{\alpha_i}(q - q_0)^{\frac{\alpha_i}{d}},$$

where d is a common divisor of Δ and Δ', θ is any root of $\theta^d = 1$, and $\frac{a_i}{d}$ is the exponent next inferior to h. The number of intersections of the two superlinear branches is then

$$N' = h\frac{\Delta\Delta'}{d} + \frac{\Delta\Delta'}{d^2}[\sigma(d-d_1) + \sigma_1(d_1-d_2) + \ldots],$$

the numbers $\sigma, \sigma_1, \sigma_2, \ldots, d_1, d_2, \ldots$ (of which in particular $\sigma = a_1$) being determined from the series of exponents $\frac{a}{d}$, in the same way that the numbers $\gamma, \gamma_1, \ldots, \Delta_1, \Delta_2, \ldots$ were determined from the series of exponents $\frac{\beta}{\Delta}$. For if θ represent a given root of the equation $\theta^d = 1$, the $\frac{\Delta}{d}$ roots of the equation $\omega^\Delta = 1$, which satisfy the equation $\omega^{\frac{\Delta}{d}} = \theta$, will give the same initial terms; and we may thus divide the equations (Δ) into $\frac{\Delta}{d}$ groups, each containing d equations; the equations of the same group differing from one another by containing different values of θ, but the different groups not differing from one another, so far as the initial terms are concerned. Similarly we may divide the equations (Δ') into $\frac{\Delta'}{d}$ groups. Considering only one group of each set, we find (by the same reasoning as before) for the order of the product of the $d \times d$ differences obtained from them, the expression

$$hd + \sigma(d-d_1) + \sigma_1(d_1-d_2) + \ldots,$$

the additional term hd appearing because we have now to take into account the d differences in which all the initial terms vanish: the result, multiplied by $\frac{\Delta}{d} \times \frac{\Delta'}{d}$, gives the value of N'.

Lastly, when two superlinear branches have not the same tangent, the number of their intersections is evidently $N'' = \Delta\Delta'$. By means of these formulae the discriminantal indices of the branches at any singular point, taken by themselves or in pairs, may always be obtained as soon as the developments appertaining to the branches have been found. The sum of these separate discriminantal indices is of course the discriminantal index of the point, or $\nu = 2\Sigma N + 2\Sigma N' + 2\Sigma N''$.

9. Every singular point of a plane curve is regarded by Professor Cayley as being equivalent in a certain manner to δ common nodes and κ common cusps; and, correlatively, every singular tangent as equivalent to τ double tangents and ι inflexional tangents. For any superlinear branch of order Δ passing through a

singular point, the *cuspidal index* κ is by definition $\Delta - 1$; thus, for a linear branch $\kappa = 0$. The cuspidal index of a singular point is the sum of the cuspidal indices of the several superlinear branches passing through it; so that, for any singular point, $\kappa = \Sigma(\Delta - 1) = \mu - \lambda$, if μ is the order (Art. 2) of the point, and λ the number of distinct linear, or superlinear, branches passing through it. The *nodal index* δ for a singular point, and for its branches, taken singly or in pairs, is defined, not directly, but by equating $2\delta + 3\kappa$ to the discriminantal index; thus, for any superlinear branch of order Δ, we have

$$2\delta = (\gamma - 3)(\Delta - \Delta_1) + (\gamma_1 - 3)(\Delta_1 - \Delta_2) + \ldots,$$

which is always even (Art. 8), and positive, except when $\gamma = 3$, $\Delta = 2$, in which case $\delta = 0$, and the superlinear branch is a common cusp.

For τ and ι we have correlative definitions.

10. Adopting these definitions, we have now to prove that the numbers $\Sigma\delta$, $\Sigma\kappa$, $\Sigma\tau$, $\Sigma\iota$ (the summations extending to all the singularities of the curve) satisfy the equations of Plücker, and further that the deficiency of the curve is correctly given by the formula

$$H = \tfrac{1}{2}(m-1)(m-2) - \Sigma\delta - \Sigma\kappa.$$

It is sufficient to establish the four equations,

$$\text{(i.)} \quad n = m(m-1) - 2\Sigma\delta - 3\Sigma\kappa,$$

$$\text{(ii.)} \quad m = n(n-1) - 2\Sigma\tau - 3\Sigma\iota,$$

$$\text{(iii.)} \quad H = \tfrac{1}{2}(m-1)(m-2) - \Sigma\delta - \Sigma\kappa,$$

$$\text{(iv.)} \quad H = \tfrac{1}{2}(n-1)(n-2) - \Sigma\tau - \Sigma\iota,$$

because the three equations (i.), (ii.), and (iii.) = (iv.) are equivalent to the six equations of Plücker. But the equation (i.) has been already proved; for we have found (Art. 2) that $n = m(m-1) - \Sigma\nu$; and by definition $\Sigma\nu = \Sigma(2\delta + 3\kappa)$. The equation (ii.) is the correlative of (i.) and needs no separate proof. In the equations (iii.) and (iv.) it is important to take a definition of H which does not involve any special supposition as to the nature of the singularities appertaining to the curve. The simplest, though not the most direct, course is to adopt the method of Riemann, and to define $2H + 1$ as the index of multiplicity of connexion of the m-leaved spirally connected surface $[Q]$, which is such that if the complex values of q be represented upon it in the usual manner, p may be regarded as a one-valued function of q. In any such surface the index of multiplicity of connexion $2H + 1$, the number of leaves m, and the number of *spires* (spiral points, *windungs-punkte*) N are connected by the equation $N = 2H + 2m - 2$. This equation Riemann himself demonstrates by comparing the values of certain

contour-integrals ('Theorie der Abelschen Functionen,' Art. 7). But he observes that it is entirely independent of considerations of magnitude, and that it belongs properly to the geometry of situation. The demonstration of it from this point of view, which has been given by M. Neumann ('Vorlesungen,' p. 309, § 99), is also independent of any supposition as to the special nature of the singularities of the curve C; and is therefore available for our present purpose. But we may observe that the algebraical demonstration of the same equation, which is given by MM. Clebsch and Gordan (in their 'Theorie der Abelschen Functionen,' p. 54, § 16), would here be inadmissible, because in that demonstration it is expressly supposed that the singular points of C are only common nodes and cusps. (See the note at p. 11, *loc. sit.*)

It is not difficult to find the number of spires N on the surface $[Q]$. There is a one-fold spire for every tangent from P to C; for, if (p_0, q_0) be the point of contact of any such tangent, we have for values of q in the vicinity of q_0 two conjugate developments of the type

$$(p-p_0) = B_1 (q-q_0)^{\frac{1}{2}} + B_2 (q-q_0)^{\frac{2}{3}} + \ldots,$$

in which B_1 is different from zero; all the other developments (Art. 3) being of the type (A), because the point P has no speciality of position. Again, there is a $(\Delta-1)$-fold spire for any singular branch which is superlinear and of order Δ; this is apparent from the form of the Δ developments appertaining to the branch (see Riemann, *loc. cit.*, Art. 6; M. Puiseux, 'Liouville,' 1st series, Vol. XV. pp. 384–404).

We have therefore $N = n + \Sigma(\Delta-1) = n + \Sigma\kappa$, and Riemann's equation becomes

$$n + \Sigma\kappa = 2H + 2(m-1);$$

or, since

$$n + \Sigma\kappa = m(m-1) - 2\Sigma\delta - 2\Sigma\kappa,$$

$$H = \tfrac{1}{2}(m-1)(m-2) - \Sigma\delta - \Sigma\kappa,$$

which is the equation (iii.) Again, it is an immediate consequence of Riemann's definition of the number H (see his 'Abelsche Functionen,' Art. 11) that this number remains unchanged by any unicursal transformation of the equation $F(p, q) = 0$. But (as has been already observed by MM. Clebsch and Gordan) any tangential equation of the curve C may be regarded as an unicursal transformation of the equation $F(p, q) = 0$, because the points and tangents of a curve correspond to one another one to one. The equation (iii.), therefore, involves the equation (iv.); a result which, as we have seen, implies that the six equations of Plücker are satisfied by the numbers $\Sigma\delta$, $\Sigma\kappa$, $\Sigma\tau$, $\Sigma\iota$.

11. The indices τ and ι appertaining to any superlinear branch at a singular

point, and the number of tangents common to two osculating superlinear branches, may be ascertained directly from the point-equations $\overline{B}$, without actually forming the corresponding line-equations. To prove this, we shall establish a relation which subsists between certain terms in the two sets of equations.

If Q and R are given constants, $p = Qq + R$ is the equation of a straight line in the system of parametric point-coordinates which we have been employing. In passing to line-coordinates, we may take Q and R as the coordinates of this straight line; and we may regard Q and R as the parameters of two ranges of points, lying on the lines PQ, PR, respectively, and represented by equations of the form

$$Q(P) + (Q) = 0, \quad R(P) + (R) = 0;$$

the line $p = Q_0 q + R_0$ or (Q_0, R_0) being the line joining the points determined in the two ranges by the values Q_0, R_0 of the parameters. If to the hypothesis of Art. 1 we add the supposition that PR is not a tangent to C, and does not pass through any singular point of C, the line-equation of C, which we may represent by $\Phi(Q, R) = 0$, will have the same sort of freedom from speciality which has been already attributed to the point-equation $F(p, q) = 0$. The parameters of the tangent to C at the point (p, q) are

$$Q = -\left(\frac{dF}{dq}\right) \div \left(\frac{dF}{dp}\right), \quad R = p - qQ.$$

Let (p, q) be a point lying on the branch $\overline{B}$, of which the point-equation is

$$p - p_0 = B_0 (q - q_0) + B_1 \omega^{\beta_1} (q - q_0)^{\frac{\beta_1}{\Delta}} + \dots;$$

and suppose (p, q) different from (p_0, q_0), but sufficiently near to it (Art. 3) to ensure the convergence of the m series $\overline{A}$ and $\overline{B}$. Writing

$$F(p, q) = M \times (p - p_0 - \overline{B}),$$

where M is a product of factors, none of which can vanish at the point (p, q), because no singular point other than (p_0, q_0) exists within the range of values attributed to q, we find

$$Q = \left(\frac{d\overline{B}}{dq}\right), \quad R = p - q\left(\frac{d\overline{B}}{dq}\right).$$

Putting

$$B_0 = Q_0, \quad p_0 - q_0 B_0 = R_0,$$

so that (Q_0, R_0) is the tangent at (p_0, q_0) to $\overline{B}$, we obtain the equations

$$Q - Q_0 = \frac{\beta_1}{\Delta} B_1 \omega^{\beta_1} (q - q_0)^{\frac{\beta_1}{\Delta} - 1} + \frac{\beta_2}{\Delta} B_2 \omega^{\beta_2} (q - q_0)^{\frac{\beta_2}{\Delta} - 1} + \dots,$$

$$R - R_0 = -q_0 (Q - Q_0) + \left(1 - \frac{\beta_1}{\Delta}\right) B_1 \omega^{\beta_1} (q - q_0)^{\frac{\beta_1}{\Delta}} + \left(1 - \frac{\beta_2}{\Delta}\right) B_2 \omega^{\beta_2} (q - q_0)^{\frac{\beta_2}{\Delta}} + \dots,$$

which determine the parameters Q, R of the tangent at any point of $\overline{B}$. If we further write

$$\omega(q-q_0)^{\frac{1}{\Delta}}=\xi, \quad Q-Q_0=\frac{\beta_1 B_1}{\Delta}\times Y^{\beta_1-\Delta}, \quad R-R_0+q_0(Q-Q_0)=Z,$$

these equations become

$$Y^{\beta_1-\Delta}=\xi^{\beta_1-\Delta}[1+\sigma_2\xi^{\beta_2-\beta_1}+\sigma_3\xi^{\beta_3-\beta_1}+\ldots], \quad \ldots\ldots \quad (\alpha)$$

$$Z=\rho_1\xi^{\beta_1}+\rho_2\xi^{\beta_2}+\ldots, \quad \ldots\ldots\ldots\ldots \quad (\beta)$$

where we have written σ_i for $\frac{\beta_i B_i}{\beta_1 B_1}$, and ρ_i for $\left(1-\frac{\beta_i}{\Delta}\right)B_i$. It will be observed that ω has disappeared from these equations, which therefore appertain equally to all the Δ branches composing the superlinear branch $\overline{B}$. To obtain the tangential equation of that branch, *i.e.*, the expansion of $R-R_0$ in a series proceeding by powers of $Q-Q_0$, we have three operations to perform. First, we have to raise each side of the equation (α) to the power $\frac{1}{\beta_1-\Delta}$; we thus obtain an expansion of the form

$$\theta Y=\xi(1+A\xi^a+B\xi^b+\ldots), \quad \ldots\ldots\ldots \quad (Y)$$

θ denoting any root of the equation $\theta^{\beta_1-\Delta}=1$. Secondly, we have to revert the series (Y), so as to obtain the series

$$\xi=\theta Y\{1+A'(\theta Y)^{a'}+B'(\theta Y)^{b'}+\ldots\}. \quad \ldots\ldots\ldots \quad (\xi)$$

Lastly, we have to substitute, in the equation (β), for ξ its value given by the series (ξ); the final result being of the form

$$Z=H_1(\theta Y)^{h_1}+H_2(\theta Y)^{h_2}+\ldots; \quad \ldots\ldots\ldots \quad (Z)$$

or, if

$$\mu=\left(\frac{\Delta}{\beta_1 B_1}\right)^{\frac{1}{\beta_1-\Delta}},$$

$$R-R_0=-q_0(Q-Q_0)+H_1\mu^{h_1}\theta^{h_1}(Q-Q_0)^{\frac{h_1}{\beta_1-\Delta}}+H_2\mu^{h_2}\theta^{h_2}(Q-Q_0)^{\frac{h_2}{\beta_1-\Delta}}+\ldots. \quad (H)$$

12. Certain of the terms of H, and indeed precisely those critical terms upon which the determination of τ and ι depends, can be assigned *à priori* by the help of the following considerations.

(i.) If $a, b, c, \ldots, l, \ldots$ are positive and integral numbers, arranged in order of magnitude, of which l is such that it cannot be formed by addition of any multiples of the numbers which precede it, the coefficient of x^l in the expansion of $[\psi(x)]^\sigma$, where σ is any real exponent, and

$$\psi(x)=1+Ax^a+Bx^b+Cx^c+\ldots+Lx^l+\ldots,$$

is σL; and, in particular, if all the numbers preceding l are multiples of any

number a, of which l is not itself a multiple, a supposition which implies that l cannot be formed by addition of multiples of $a, b, c, \ldots$, l is the least exponent in the development of $[\psi(x)]^a$, which is not divisible by a.

(ii.) If the series $y = x\psi(x)$ be reverted so as to obtain the equation

$$x = y\psi_1(y) = y(1 + A_1 y^{a_1} + B_1 y^{b_1} + \ldots),$$

the exponents $a_1, b_1, c_1, \ldots$ are all formed by addition of multiples of $a, b, c, \ldots$. For, if this is not so, let h_1 be the least exponent in $\psi_1(y)$, which cannot be formed by adding multiples of $a, b, c, \ldots$; on substituting $x\psi(x)$ for y in $y\psi_1(y)$, a substitution which ought to have x for its result, we find that the coefficient of x^{h_1+1} is H_1; *i.e.* $H_1 = 0$, or the exponent h_1 does not occur in $\psi_1(y)$. Again, if the exponent l in $\psi(x)$ cannot be formed by adding multiples of the exponents which precede it, the coefficient L_1 of y^l in $\psi_1(y)$ is $-L$; for, on making the same substitution as before, the coefficient of x^{l+1} is found to be $L_1 + L$; *i.e.* $L_1 = -L$. And, in particular, if l is the lowest exponent in $\psi(x)$ which is not divisible by a, l is also the lowest exponent in $\psi_1(y)$ which is not divisible by a.

Let $\beta_i = \gamma_s$ be one of the critical exponents $\gamma, \gamma_1, \ldots$ considered in Art. 8; then all the differences $\beta_2 - \beta_1, \beta_3 - \beta_1, \ldots$ up to $\beta_{i-1} - \beta_1$, are divisible by Δ_{s-1}; but $\beta_i - \beta_1$ is not divisible by Δ_{s-1}. Therefore, by (i.), the coefficient of $\xi^{\beta_i - \beta_1}$ in the development of $\dfrac{\theta Y}{\xi}$ is $\dfrac{\sigma_i}{\beta_1 - \Delta}$; by (ii.) the coefficient of $(\theta Y)^{\beta_i - \beta_1}$ in the expansion of $\dfrac{\xi}{\theta Y}$ is $-\dfrac{\sigma_i}{\beta_1 - \Delta}$, and $\beta^i - \beta_1$ is the least exponent in that expansion which is not divisible by Δ_{i-1}; finally, on substituting in the equation (β), we see that the term $H(\theta Y)^{\beta_i}$ in the development of Z can arise only from the terms $\rho_1 \xi^{\beta_1}$ and $\rho_i \xi^{\beta_i}$ in (β); its coefficient H is therefore $-\rho_1 \dfrac{\beta_1 \sigma_i}{\beta_1 - \Delta} + \rho_i = B_i$; and the coefficient of $\theta^{\beta_i}(Q - Q_0)^{\frac{\beta_i}{\beta_1 - \Delta}}$, or $\theta^{\gamma_s}(Q - Q_0)^{\frac{\gamma_s}{\gamma - \Delta}}$, in the expansion of $R - R_0$ is $\mu^{\beta_i} B_i$. Nor can any exponent preceding $\dfrac{\gamma_s}{\gamma - \Delta}$ have a numerator which is not divisible by Δ_{s-1}.

Observing that the greatest common divisor of $\beta_1 - \Delta$, and β_1, is the same as that of Δ and β_1, we infer from this result that the numbers $\Delta_1, \Delta_2, \Delta_3, \ldots$; $\gamma, \gamma_1, \gamma_2, \ldots$ are the same for the series H as for the series B; and since the numbers $\gamma, \gamma_1, \gamma_2, \ldots$ have no common divisor with Δ, neither have they with $\beta_1 - \Delta = \gamma - \Delta$; *i.e.*, $\gamma - \Delta$ is the least common denominator of the exponents of H. Hence we have, writing $\gamma - \Delta = \Delta_s$,

$$\iota = \Delta_{,} - 1, \quad \gamma = \Delta + \Delta_{,} = \iota + \kappa + 2,$$
$$2\tau + 3\iota = \gamma(\Delta_{,} - \Delta_1) + \gamma_1(\Delta_1 - \Delta_2) + \gamma_2(\Delta_2 - \Delta_3) + \dots;$$

or, subtracting the discriminantal index $2\delta + 3\kappa$,

$$(2\tau + 3\iota) - (2\delta + 3\kappa) = \Delta_{,}^2 - \Delta^2,$$
$$\tau - \delta = \tfrac{1}{2}(\iota - \kappa)(\iota + \kappa - 1),$$

an equation which establishes a relation between the four indices of the superlinear branch.

13. If we consider any term whatever in the series B, for example the term $(i) = B_i \omega^{\beta_i} (q - q_0)^{\frac{\beta_i}{\Delta}}$, we shall in general find a corresponding term (I) in the series H, containing $Q - Q_0$ raised to the power $\frac{\beta_i}{\beta_1 - \Delta}$: (I) may be considered as the sum of two parts, I_1 and I_2, of which the first, I_1, arises from the term (i) itself, the other, I_2, from the terms preceding (i); (I) being in no way affected by the terms following (i). If β_i is one of the critical exponents, we have just seen that $I_2 = 0$, $I_1 = \mu^{\beta_i} B_i (Q - Q_0)^{\frac{\beta_i}{\beta_1 - \Delta}}$. If β_i is not one of the critical exponents, the first of these equations ceases to subsist, but the second remains true, and its proof requires only a slight modification of the reasoning in Art. 12. Now let two series $\overline{B}$, appertaining to two different superlinear branches, which have a common tangent, coincide as far as the term (i), but exclusively of it; the two corresponding series $\overline{H}$ will coincide as far as the term (I), but exclusively of it; we suppose $i > 0$. That all terms preceding (I) will coincide in the two developments $\overline{H}$ is evident, for these terms arise solely from the terms preceding (i), which are identical in the two developments $\overline{B}$. And the terms (I) themselves are different: for the difference of the two terms (i) is $(B_i - B_i') \omega^{\beta_i} (q - q_0)^{\frac{\beta_i}{\Delta}}$, where one of the two B_i, B_i' may be zero, but the difference $B_i - B_i'$ is by hypothesis not zero; and the difference of the two terms (I) is

$$\mu^{\beta_i} (B_i - B_i') \theta^{\beta_i} \times (Q - Q_0)^{\frac{\beta_i}{\beta - \Delta}} = I_1 - I_1',$$

for these two terms have the same part I_2.

Let $\overline{D}$ be the number of points, $\overline{T}$ the number of tangents common to the two branches $\overline{B}$ at the point (p_0, q_0); $\overline{T}$ is given by the formula

$$\overline{T} = \beta_i \frac{\gamma' - \Delta'}{\sigma - d} + \frac{(\gamma - \Delta)(\gamma' - \Delta')}{(\sigma - d)^2} \{\sigma(\sigma - d - d_1) + \sigma_1(d_1 - d_2) + \dots\},$$

which is derived from the expression for N' in Art. 8, by writing $\gamma - \Delta$ for Δ,

$\gamma' - \Delta'$ for Δ', and $\sigma - d$ for d. Observing that $\frac{\gamma}{\Delta} = \frac{\gamma'}{\Delta'} = \frac{\sigma}{d}$, whence

$$\frac{\gamma' - \Delta'}{\sigma - d} = \frac{\Delta'}{d}, \quad \frac{\gamma - \Delta}{\sigma - d} = \frac{\Delta}{d}, \quad \frac{\Delta}{d}\sigma = \gamma = \iota + \kappa + 2,$$

$$\frac{\Delta'}{d}\sigma = \gamma' = \iota' + \kappa' + 2, \quad \frac{\iota + 1}{\kappa + 1} = \frac{\iota' + 1}{\kappa' + 1},$$

we find
$$\overline{T} - \overline{D} = \frac{\Delta\Delta'}{d^2}\sigma(\sigma - 2d)$$
$$= (\iota - \kappa)(\iota' + \kappa' + 2) = (\iota' - \kappa')(\iota + \kappa + 2)$$
$$= (\iota + 1)(\iota' + 1) - (\kappa + 1)(\kappa' + 1) = \Delta_1\Delta_1' - \Delta\Delta'.$$

We have supposed in the demonstration that $i > 1$, or that the two developments of $p - p_0 - B_0(q - q_0)$ coincide for at least one term. But, for the validity of the formulæ, it is only necessary that the first exponent should be the same in the two developments; and indeed the last two expressions for $\overline{T} - \overline{D}$ hold universally for any two superlinear branches having a common tangent.

14. The *species* of a superlinear singularity may be regarded as defined by the series of numbers Δ and Δ_1; Δ_1, Δ_2, ..., γ_1, γ_2, ..., so that two superlinear singularities, for which these indices have the same values, may be considered as belonging to the same species. A rougher classification, however, which is sometimes useful, may be obtained in the following way. Leaving out of sight the case in which two superlinear singularities present themselves as conjugate imaginaries, and attending only to the case of a real superlinearity, we may distinguish four varieties differing from one another in the appearance which they present to the eye. (See a Memoir by M. Stolz, 'Mathematische Annalen,' Vol. VIII., p. 440.)

(i.) Δ uneven, Δ_1 uneven; no apparent cusp or inflexion.

(ii.) Δ even, Δ_1 uneven; an apparent cusp, no apparent inflexion.

(iii.) Δ uneven, Δ_1 even; an apparent inflexion, no apparent cusp.

(iv.) Δ even, Δ_1 even; an apparent cusp, and an apparent inflexion.

The form of (ii.) is that of the common or keratoid cusp; (iv.) has the form of the cusp of the second species, or rhamphoid cusp. There is an apparent inflexion at the rhamphoid cusp, because, if a person describing the curve continuously passes through the cusp, the concavity of the curve is to his right after he has passed through the cusp, if it was to his left hand before, and *vice versâ*. We may further observe that, in case (iv.), Δ and Δ_1, being both even, have a common

measure; thus $\Delta_2 > 1$, and the superlinearity is *composite*. The cases (ii.) and (iii.) are correlative; the cases (i.) and (iv.) are their own correlatives.

15. The curvature of a curve at two points infinitely near to a given superlinear point, and at equal distances from it on either side, is always the same; and is infinite, finite, or zero, according as $\Delta > \Delta_1$, $\Delta = \Delta_1$, or $\Delta < \Delta_1$. Thus, in each of the cases (i.) and (iv.), there are three sub-varieties of form; and two in each of the cases (iii.) and (ii.). The following are the simplest examples of each of these sub-varieties: for the sake of completeness, the cases in which either of the two numbers Δ or Δ_1 is unity, are included.

(i.) Δ and Δ_1 uneven,

$\Delta > \Delta_1$: $y = x^{\frac{4}{3}}$; $y = x^{\frac{8}{5}}$.

$\Delta = \Delta_1$: $y = x^2$; $y = x^2 + x^{\frac{7}{3}}$.

$\Delta < \Delta_1$: $y = x^4$; $y = x^{\frac{8}{3}}$.

(ii.) Δ even, Δ_1 uneven,

$\Delta > \Delta_1$: $y = x^{\frac{3}{2}}$; $y = x^{\frac{7}{4}}$.

$\Delta < \Delta_1$: $y = x^{\frac{5}{2}}$.

(iii.) Δ uneven, Δ_1 even,

$\Delta > \Delta_1$: $y = x^{\frac{5}{3}}$.

$\Delta < \Delta_1$; $y = x^3$; $y = x^{\frac{7}{3}}$.

(iv.) Δ and Δ_1 even,

$\Delta > \Delta_1$: $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$.

$\Delta = \Delta_1$: $y = x^2 + x^{\frac{5}{2}}$.

$\Delta < \Delta_1$: $y = x^3 + x^{\frac{7}{2}}$.

It should be noticed that in the equation $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$, the only independent radical is $x^{\frac{1}{4}}$, and that $x^{\frac{1}{2}}$ is to be interpreted as $(x^{\frac{1}{4}})^2$. Thus, supposing x positive, and understanding by $\sqrt{x^3}$ and $\sqrt[4]{x^7}$ the real and positive values of the radicals, we have for the four partial branches the equations

$$y = \sqrt{x^3} + \sqrt[4]{x^7}, \quad y = \sqrt{x^3} - \sqrt[4]{x^7},$$
$$y = -\sqrt{x^3} - i\sqrt[4]{x^7}, \quad y = -\sqrt{x^3} + i\sqrt[4]{x^7},$$

of which the first two appertain to a real rhamphoid cusp. If we were to change the sign of $\sqrt{x^3}$, we should pass from the equation

$$U = (y^2 + x^3)^2 - x^3(x^2 + 2y)^2 = 0,$$

which is the rationalised equivalent of $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$, to the equation

$$V = (y^2 + x^3)^2 - x^3(x^2 + 2y)^2 = 0,$$

which is the rationalised equivalent of $y = -x^{\frac{3}{2}} + x^{\frac{7}{4}}$. It is, of course, quite possible that two developments, such as $y = \pm x^{\frac{3}{2}} + x^{\frac{7}{4}} + \ldots$, may both belong to the same curve (as indeed they do both belong to the curve $UV + x^{15}\phi(x, y) = 0$), but such a curve would have two distinct superlinear branches touching one another at the point $x = 0$, $y = 0$.

16. Let O be any point whatever on a curve line; let the arc $OP = \sigma$, P being a point on the curve infinitely near to O; let M be the orthogonal projection of P on the tangent at O; and let the tangents at O and P intersect at T, making the infinitesimal angle ω. Then it will be found that

$$\frac{\Delta'}{\Delta} = \frac{\log \omega}{\log \sigma} = \frac{OT}{TP} = \frac{\angle TPO}{\angle TOP} = \frac{\log MP}{\log OM} - 1.$$

The fraction $\frac{\Delta'}{\Delta}$ which admits of these various geometrical interpretations may perhaps be called the *logarithmic curvature* of the curve at the point O. At any ordinary point it is unity; and in a geometrical curve it is always rational, but in a transcendental curve it may have any value rational or irrational.

Since Δ or $\kappa + 1$ is the number of points in which the superlinear branch is cut by any line passing through O, other than its tangent at the point O, we infer that, correlatively, $\iota + 1$ or Δ' is the number of tangents drawn to the superlinear branch from any point on the tangent at O, other than O itself. Thus, if d be the discriminantal index of O, or the number of points in which the curve is cut at O by the polar of any arbitrary point, $d + \Delta'$ is the number of points in which the curve is cut at O by the polar of any point on the tangent at O, other than O itself; there is, of course, a correlative definition of $d + \Delta$. Lastly, since $\Delta + \Delta'$ is the number of points common at O to the tangent and the curve, it is also, correlatively, the number of tangents drawn from O to touch the curve at that point. Thus the polar of the point O intersects the curve at O in $d + \Delta + \Delta'$ points, and the tangent at O counts as $d + \Delta + \Delta'$ tangents common to the curve, and to the tangential polar of OT with regard to the curve. For the numbers $\Delta_1, \Delta_2, \ldots, \gamma_1, \gamma_2, \ldots$ no simple geometrical definition has as yet presented itself.

17. The proof of Plücker's formulæ, which is indicated in Art. 10, may appear very indirect. Some further observations on these formulæ, and on the various modes of demonstrating them, may not be out of place.

(1.) If we write $D = \Sigma(2\delta + 3\kappa)$, $T = \Sigma(2\tau + 3\iota)$, $I = \Sigma\iota$, $K = \Sigma\kappa$, $\frac{1}{3}(I - K) = \Omega$, Plücker's formulæ become

$$n = m(m-1) - D,$$
$$m = n(n-1) - T;$$

$$\Omega = m(m-2) - D,$$
$$-\Omega = n(n-2) - T;$$

giving
$$T - D = n^2 - m^2, \quad \Omega = 3(n-m),$$
$$\Omega^4 - 2\Omega^2(T+D) - 4\Omega(T-D) + (T-D)^2 = 0.$$

It is thus apparent that Plücker's equations do not contain either K or I separately, but only the difference $I-K$.

(2.) The discriminantal index $d = 2\delta + 3\kappa$ of any given point is defined geometrically as the number of intersections of the polar of an arbitrary point with the curve at the given point. But the definition which we have given in Art. 9 of the cuspidal index κ is an analytical one, and does not readily admit of interpretation in coordinate geometry. The Hessian does not serve to define either ι or κ, for in all the cases that have as yet been rigorously investigated, it has been found that the number of intersections of the Hessian with the curve at a point of discriminantal index d is $3d + \iota - \kappa$, so that, even if the number of these intersections at any singular point should be determined by a general method, we should only obtain a definition of the difference $\iota - \kappa$. Again, if several superlinear branches have a common tangent OT at the point O, it will be seen that the geometrical definitions of Art. 16 only give the numbers $\Sigma(\iota+1)$ and $\Sigma(\kappa+1)$; viz., if d is the total discriminantal index of all the branches intersecting at O, the first polar of any point on OT (other than O) intersects the curve at O in $d + \Sigma(\iota+1)$ points; the polar of O intersects the curve at O in $d + \Sigma(\iota+\kappa+2)$ points; and there are correlative definitions of the numbers $d + \Sigma(\kappa+1)$, and $d + \Sigma(\iota+\kappa+2)$. By combining these definitions, we obtain a geometrical definition of the difference $\Sigma(\iota-\kappa)$, the summation extending to all the branches which touch one another at O. But here it is to be observed (1) that to deduce the values of $\Sigma\iota$ and $\Sigma\kappa$ from those of $\Sigma(\iota+1)$ and $\Sigma(\kappa+1)$, we should require to determine the number λ of distinct superlinear branches which touch OT at O; and (2) that, even if $\Sigma\iota$ and $\Sigma\kappa$ were known, it would still remain to determine the decomposition of these sums, and to assign the partial indices appertaining to each of the λ branches; whereas no determination of the number λ, or of the indices ι and κ of each separate superlinear branch, has as yet been obtained by considering the intersections of the given curve with any concomitant or system of concomitants.

(3.) The difficulty, which thus presents itself in obtaining a definition of the indices ι and κ, ceases to exist when we leave the domain of coordinate geometry, and consider either the analytical expansions, or the geometrical representations

(depending on principles foreign to coordinate geometry) which correspond to those expansions. If several superlinear branches touch one another at a given point, the analytical expansions separate them, and assign the cuspidal and inflexional indices proper to each of them. If we apply to the equation $F(p, q) = 0$ the geometrical methods of double algebra, the cuspidal indices appear in the cycles of values of p, which present themselves at the points answering to the discriminantal values of q. (See the memoir of M. Puiseux, 'Liouville,' Vol. XV., p. 384.) If, instead of the simple plane of double algebra, we use the multiple plane of Riemann, the cuspidal indices are represented by the spires which connect the leaves of the multiple plane. But it is important to remember that, in employing the methods of double algebra, and *à fortiori* in employing the surfaces of Riemann, we are entirely abandoning the methods of coordinate and projective geometry. The present question is perhaps not directly affected by the fundamental distinction between the 'infinite' of double algebra, which is a point, and the infinite of projective geometry, which is a straight line. But the duality, characteristic of projective geometry, is lost in double algebra; so that, when the complex values of p and q which satisfy the equation $F(p, q) = 0$ are regarded as developed on a plane, or on one of Riemann's surfaces, we do indeed obtain a direct representation of the cuspidal index κ, but no corresponding representation (unless we first transform the equation into its reciprocal) of the correlative index ι. Indeed, it may be asserted that, whereas the character of any given superlinearity mainly depends on a series of indices $\Delta = \kappa + 1$, $\Delta_{,} = \iota + 1$, $\Delta_1, \Delta_2, \ldots, \gamma_1, \gamma_2, \ldots$, the modes of geometrical representation, to which we are here referring, offer a sensible image of the first of these indices only. If we employ a simple plane, any one of the Δ values of p, which come to coincide with one another at the discriminantal point, must describe Δ elementary contours around that point before it acquires again its original value. If for simplicity we suppose that $\Delta_2 = 1$, the Δ values of p, which form the cycle, will divide themselves into Δ_1 sub-cycles, each containing $\frac{\Delta}{\Delta_1}$ values; and any value, belonging to one of these sub-cycles, will acquire approximately its original value, after describing $\frac{\Delta}{\Delta_1}$ elementary contours around the discriminantal point, the order of the error being $\frac{\gamma_1}{\Delta}$ if the order of the infinitesimal radius be taken as unity. And upon this approximate return to the original value depends the only indication which the method affords of the existence of sub-cycles, and of the values of the

numbers Δ_1 and γ_1. If we employ the multiple plane of Riemann, we may perhaps represent the relations of the Δ expansions to one another by taking a Δ_1-leaved plane, repeated $\frac{\Delta}{\Delta_1}$ times, and having a spire of order $\Delta - 1$, so arranged that after $\frac{\Delta}{\Delta_1}$ revolutions we return to the same Δ_1-leaved plane upon which we were when we set out, but not to the same leaf of that plane. And we can give to this image a certain amount of clearness by supposing that the Δ_1 leaves of any Δ_1-leaved plane are infinitely nearer to one another than are any two of the $\frac{\Delta}{\Delta_1}$ repetitions of the Δ_1-leaved plane.

(4.) The demonstrations of Plücker's formulae, which are usually given, apply only to the case in which the singularities are simple; the cases of multiple points, or multiple tangents, or of branches having contact with one another of any order, being made to depend, by the method of limits, on the simple cases of double points, or double tangents (see Dr. Salmon's 'Higher Plane Curves,' p. 53). But these demonstrations do not admit of immediate extension to the case of the higher singularities properly so called, because it has not as yet been established, in any general manner, that a higher singularity may be regarded as the limit of an equivalent number of lower singularities situated infinitely near to one another. It would seem that Plücker himself was well aware of the incompleteness (in this respect) of the demonstration of his equations; for he supplements that demonstration by separately considering the case of a common cusp of the second species. Assuming the equation $n = m(m-1) - D$, and its reciprocal, (about the rigorous proof of which there is no doubt,) we have only to establish one other equation of the system. Two different methods are given by Plücker ('Theorie der Algebraischen Curven,' Part ii., Arts. 77–81): (i.) He establishes directly the theorem that, at a cusp of the second species, the curve

$$\frac{d^2F}{dp^2}\left(\frac{dF}{dq}\right)^2 - 2\frac{d^2F}{dp\,dq}\frac{dF}{dp}\frac{F}{dq} + \frac{d^2F}{dq^2}\left(\frac{dF}{dp}\right)^2 = 0$$

(which may be used for our present purpose instead of the Hessian) intersects the given curve in $3d + \iota - \kappa = 15$ points. We have already stated that, in all the cases which have been examined hitherto, the number of intersections of the Hessian with the curve at any point has been found to be $3d + \iota - \kappa$; but no general demonstration of this theorem has as yet been given. The only method at present known for determining the number of intersections of two curves at a point which is singular on each of them, consists in obtaining the developments

of the various branches of the two curves at the point, and in comparing these developments with one another. The discussion in Art. 18 of the development of the polar curve in the vicinity of a superlinear branch, may serve to show that the corresponding enquiry in the case of the Hessian is one of considerable intricacy. (ii.) The other method employed by Plücker depends on a determination of the number of double tangents lost by a curve of the fourth order in consequence of the presence of a cusp of the second species. In the absence of any demonstration that a higher singularity can be regarded as the limit of simple singularities existing infinitely near to one another, it is difficult to see how this mode of proof can be rendered universally applicable.

(5.) We have seen (Art. 10) that the theorem of the invariance of the number $\frac{1}{2}(m-1)(m-2)-\Sigma\delta-\Sigma\kappa$ in any unicursal transformation of the curve suffices to establish the equation

$$\tfrac{1}{2}(m-1)(m-2)-\delta-\kappa=\tfrac{1}{2}(n-1)(n-2)-\tau-\iota, \quad . \quad . \quad . \quad . \quad (A)$$

and thus to complete the proof of the formulae of Plücker. Among the demonstrations of this theorem which have been given in recent times that of MM. Bertini and Zeuthen ('Giornale di Mathematica,' Vol. VII., p. 105; 'Mathematische Annalen,' Vol. III., p. 150; Dr. Salmon's 'Higher Plane Curves,' p. 314) is remarkable for its simplicity; and appears, as we shall now attempt to show, to admit of extension to the case in which the curves have any singularities whatever. We begin by assuming that when a curve is subjected to an unicursal, or one-to-one transformation, the continuity of its branches is invariably preserved, even when the position of these branches with regard to one another has undergone great distortion. For example, if a curve have two branches intersecting at the point O, these two branches will certainly be represented by two corresponding branches in the transformed curve; but these two branches may have no point of intersection, and the point O may be represented by two different points one on each of the two branches. Again, two branches which osculate one another with any degree of approximation may be transformed into branches having no contact and no point in common. But a superlinear branch behaves as one branch, and always is transformed into one branch and one only. Consider, for example, a real branch which is superlinear at O, and suppose for simplicity that no other branch passes through O; whatever be the nature of the superlinearity, we have one continuous branch passing through O, and if a point describe this branch, the track of the image point in the transformed figure cannot be anything but one continuous branch.

Let C_1, C_2 be two curves of the orders m_1, m_2, and of the classes n_1, n_2 lying

in the same plane and corresponding to one another unicursally; and let P_1, P_2 be points upon them corresponding unicursally. Taking two arbitrary points S_1, S_2, we consider, with M. Zeuthen, the locus Γ of the intersection of the rays S_1P_1, S_2P_2; and we propose to determine the number of tangents that can be drawn to Γ from each of the two points S_1 and S_2. We may suppose that S_1S_2 cuts each of the two curves in points which do not have singular points of the other curve for their corresponding points; then it is evident that Γ will have m_2 ordinary branches passing through S_1, and m_1 ordinary branches passing through S_2. We may further suppose that the n_1 tangents drawn from S_1 to C_1 are none of them singular tangents, and that to the points of contact of these n_1 tangents there answer on C_2 points having no singularity: each of these tangents will then be a tangent of Γ, but not a singular tangent of that curve. Beside the $2m_2+n_1$ tangents, which we have now drawn from S_1 to Γ, there may be others, coinciding in direction with the rays running from S_1 to the singular points of C_1. Let X_1 be a superlinear point on C_1, having the cuspidal index κ_1; and to X_1 let X_2 answer on C_2, the cuspidal index of X_2 being κ_2, where $\kappa_2 \geqq 0$. We may suppose at first that only one branch passes through X_1 and only one through X_2. The ray S_1X_1 meets C_1 at X_1 in precisely κ_1+1 coincident points, because S_1X_1 is not a tangent at X_1; similarly S_2X_2 is not a tangent at X_2, but meets C_2 in precisely κ_2+1 points at X_2, since we may attribute to S_2 the requisite generality of position with regard to C_2. Thus, if Q is the intersection of S_1X_1, S_2X_2, the locus Γ is intersected at Q κ_1+1 times by S_1X_1, and κ_2+1 times by S_2X_2. The points of the curve Γ answer, one to one, to the points of C_1 or C_2; thus at Q there is but one branch answering to the one branch at X_1, or to the one branch at X_2. If $\kappa_1=\kappa_2$, the cuspidal index of this branch is $\kappa_1=\kappa_2$, while its inflexional index remains unknown. If $\kappa_1>\kappa_2$, its cuspidal index is κ_2, its inflexional index is $\kappa_1-\kappa_2-1$; similarly, if $\kappa_2>\kappa_1$, these indices are κ_1 and $\kappa_2-\kappa_1-1$; *i.e.*, in the first case, S_1X_1 counts $\kappa_1-\kappa_2$ times as a tangent to Γ at Q, and S_2X_2 is not a tangent at all; in the second case, S_2X_2 counts $\kappa_2-\kappa_1$ times as a tangent, and S_1X_1 is not a tangent at all. When $\kappa_1=\kappa_2$, neither S_1X_1 nor S_2X_2 are tangents. The preceding reasoning will not be affected, if we now introduce the supposition that several linear or superlinear branches intersect or osculate at X_1, and that branches corresponding to some or all of them pass through X_2. Several branches will now pass through Q, but each of them may be considered separately, and the number of times that it is touched by S_1Q or S_2Q may be ascertained as above. Equating the results appertaining to the points S_1 and S_2, we now obtain

$$2m_2+n_1+\Sigma'(\kappa_1-\kappa_2)=2m_1+n_2+\Sigma'(\kappa_2-\kappa_1);$$

where Σ' extends only to those differences which are positive. Written in the form

$$n_1+\Sigma\kappa_1-2m_1=n_2+\Sigma\kappa_2-2m_2,$$

this equation coincides with the formula

$$\tfrac{1}{2}(m_1-1)(m_1-2)-\Sigma(\delta_1+\kappa_1)=\tfrac{1}{2}(m_2-1)(m_2-2)-\Sigma(\delta_2+\kappa_2),$$

which it was required to prove.

The assumption, which we have explicitly made, that a linear or superlinear branch is always transformed by a one-to-one transformation into one branch, and one only, is indispensable in the preceding proof; as upon it depends the determination of the number of times that Γ is touched by S_1X_1 or S_2X_2. In the case of a real branch transformed by a real transformation, the assumption may be regarded as evident; in the general case, we should have to consider, instead of two plane curves, the two corresponding surfaces of Riemann. For our immediate purpose, however, we do not need to establish the assumption as universally true in all cases; because the only one-to-one transformation (beside that of C_1 or C_2 into Γ) which is here employed is the transformation by polar reciprocation; and the investigation of Art. 11 affords a direct proof that in this transformation any one linear or superlinear branch is always transformed into one branch (linear or superlinear).

(6.) Abandoning for a time the hypotheses of Art. 1, let us suppose that P is a singular point on the curve C, Q retaining its generality of position. And first let P be a point through which only one superlinear branch passes, having the indices $\kappa=\Delta-1$, $\iota=\Delta,-1$; let us also suppose that no singular tangent of C (other than the tangent at P) passes through P. The order of p in the equation $F(p,q)=0$ is now $m-\Delta$, instead of m; and the number of tangents that can be drawn from P to the curve C (other than the coincident tangents at P itself) is $n-\Delta-\Delta$, (see Art. 16), instead of n. To all the singular points of C, other than P, there will appertain developments of precisely the same form as in the case in which P has no speciality of position. Let q_0 be the value of q corresponding to the tangent at P; the parameters of the point P are $p=\infty$, $q=q_0$. We cannot therefore, in examining the superlinear branch at P, develope p in a series proceeding by powers of $q-q_0$; but we may so develope $\dfrac{1}{p}$, or any linear function of p, such as $\dfrac{c+dp}{a+bp}$, which assumes a finite value $p_0=\dfrac{d}{b}$, when $p=\infty$. The exponents in any such development will have Δ, instead of Δ, for their least com-

mon denominator, because the tangent to C at P meets the curve (Art. 16) in $\Delta+\Delta$, points, so that, if $q=q_0$, Δ, of the $m-\Delta$ values of $\frac{c+dp}{a+bp}$ become equal to p_0. Setting out from the given equation $F(p,q)=0$, let us form the developments appertaining to all the *singular* discriminantal values of q; and in each group of conjugate developments let us consider the greatest common divisor θ of its exponents. The sum $\Sigma(\theta-1)$ will be equal to $\Sigma\kappa+\Delta_1-\Delta$,, instead of $\Sigma\kappa$; and the three numbers, by which we have now replaced m, n, and $\Sigma\kappa$, will satisfy the equation

$$(n-\Delta-\Delta,)+(\Sigma\kappa+\Delta,-\Delta)-2(m-\Delta)=n+\Sigma\kappa-2m.$$

The cases in which (α) more than one branch passes through P, (β) one or more singular tangents pass through P, (γ) Q as well as P has some speciality of position with regard to C, may all be treated by the same method. In any of these cases, let $E(p)$ be the highest exponent of p in the equation $F(p,q)=0$; and let $\omega(p)=\Sigma(\theta-1)$, the sign of summation now extending to *all* the discriminantal values of q, so that $\Sigma(\theta-1)$ contains an unit for every ordinary tangent that can be drawn from P to touch the curve elsewhere. If any of the discriminantal values of q, or any of the corresponding equal values of p, are infinite, we are to employ linear functions of p and q themselves, in forming the developments from which we are to infer the numbers θ. We shall thus obtain the equation

$$\omega(p)-2E(p)=\omega(q)-2E(q)=n+\Sigma\kappa-2m,. \quad . \quad . \quad . \quad . \quad (B)$$

from which, as Clebsch has shown, the general theorem of the invariance of the deficiency may be immediately deduced. (See a Memoir by M. Nöther, 'Mathematische Annalen,' Vol. VIII., p. 497.)

In the memoir to which we have just referred, M. Nöther offers a demonstration of the equation (B). But this demonstration is perhaps not wholly free from obscurity. (See the words, p. 499, *loc. cit.*, 'Dieses findet . . . ergiebt,' with the accompanying reference to the Göttingen 'Nachrichten.') A similar remark applies to a second demonstration, in the same memoir, of the invariance of the deficiency. [See p. 501, 'Man hat aber dann . . . das Glied $\Sigma i_1(i_1-1)$.']

M. Nöther has returned to the same subject, in a recent memoir of great interest ('Mathematische Annalen,' Vol. IX., p. 166), in which he considers the resolution of a higher singularity by successive applications of a simple quadratic transformation, and infers (though by a method which can hardly be accepted as rigorous) that any higher singularity may be regarded as the limit of a certain number of lower singularities situated infinitely near to one another. We may

observe (α) that the use of a quadratic transformation for the resolution of complicated singularities is due to Cramer ('Analyse des Lignes Courbes'); (β) that to establish the complete system of the formulae of Plücker, M. Nöther selects the same three equations, which we have been led to employ in the present paper [viz., the equations (i.), (ii.), and (iii.) = (iv.), of Art. 10].

18. The expansions of Arts. 3 and 4 enable us to examine the relation of a curve at a singular point to its polar curves. Putting for brevity $p-p_0=\eta$, $q-q_0=\xi$, $F(p, q)=F_1(\eta, \xi)$, we have $F_1(\eta, \xi)=\Pi(\eta-\overline{A})\times\Pi(\eta-\overline{B})$, $\dfrac{dF}{dp}=\dfrac{dF_1}{d\eta}$. From the expression of $F_1(\eta, \xi)$ as a product of m factor-series, we infer that if, on writing $K_1\xi$ for η in $F_1(\eta, \xi)$, we obtain a result of which the order of evanescence with ξ is higher than μ, $\eta=K_1\xi+\ldots$ is the beginning of one at least of the expansions $\overline{B}$. Again, let us substitute for η in $F_1(\eta, \xi)$ an expression of the form

$$K=K_1\xi+K_2\xi^{a_2}+K_3\xi^{a_3}+\ldots+K_\nu\xi^{a_\nu},$$

in which $1<a_2<a_3<\ldots<a_\nu$. If the order of evanescence of $F_1(K, \xi)$ with ξ experiences an abrupt diminution when either a_ν or K_ν (the exponent and coefficient of the last term of K) is affected by any small variation, the terms K are the initial terms of one at least of the expansions $\overline{B}$. This observation (which admits of some useful applications) enables us to deduce the developments appertaining to the polar curve $\dfrac{dF}{dp}$, in the vicinity of the point (p_0, q_0), from the developments appertaining to C.

Let k of the developments $\overline{B}$ coincide with one another and with K, as far as the term $K_\nu\xi^{a_\nu}$ inclusively, so that for any one of these k developments we have

$$\overline{B}_i=K+L_i\xi^{l_i}, \quad l_i>a_\nu, \quad \ldots\ldots\ldots \quad (K)$$
$$L_i=\lambda_i+\lambda_i'\xi^{l_i}+\ldots,$$

the terms $\lambda_i\xi^{l_i}$ not being all identical.

Put $$V=\eta-K, \quad \Pi_1^k(V-L_i\xi^{l_i})=\phi(V);$$

then $$F_1(\eta, \xi)=M\times\phi(V), \quad \text{and} \quad \frac{dF_1}{d\eta}=\frac{dM}{d\eta}\phi(V)+M\phi'(V),$$

M being a product of $m-k$ factors, viz., of the $m-\mu$ factors $\eta-\overline{A}$, and of those $m-k$ factors $\eta-\overline{B}$ which do not coincide with $\eta-K$ as far as the term $K_\nu\xi^{a_\nu}$ inclusively. Suppose, at first, that $l_1, l_2, \ldots, l_k$ are all unequal, and arranged in order of magnitude; it is easily ascertained that the first terms in the expansions

of the roots of $\phi'(V)=0$ are

$$V_1=\frac{k-1}{k}\lambda_1\xi^{l_1},\quad V_2=\frac{k-2}{k-1}\lambda_2\xi^{l_2},\quad V_3=\frac{k-3}{k-2}\lambda_3\xi^{l_3},\ \ldots,\ V_{k-1}=\tfrac{1}{2}\lambda_{k-1}\xi^{l_{k-1}}.$$

Substitute for η in $\dfrac{dF_1}{d\eta}$ an expression of the form $\eta=K+H\xi^h$, where $h>a_\nu$, and H is independent of ξ. If $H\xi^h$ is not the same as any one of the quantities V_1, V_2, ..., V_{k-1}, the order of evanescence of $\phi(V)\dfrac{dM}{d\eta}$ surpasses that of $M\phi'(V)$; for the order of evanescence of M cannot surpass that of $\dfrac{dM}{d\eta}$ by a number greater than a_ν, whereas the order of $\phi(V)$, on the supposition that none of the equations $H\xi^h=V_i$ is satisfied, surpasses the order of $\phi'(V)$, at least by one of the numbers $l_1, l_2, \ldots, l_k$. If we now suppose H and h to vary continuously, the order of evanescence of $\phi'(V)$ is abruptly increased when $H\xi^h$ comes to coincide with any one of the roots $V_0, V_1, \ldots, V_{k-1}$; and, since the order of evanescence of M remains unchanged, that of $\dfrac{dF}{d\eta}$ is also increased abruptly. Hence $k-1$ of the developments appertaining to $\dfrac{dF}{d\eta}$ are of the type

$$\eta=K+\frac{k-1}{k}\lambda_1\xi^{l_1}+\ldots,\quad \eta=K+\frac{k-2}{k-1}\lambda_2\xi^{l_2}+\ldots,\quad \eta=K+\tfrac{1}{2}\lambda_{k-1}\xi^{l_{k-1}}+\ldots.$$

Again, suppose that s of the indices l are equal; let, for example, the s lowest indices be equal; then s roots of the equation $\phi'(V)=0$ are of the form $H_i\xi^l+\ldots$, where $l=l_1=l_2=\ldots=l_s$; and if

$$\psi(\theta)=(\theta-\lambda_1)(\theta-\lambda_2)\ldots(\theta-\lambda_s),$$

the s coefficients H_i are the roots of the equation

$$(k-s)\,\psi(\theta)+\theta\psi'(\theta)=0. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (\theta)$$

If the s equal indices $l_1 \ldots l_2$ are followed by another set of s' indices equal to one another and to l', l' being $>l$, put

$$(\theta-\lambda_{s+1})(\theta-\lambda_{s+2})\ldots(\theta-\lambda_{s+s'})=\psi_1(\theta);$$

then the equation $\phi'(V)=0$ has s' roots of the form $H_i'\xi^{l'}+\ldots$, the coefficients H_i' being the roots of the equation

$$(k-s-s')\,\psi_1(\theta)+\theta\psi_1'(\theta)=0, \quad . \ . \ . \ . \ . \ . \ . \ . \quad (\theta')$$

and so on continually. Lastly, considering any group of equal indices l, for

example the group $l_{s+1}, l_{s+2}, \ldots, l_{s+s'}$, let σ of the corresponding coefficients λ be supposed equal (in which case σ of the developments K coincide with one another for one term at least after $K_\nu \xi^{a_\nu}$); the corresponding equation (θ') will have $\sigma - 1$ roots (and no more) equal to one another and to the equal coefficients λ; so that $\sigma - 1$ of the developments appertaining to the polar will coincide, as far as the term next after $K_\nu \xi^{a_\nu}$, with the σ developments appertaining to C. To carry on these $\sigma - 1$ developments until their complete separation from one another, we must repeat the preceding process as often as may be necessary, using in the first instance $K + \lambda \xi^l$ instead of K, and confining our attention to the σ developments, appertaining to C, in which $K + \lambda \xi^l$ are the initial terms.

As the roots of the equations $\psi(\theta) = 0$, $\psi_1(\theta) = 0$, ... are all different from zero, so also are the roots of the equations (θ), (θ'), ..., except when the highest index l is one of a group of equal indices. In this case, if $\psi(\theta) = \Pi(\theta - \lambda)$, the sign of multiplication extending only to those coefficients λ_i which occur in terms having the greatest exponent l, the last of the equations (θ) is of the form $\psi'(\theta) = 0$, and r of its roots may be equal to zero. When this happens, in the r polar developments corresponding to the zero roots, the terms K are not followed by a term of the form $H\xi^l$, but by a term of higher exponent. To determine this term in each of the r developments, we must use, in forming $\psi(\theta)$, not simply the quantities λ_i, but as many terms of the series $\lambda_i + \lambda_i' \xi^{l_i} + \ldots$ as may be necessary. The zero roots of $\psi'(\theta) = 0$ are then replaced by roots of the form $H\xi^a$, a being positive, and the initial terms of the r polar developments are given by the formula $K + H\xi^{l+a}$.

We shall employ the preceding method to examine the nature of the polar branches in the vicinity of a superlinear branch. We suppose the superlinear branch to be of the type

$$[\Delta, \Delta_1, \Delta_2, \ldots, \Delta_s, \Delta_{s+1} = 1;\ \gamma_1, \gamma_2, \ldots, \gamma_s];$$

and we consider only the case in which this superlinear branch (Δ) is not touched by any other branch. The polar has $\Delta - 1$ branches (Δ') touching the superlinear branch. Their developments coincide with one another, and with those of (Δ), as far as the term $[x^{\frac{\gamma}{\Delta}}]$ exclusively. But at this term $\frac{\Delta_1}{\Delta} - 1$ of them cease to osculate any branch of (Δ); they do not contain the term $[x^{\frac{\gamma}{\Delta}}]$, which is replaced in each of them by a term of higher exponent, yet so that the aggregate of the $\frac{\Delta}{\Delta_1} - 1$ exponents cannot exceed $\frac{\gamma}{\Delta_1} - 1$. The remaining $\frac{\Delta}{\Delta_1}(\Delta_1 - 1)$ branches

divide themselves into $\frac{\Delta}{\Delta_1}$ groups of $\Delta_1 - 1$ each. The $\Delta_1 - 1$ branches of each group are identical with one another, and with Δ_2 of the branches (Δ), as far as the term $[x^{\frac{\gamma_1}{\Delta}}]$ exclusively. At this term $\frac{\Delta_1}{\Delta_2} - 1$ branches out of each group cease to osculate any branch of (Δ), and the remaining $\frac{\Delta_1}{\Delta_2}(\Delta_2 - 1)$ divide themselves, in the same way as before, into $\frac{\Delta_1}{\Delta_2}$ groups of $\Delta_2 - 1$ each; the branches of each group being identical with one another, and with Δ_2 of the branches of (Δ), as far as the term $(x^{\frac{\gamma_2}{\Delta}})$ exclusively. In this way we obtain the following theorem in which i is to have every value from 0 to s, both inclusively.

'The polar curve of an arbitrary point has $\frac{\Delta}{\Delta_{i+1}} - \frac{\Delta}{\Delta_i}$ branches which form $\frac{\Delta_i}{\Delta_{i+1}} - 1$ superlinear branches of the type

$$\left[\Delta' = \frac{\Delta}{\Delta_i},\ \Delta'_1 = \frac{\Delta_1}{\Delta_i},\ \ldots,\ \Delta'_{i-1} = \frac{\Delta_{i-1}}{\Delta_i},\ \Delta'_i = 1;\ \gamma' = \frac{\gamma}{\Delta_i},\ \gamma'_1 = \frac{\gamma_1}{\Delta_i},\ \ldots,\ \gamma'_{i-1} = \frac{\gamma_{i-1}}{\Delta_i}\right].$$

These superlinear branches coincide with one another, and with the branches of (Δ) as far as the term $[x^{\frac{\gamma_i}{\Delta}}]$ exclusively; instead of the term $[x^{\frac{\gamma_i}{\Delta}}]$ each of them contains a term of higher exponent; the $\frac{\Delta_i}{\Delta_{i+1}} - 1$ superlinear branches may, but do not necessarily, group themselves into higher superlinear branches.'

19. The development appertaining to a superlinear branch can always be obtained from the equation of the curve by successive applications of the 'analytical triangle.' The process has been described by M. Puiseux in his important memoir 'Recherches sur les fonctions algébriques.' ('Liouville,' Vol. XV., p. 384; see also a paper by M. de la Gournerie, *ibid.*, 2nd series, Vol. XIV., p. 425, Vol. XV., p. 1.) We propose to conclude the present paper by showing how the numbers $\gamma, \gamma_1, \ldots, \Delta, \Delta_1, \ldots$ present themselves in the course of the operation. Putting, as in Art. 18, η for $p - p_0$, ξ for $q - q_0$, we first of all write the equation $F_1(\eta, \xi) = 0$ in the form $u_\mu + u_{\mu+1} + \ldots$, where u_μ is a homogeneous function of ξ and η of the order μ, which is that of the singular point. If $(\eta - B_0\xi)^\alpha$ is a multiple factor of u_μ the line $\eta - B_0\xi$ is touched by branches (linear or superlinear) of which the aggregate order is α. Put $\eta - B_0\xi = v$; the resulting equation between v and ξ will give precisely α values of v in which the order of v surpasses that of ξ. Form, by the analytical triangle, the equations (of the

aggregate order α in v) which give the initial terms of the expansions of these α values. These equations are of the type

$$(v^\nu - K\xi^\lambda)^{\alpha_1} = 0,$$

where λ and ν are relatively prime, $\lambda > \nu$, and $\Sigma \alpha_1 \nu = \alpha$; they are always obtained linearly, except when there are s of them in which the numbers α_1, λ, ν are all the same; in which case the analytical triangle determines an equation, of order s, having constant coefficients, of which the roots are the s quantities K. There are four cases to be considered: (i.) $\alpha_1 = 1$, $\nu = 1$; (ii.) $\alpha_1 = 1$, $\nu > 1$; (iii.) $\alpha_1 > 1$, $\nu = 1$; (iv.) $\alpha_1 > 1$, $\nu > 1$. (i.) To the equation $v - K\xi^\lambda = 0$, $\lambda > 1$, answers a linear branch which, considered by itself, has no point-singularity (if λ is > 2, it is an inflexion). (ii.) To the equation $v^\nu - K\xi^\lambda = 0$ answers a superlinear branch of which the character is defined by the equations $\Delta = \nu$, $\Delta_1 = 1$, $\gamma = \lambda$; its development proceeds by integral powers of $\xi^{\frac{1}{\nu}}$, and the successive terms are obtained linearly by the analytical triangle. (iii.) To the equation $(v - K\xi^\lambda)^{\alpha_1} = 0$ answer α_1 branches, which may be all linear, but which also may group themselves in whole or in part into superlinear branches; if Δ, Δ', Δ'', ... are the orders of these separate linear or superlinear branches, we have $\Sigma\Delta = \alpha_1$. (iv.) To the equation $(v^\nu - K\xi^\lambda)^{\alpha_1} = 0$ answer $\alpha_1 \nu$ branches, which may belong to α_1 distinct superlinear branches of the type $(\Delta = \nu, \Delta_1 = 1, \gamma = \lambda)$; these superlinear branches may however themselves be grouped, wholly or in part, into branches of higher superlinearity; if Δ, Δ', Δ'', ... are the orders of the distinct superlinear branches, these numbers are all divisible by ν, and $\Sigma \frac{\Delta}{\nu} = \alpha_1$; we have also for every one of them $\frac{\Delta}{\Delta_1} = \nu$, $\frac{\gamma}{\Delta_1} = \lambda$, Δ_1 having to be determined subsequently for each of them separately. With the cases (i.) and (ii.) we have nothing further to do; the case (iii.) may be regarded as included under (iv.); we therefore continue the process in this last case only. Put $v - K^{\frac{1}{\nu}}\xi^{\frac{\lambda}{\nu}} = v_1$, $\xi^{\frac{1}{\nu}}$ representing any one determinate value of the radical; and form by the analytical triangle equations of the type

$$(v_1^{\nu_1} - K_1\xi^{\frac{\lambda_1}{\nu}})^{\alpha_2} = 0,$$

of which the aggregate order in v_1 is α_1, and which give the initial terms of those α_1 values of v_1, of which the order surpasses that of $\xi^{\frac{\lambda}{\nu}}$; we have of course $\frac{\lambda_1}{\nu\nu_1} > \frac{\lambda}{\nu}$, or $\frac{\lambda_1}{\nu_1} > \lambda$; λ_1 and ν_1 are relatively prime, but we observe that λ_1 is not

necessarily prime to ν. We consider the same four cases as before. (i.) To the equation $v_1 - K_1 \xi^{\frac{\lambda_1}{\nu}} = 0$, or more properly to the ν equations comprehended in it, answers a superlinear branch of the type $(\Delta = \nu, \Delta_1 = 1, \gamma = \nu)$. (ii.) To the ν equations $v_1^{\nu_1} - K_1 \xi^{\frac{\lambda_1}{\nu}} = 0$ there also answers a single superlinear branch for which $\Delta = \nu\nu_1$, $\Delta_1 = \nu_1$, $\Delta_2 = 1$; $\gamma = \lambda\nu_1$, $\gamma_1 = \lambda_1$; *i.e.*, a superlinear branch of the type

$$\left(\frac{\Delta}{\Delta_1} = \nu, \ \frac{\Delta_1}{\Delta_2} = \nu_1, \ \Delta_2 = 1; \ \gamma = \lambda\Delta_1, \ \gamma_1 = \lambda_1\Delta_2\right).$$

In this case, as well as in (i.), the discussion of the superlinearity is complete. (iii.) To the ν equations $(v_1 - K_1 \xi^{\frac{\lambda}{\nu}})^{a_2} = 0$ there may answer α_2 superlinear branches of the type $\left(\frac{\Delta}{\Delta_1} = \nu, \ \Delta_1 = 1; \ \gamma = \lambda\Delta_1\right)$; or these may group themselves in any manner into higher superlinear branches for each of which $\frac{\Delta}{\Delta_1} = \nu$, $\gamma = \lambda\Delta_1$; the numbers Δ_1 (which have to be determined for each branch separately), satisfying the condition $\Sigma\Delta_1 = \alpha_2$. (iv.) To the equations $(v_1^{\nu_1} - K_1 \xi^{\frac{\lambda_1}{\nu}})^{a_2}$ answer a certain number of superlinear branches, for each of which

$$\frac{\Delta}{\Delta_1} = \nu, \ \frac{\Delta_1}{\Delta_2} = \nu_1; \ \gamma = \lambda\Delta_1, \ \gamma_1 = \lambda_1\Delta_2;$$

while Δ_2 and the subsequent numbers of the series have still to be determined, and may be different for each of them; we have however the equation

$$\Sigma\Delta_2 = \Sigma\frac{\Delta_1}{\nu_1} = \alpha_2.$$

The process, which we need not follow further, may be considered to terminate for any particular development, when that development is separated from every other, and can be continued linearly. This will happen when, in the series $\alpha\alpha_1\alpha_2\ldots$, we arrive at a term equal to unity. And we shall eventually arrive at such a term; for, though the second of two consecutive indices α may be as great as the first (the equation $(v - K\xi^{\frac{\lambda}{\nu}})^{a_1}$ may, for example, at the next step in the process, lead to only one equation; and this may be of the type

$$(v_1 - K_1 \xi^{\frac{\lambda_1}{\nu}})^{a_1} = 0,$$

so that we should have $\alpha_2 = \alpha_1$), yet it is impossible for two branches to osculate one another indefinitely, because the discriminantal index is necessarily finite.

If a_s be the first of the indices a which is equal to unity, we have

$$\frac{\Delta}{\Delta_1}=\nu,\quad \frac{\Delta_1}{\Delta_2}=\nu_1,\ \ldots,\ \frac{\Delta_s}{\Delta_{s+1}}=\nu_s,\quad \Delta_{s+1}=1;$$

and the development appertaining to the superlinear branch is of the type

$$(\Delta=\nu\nu_1\nu_2\ldots\nu_s,\quad \Delta_1=\nu_1\nu_2\ldots\nu_s,\ \ldots\ldots,\ \Delta_s=\nu_s,\quad \Delta_{s+1}=1,$$
$$\gamma=\lambda\Delta_1,\quad \gamma_1=\lambda_1\Delta_2,\ \ldots\ldots,\ \gamma_{s-1}=\lambda_{s-1}\Delta_s,\quad \gamma_s=\lambda_s).$$

XXVII.

MATHEMATICAL NOTES.

[Proceedings of the London Mathematical Society, vol. vii. pp. 237–238. Read December 9, 1875. First printed in the Messenger of Mathematics, vol. v. pp. 143–144 (January 1876)].

(i.) *On a Problem of Eisenstein's.* If p is an uneven prime, the function $4\frac{x^p-1}{x-1}=Z$ can always be expressed in the form $Y^2-(-)^{\frac{1}{2}(p-1)}pX^2$, where X and Y are rational and integral functions of x having integral coefficients. This is a theorem of Gauss. Eisenstein's problem ('Crelle's Journal,' vol. xxvii., p. 83) is 'To determine the cases in which the equation $Z=Y^2-(-)^{\frac{1}{2}(p-1)}pX^2$ admits of a multiplicity of solutions, and to ascertain the law connecting the various solutions, when there is more than one.' The solution of this problem is as follows: If $[T, U]$ is *any* solution whatever in integral numbers of the equation $T^2-(-)^{\frac{1}{2}(p-1)}pU^2=4$, and $[X, Y]$ is any one given solution of Gauss' equation, then all the solutions of Gauss' equation are comprised in the formula

$$[\tfrac{1}{2}\{TX+(-)^{\frac{1}{2}(p-1)}pUV\},\ \ \tfrac{1}{2}(UX+TY)].$$

Thus, if $p=4n+3$, the equation admits of but one solution (the four solutions $[\pm X, \pm Y]$ being regarded as but one) except in the case $p=3$, when it admits of three; if $p=4n+1$, the equation admits of an infinite number of solutions. That the functions $[\frac{1}{2}(TX+pUY), \frac{1}{2}(UX+TY)]$ are all of them solutions of Gauss's equation, is evident; the proof that this formula comprises all the solutions of the equation is less elementary, because it depends on the irreducibility of the function Z. There exists a general theory of the representation of rational and integral functions of x by quadratic forms; such representation being, of course, only possible when the given function of x is capable of resolution into two factors by the adjunction of a quadratic surd.

(ii.) *On the Joint Invariants of Two Conics or Two Quadrics.* Let P and Q be two conics, and let 123 be any triangle self-conjugate with regard to P. Let also P_1, P_2, P_3 be the rectangles of the points 1, 2, 3 with regard to the conic P, these rectangles being taken upon transversals measured in any fixed direction; and let Q_1, Q_2, Q_3 have similar meanings with regard to the conic Q, the direction of the transversals being also fixed. Then the expression $\frac{Q_1}{P_1} + \frac{Q_2}{P_2} + \frac{Q_3}{P_3}$ has the same value for all self-conjugate triangles of P, and is, in fact, that invariant of P, Q which is linear with regard to Q and quadratic with regard to P, and the evanescence of which expresses that Q harmonically circumscribes P. The corresponding theorem in the geometry of the straight line is 'If $Q_1 Q_2$, $P_1 P_2$ are two pairs of fixed points on a line, and if $A_1 A_2$ is any pair of harmonic conjugates of $P_1 P_2$, the value of the expression $\frac{A_1 Q_1 . A_1 Q_2}{A_1 P_1 . A_1 P_2} + \frac{A_2 Q_1 . A_2 Q_2}{A_2 P_1 . A_2 P_2}$ is independent of the particular pair $A_1 A_2$ considered.' From this theorem the result given above for two conics follows immediately; from it the corresponding property for two quadrics may be inferred, viz. $\frac{Q_1}{P_1} + \frac{Q_2}{P_2} + \frac{Q_3}{P_3} + \frac{Q_4}{P_4} = \text{constant}$; and so on for quadratic functions containing any number of indeterminates.

(iii.) *On the Equation $P \times D = \text{constant}$, of the Geodesic Lines of an Ellipsoid.* From this equation (in which P is the perpendicular from the centre upon the tangent plane at any point of the geodesic, and D is the semi-diameter parallel to the tangent line of the geodesic), it is convenient to be able to infer directly the principal properties of the geodesic line, without having first to transform the equation into M. Liouville's form $\mu^2 \cos^2 i + \nu^2 \sin^2 i = a^2$. In Dr. Salmon's 'Geometry of Three Dimensions,' the theorem of the constancy of the sum or difference of the geodesic radii vectores, drawn from any point of a line of curvature to two umbilics, is thus demonstrated. And it is worth while to add (though it is very improbable that the point has not been noticed before), that a proof of the theorem, that two geodesic tangents of a line of curvature, which intersect at right angles, intersect on a sphero-conic, may similarly be obtained without transforming the equation. Let Q be the point where the two geodesic tangents intersect at right angles, O the centre of the ellipsoid; let $c = OQ$, and let a, b be the semi-axes of the central section parallel to the tangent plane at Q. The two geodesics make angles of 45° with the lines of curvature at Q; hence, for either of these geodesic lines, $D^2 = \frac{2a^2 b^2}{a^2 + b^2}$. Let Q' be a second point

where two geodesic tangents to the same line of curvature intersect at right angles; then $\frac{2P^2a^2b^2}{a^2+b^2} = \frac{2P'^2a'^2b'^2}{a'^2+b'^2}$, because $P \times D$ has the same value for all geodesic lines touching the same line of curvature. But $P^2a^2b^2 = P'^2a'^2b'^2$ because parallelepipeds circumscribing an ellipsoid with their faces parallel to conjugate diametral planes are equal. Hence $a^2+b^2 = a'^2+b'^2$. But also

$$a^2+b^2+c^2 = a'^2+b'^2+c'^2;$$

therefore $c = c'$ and Q and Q' lie on the same sphero-conic.

XXVIII.

NOTE ON CONTINUED FRACTIONS.*

[Messenger of Mathematics, Ser. II. vol. vi. pp. 1–14 (May 1876)].

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1. LET $\frac{p}{q} = \mu_1 + \frac{1}{\mu_2 +} \frac{1}{\mu_3 +} \dots \frac{1}{\mu_s}$, p and q being two numbers relatively prime, of which p is the greater. Writing, for convenience, $P = \frac{1}{p}$ and $Q = \frac{1}{q}$, we divide a line 01 of unit length (measured from left to right) into p equal parts at the points $1P$, $2P$, $3P$, ..., $(p-1)P$; and also into q equal parts at the points $1Q$, $2Q$, $3Q$, ..., $(q-1)Q$. We do not reckon 0 either as a point P or as a point Q, but we reckon 1 both as a point P and as a point Q, so that we have in all p points P, and q points Q, of which none are coincident, excepting the two extreme points, which coincide at 1.

2. It is the purpose of this note to show that the arrangement of the points P and Q upon the line 01, or, which is the same thing, the arrangement in order of magnitude of the proper fractions $\frac{x}{p}$ and $\frac{y}{q}$, may be inferred from the development of $\frac{p}{q}$ in a continued fraction; and that, *vice versâ*, the development of $\frac{p}{q}$ may be inferred from an inspection of the arrangement of the points. An example will serve to explain the nature of the relation which we have to establish.

3. Let $p = 39$, $q = 17$, so that we have the development $\frac{39}{17} = 2 + \frac{1}{3 +} \frac{1}{2 +} \frac{1}{2}$; the arrangement of the points P and Q is indicated in the following scheme, in

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* The substance of this note was commnnicated to the Mathematical Section of the British Association, at the Bristol meeting in 1875.

which transverse lines are placed at the close of each of the *sequences** to be presently defined.

$$P, 2P, Q \mid 3P, 4P, 2Q \mid 5P, 6P, 3Q \mid 7P, \| 8P, 9P, 4Q \mid,$$
$$10P, 11P, 5Q \mid 12P, 13P, 6Q \mid 14P \| 15P, 16P, 7Q \|\|,$$
$$17P, 18P, 8Q \mid 19P, 20P, 9Q, \mid 21P, 22P, 10Q \mid 23P \|$$
$$24P, 25P, 11Q \mid 26P, 27P, 12Q \mid 28P, 29P, 13Q \mid 30P, \|\|$$
$$31P, 32P, 14Q \|\| 33P, 34P, 15Q \mid 35P, 36P, 16Q \mid$$
$$37P, 38P, 17Q \mid 39P \|\|.$$

In this scheme, because $\mu_1 = 2$, we have two points P before we come to a point Q; the sequence PPQ, which consists of μ_1 points P followed by a point Q, we term a sequence of order 1; this sequence is repeated three times, because $\mu_2 = 3$, and is then followed by a single point P (which is a sequence of order zero); a sequence, such as $PPQ \mid PPQ \mid PPQ \mid P$, consisting of μ_2 sequences of order 1, followed by a sequence of order zero, we term a sequence of order 2; it contains $\mu_1\mu_2 + 1$ points P, μ_2 points Q; this sequence of order 3 is, in the scheme before us, repeated twice, because $\mu_3 = 2$, and is then followed by a sequence of order 1; the sequence thus obtained, consisting of μ_3 sequences of order 2, followed by a sequence of order 1, we term a sequence of order 3. This sequence, containing $\mu_1\mu_2\mu_3 + \mu_1 + \mu_3 = 16$ points P, and $\mu_2\mu_3 + 1 = 7$ points Q, is in the

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* These sequences have been already noticed by M. Christoffel, in an interesting paper entitled 'Observatio Arithmetica,' (*Annali di Mathematica*, 2nd series, vol. vi., p. 148), with which I unfortunately did not become acquainted until my own investigation was completed. M. Christoffel considers the least positive remainders of the series of numbers $q, 2q, 3q, \ldots$ for the modulus p, and designates any remainder by the symbol c or d, according as it is less or greater than the remainder immediately following. It is easily seen that the sequences of the symbols c and d coincide with the sequences of the points P and Q. For if the remainder of sq is greater than the remainder of $(s+1)q$, we shall have, for some integral value of h, the inequalities

$$(h-1)p < sq < hp < (s+1)q < (h+1)p,$$

whence

$$\frac{s}{p} < \frac{h}{q} < \frac{s+1}{p},$$

or the point hQ lies between the points sP and $(s+1)P$. And, again, if the remainder of sq is less than the remainder of $(s+1)q$, we have the inequalities

$$(h-1)p < sq < hp, \quad (h-1)p < (s+1)q < hp,$$

which give immediately

$$\frac{h-1}{q} < \frac{s}{p}, \quad \frac{h}{q} > \frac{s+1}{p},$$

proving that no point Q can lie between the points sP and $(s+1)P$.

scheme repeated twice, because $\mu_4 = 2$, and is followed by a sequence of order 2. We thus obtain a sequence of order 4, consisting of μ_4 sequences of order 3, followed by a sequence of order 2, and containing

$$\mu_1\mu_2\mu_3\mu_4 + \mu_1\mu_2 + \mu_1\mu_4 + \mu_3\mu_4 + 1 = 39 \text{ points } P,$$

and

$$\mu_2\mu_3\mu_4 + \mu_2 + \mu_4 = 17 \text{ points } Q.$$

This sequence, in the instance which we are considering, exhausts the whole system of points. We observe that all sequences begin with P, and that sequences of an uneven order end with PQ, sequences of an even order with QP.

4. In general, when the continued fraction is given, and it is required to obtain the arrangement of the points P and Q, we denote a sequence of order i by S_i, and we then find successively $S_1 = P^{\mu_1} Q$, $S_2 = S_1^{\mu_2} P$, $S_3 = S_2^{\mu_3} S_1$, ..., the final sequence (which exhausts the whole series of points) being $S_s = S_{s-1}^{\mu_s} S_{s-2}$.

Vice versâ, when the arrangement of the points is given, and it is required to infer from it the development in a continued fraction, we count the points P till we come to the first point Q; if there are μ_1 of them, μ_1 is the first quotient, and $S_1 = P^{\mu_1} Q$. If we can repeat this sequence μ_2 times, without departing from the given arrangement, the second quotient is μ_2, and the sequence of order 2 is $S_2 = S_1^{\mu_2} P$. This sequence we now repeat as often as we can do so without departing from the given arrangement, observing, however, that the last repetition of S_2 is to be followed by a sequence S_1. If, subject to this condition, we can repeat S_2 μ_3 times, the third quotient is μ_3, and the sequence of order 3 is $S_2^{\mu_3} S_1$. The subsequent quotients and sequences are to be determined in the same manner; and, if $\frac{p_i}{q_i}$ is the convergent $\mu_i + \cfrac{1}{\mu_2 + \cfrac{}{\ddots\ \mu_i}}$, p_i and q_i are respectively the numbers of points P and points Q in the sequence S_i.

5. Since $\mu_1 < \frac{p}{q} < \mu_1 + 1$, or $\mu_1 P < Q < (\mu_1 + 1) P$, it is evident that the arrangement of the first $\mu_1 + 1$ points of the series is represented correctly by the sequence $S_1 = P^{\mu_1} Q$. We therefore proceed to show that the arrangement of the first $\mu_1\mu_2 + 1$ points P, and the first μ_2 points Q is correctly represented by the sequence $S_2 = S_1^{\mu_2} P$. Since

$$\mu_1 + \frac{1}{\mu_2} > \frac{p}{q} > \mu_1 + \frac{1}{\mu_2 + 1},$$

we have $(\mu_1 k + 1)\ P > kQ$, for all values of $k \leqq \mu_2$, but $(\mu_1 k + 1) P < hQ$, if $k > \mu_2$.

If we write down the sequence $P^{\mu_1} Q$, $1+\mu_2$ times over, so as to obtain the series

$$\begin{array}{cccc}
1P,\ 2P, & \ldots & \mu_1 P,\ Q, \\
(1+\mu_1)P,\ (2+\mu_1)P, & \ldots & 2\mu_1 P,\ 2Q, \\
(1+2\mu_1)P,\ (2+2\mu_1)P, & \ldots & 3\mu_1 P,\ 3Q, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\
(1+[\mu_2-1]\mu_1)P,\ (2+[\mu_2-1]\mu_1)P, & \ldots & \mu_2\mu_1 P,\ \mu_2 Q, \\
(1+\mu_2\mu_1)P,\ (2+\mu_2\mu_1)P, & \ldots & (1+\mu_2)\mu_1 P,\ (1+\mu_2)Q, \\
(1+[\mu_2+1]\mu_1)P, & \ldots & \ldots\ldots\ldots.,
\end{array}$$

the inequalities $k\mu_1 P < kQ < (k\mu_1+1)P$, which hold as long as $k \leqq \mu_2$, show that all these points, with the exception of the last of them $(1+\mu_2)Q$, succeed one another in the proper order. But the last is in error, for, putting $k = 1+\mu_2$, $(1+\mu_1+\mu_1\mu_2)P < (1+\mu_2)Q$, and consequently $(1+\mu_2)Q$ does not follow immediately after $(1+\mu_2)\mu_1 P$. We conclude, therefore, that we can repeat the sequence $P^{\mu_1} Q$ μ_2 times, but that we cannot repeat it $1+\mu_2$ times. And, since two points Q cannot come together, the series $(P^{\mu_1} Q)^{\mu_2}$ is necessarily followed by a point P, so that the sequence $S_2 = S_1^{\mu_2} P$ correctly represents, as far as it goes, the arrangement of the points.

6. We have thus shown that the relation between the continued fraction and the sequences S_1, S_2, S_3, ... holds as far as S_2. Assuming, therefore, that it holds as far as S_i, where $i \geqq 2$, we have to prove that it holds as far as S_{i+1}. The proof depends on an elementary theorem relating to continued fractions, which was first established by Lagrange.

'If $\frac{p_{i-1}}{q_{i-1}}$, $\frac{p_i}{q_i}$ are consecutive convergents to the same rational or irrational quantity θ, $p_{i-1} - q_{i-1}\theta$ is less in absolute magnitude than any quantity of the form $y - x\theta$, where x and y are positive integers, of which x is less than q_i.'

Supposing, for brevity, that i is uneven, we infer from this principle that the least segment in the sequence S_i is its last segment $q_i Q - p_i P$, and that the next least segment in S_i is the last segment of S_{i-1}, viz. $p_{i-1} P - q_{i-1} Q$. We have to add that $p_{i-1} P - q_{i-1} Q$ is also less than the segment $P(1+p_i) - q_i Q$ which immediately follows S_i. For if

$$q_{i+1} > 1 + \mu_{i+1},$$

a condition which is certainly satisfied when $i > 1$, we have $\frac{p_i - p_{i-1} + 1}{q_i - q_{i-1}} > \frac{p_{i+1}}{q_{i+1}}$,

i.e. $> \frac{p}{q}$, because $i+1$ is even. Let us write down the sequence S_i $1+k$ times over, and let $yQ-xP$ be any segment of S_i contained between two consecutive points P and Q, of which Q is to the right of P; the corresponding segment in $(1+k)\,S_i$ will be

$$(kq_i+y)\,Q-(kp_i+x)\,P=k\,(q_i\,Q-p_i\,P)+(yQ-xP)\,;$$

i.e. Q will be still further to the right of P, and the distance between P and Q be increased. Next, let $xP-yQ$ be a segment of S_i, contained between two consecutive points P and Q, of which P lies to the right of Q; or, again, let $xP-yQ$ represent the segment $(1+p_i)\,P-q_i\,Q$, which immediately follows S_i. The corresponding segment in $(k+1)\,S_i$, or immediately following $(k+1)\,S_i$, will be

$$(kp_i+x)\,P-(kq_i+y)\,Q=xP-yQ-k\,(q_i\,Q-p_i\,P)\,;$$

so that, if k be not too great, the two new points P and Q will lie in the same relative position with regard to one another as the two points originally considered, the distance between them being diminished; but, for values of k which surpass a certain limit, the point Q will be shifted to the right of P, and the segment QP will be replaced by a segment PQ. As long as this interchange of places between two consecutive points Q and P does not occur, so long the successive repetitions of S_i will represent with accuracy the arrangement of the points P and Q. Now the least of the segments $xP-yQ$ is $p_{i-1}\,P-q_{i-1}\,Q$, and

$$p_{i-1}\,P-q_{i-1}\,Q-\mu_{i+1}\,(q_i\,Q-p_i\,P)$$

is still positive; therefore we may repeat S_i $1+\mu_{i+1}$ times, but we cannot repeat it $2+\mu_{i+1}$ times, for

$$p_{i-1}\,P-q_{i-1}\,Q-(1+\mu_{i+1})\,(q_i\,Q-p_i\,P)$$

is negative. The sequence $S_i^{\mu_{i+1}}\,S_{i-1}$ will therefore truly represent, as far as it goes, the arrangement of the points P and Q; but the sequence $S_i^{1+\mu_{i+1}}\,S_{i-1}$ would fail to do so. We should in fact come to an error in the last two points of S_{i-1}, which, according to the law of that sequence, we should have to write down as QP, whereas the true arrangement of these points is PQ. This suffices to establish the general theorem of Art. 4; but it is of interest to add, that the error which we have just shown must occur in the last two points of the sequence $S_i^{1+\mu_{i+1}}\,S_{i-1}$, is the only error that can occur in that sequence. And this is certain; for, in the first place, we have seen that there is no error in $S_i^{1+\mu_{i+1}}$; and, in the second place, if $xP-yQ$ be any segment of S_{i-1} of the same positive sign as $p_{i-1}\,P-q_{i-1}\,Q$, $xP-yQ-(1+\mu_{i+1})\,(q_i\,Q-p_i\,P)$ is necessarily positive; for, by

the theorem of Lagrange, $xP - yQ > q_{i-2}Q - p_{i-2}P$; and

$$xP - yQ + p_iP - q_iQ > (p_i - p_{i-2})P - (q_i - q_{i-2})Q > p_{i-1}P - q_{i-1}Q$$

by the same theorem; whence

$$xP - yQ - (1 + \mu_{i+1})(q_iQ - p_iP)$$

is positive, because

$$p_{i-1}P - q_{i-1}Q - \mu_{i+1}(q_iQ - p_iP)$$

is positive.

7. It will be noticed that the sequence S_1 can only be repeated μ_2 times, whereas any subsequent sequence S_i can be repeated $1 + \mu_{i+1}$ times. The exception in the case of S_1 is apparent rather than real, and arises from the fact, that the period S_0 consists of only one term. If we were to attempt to repeat the sequence S_1 $2 + \mu_2$ times, the sequence S_0, which commences the last repetition of S_1, ought, according to the general theory, to be in error; viz. its last point must be interchanged with the preceding point; and, as S_0 contains but one point, this interchange vitiates the sequence S_1 immediately preceding.

8. Any finite continued fraction may be written either with an even or with an uneven number of quotients, because the last quotient may be made either equal to unity or greater than unity. If the number of quotients be even, the two extreme points P and Q, which coincide with 1, must be written in the order QP; if the number of quotients be uneven these points must be written in the order PQ.

9. If we omit these two last coincident points, the remaining $p-1$ points P and $q-1$ points Q evidently form a symmetric series, being similarly distributed on either side of the middle point of the line. And, similarly, if we remove from any sequence whatever its two final points, we obtain a symmetrical series, because the sequence S_i corresponds to the division of a line into p_i equal parts and also into q_i equal parts.

10. If we wish, from the arrangement of the points P and Q, to infer the arrangement corresponding to the fraction $\mu_{i+1} + \dfrac{1}{\mu_{i+2}} + \ldots$, obtained from the fraction $\dfrac{p}{q}$ by omitting its first i quotients, we have only to replace the sequences S_i and S_{i-1} by single points. Thus, in the example of Art. 3, if we put $S_1 = A$, $S_0 = B$, we find

$$A,\ 2A,\ 3A,\ B \mid 4A,\ 5A,\ 6A,\ 2B \mid 7A \parallel 8A,\ 9A,\ 10A,\ 3B \mid$$
$$11A,\ 12A,\ 13A,\ 4B \mid 14A \parallel 15A,\ 16A,\ 17A,\ 5B \parallel\!\mid,$$

corresponding to $\frac{17}{5} = 3 + \cfrac{1}{2 + \cfrac{1}{2}}$. And, again, if we wish to obtain the arrangement corresponding to the fraction $\mu_i + \cfrac{1}{\mu_{i+1} + \cdots \cfrac{1}{\mu_{i+j}}}$, where $i+j<s$, we first replace S_i, S_{i-1} by single points, and then consider in the resulting arrangement the sequence of order j. Thus the arrangement

$$A,\ 2A,\ 3A,\ B\,|\,4A,\ 5A,\ 6A,\ 2B\,|\,7A\,\|$$

corresponds to the fraction $3+\frac{1}{2}$.

Addition to the preceding note.

11. The theorem of Lagrange, on which the demonstration in the preceding note depends, will be found in the second paragraph of his 'Additions to Euler's Algebra.' But as this theorem is no longer included in elementary treatises, we shall here place Lagrange's demonstration of it.

If ϕ_i is the complete quotient of order i in the development of θ, we have

$$\theta = \frac{\phi_i p_{i-1} + p_{i-2}}{\phi_i q_{i-1} + q_{i-2}}, \quad \text{or} \quad \phi_i = -\frac{p_{i-2} - q_{i-2}\theta}{p_{i-1} - q_{i-2}\theta}.$$

But ϕ_i is positive and greater than unity; hence, $p_{i-2} - q_{i-2}\theta$, and $p_{i-1} - q_{i-1}\theta$ are of opposite signs, and $p_{i-1} - q_{i-1}\theta$ is less in absolute magnitude than $p_{i-2} - q_{i-2}\theta$.

Again, since $p_i q_{i-1} - p_{i-1} q_i = (-1)^i$, we can always find, whatever the given integral numbers x and y may be, two integral numbers λ and μ satisfying the equations

$$x = \lambda q_{i-1} + \mu q_i, \quad y = \lambda p_{i-1} + \mu p_i,$$

whence we obtain

$$y - x\theta = \lambda (p_{i-1} - q_{i-1}\theta) + \mu (p_i - q_i\theta).$$

As $p_{i-1} - q_{i-1}\theta$ and $p_i - q_i\theta$ are of opposite signs, if $y - x\theta$ is less than $p_{i-1} - q_{i-1}\theta$, λ and μ must be of the same sign; that is to say, x and y are either respectively equal to q_i and p_i, or else they are respectively greater than q_i and p_i.

12. In the same place Lagrange has also established the converse theorem, that if $b - a\theta$ is a minimum difference, *i.e.* if $b - a\theta$ is less in absolute magnitude than any difference $y - x\theta$, in which x is less than a, $\frac{b}{a}$ is a convergent to θ. Writing p_{i-1} for b, and q_{i-1} for a, we first determine the positive numbers p_{i-2} and q_{i-2}, respectively less than p_{i-1} and q_{i-1}, which satisfy the equation $p_{i-1} q_{i-2} - p_{i-2} q_{i-1} = \epsilon$, ϵ denoting an unit of the same sign as $p_{i-1} - q_{i-1}\theta$. If

we write $p_{i-1}-q_{i-1}\theta=u_{i-1}$, $p_{i-2}-q_{i-2}\theta=u_{i-1}$, we find, on eliminating θ,

$$\frac{\epsilon}{q_{i-1}q_{i-2}}=\frac{u_{i-1}}{q_{i-1}}-\frac{u_{i-2}}{q_{i-2}}.$$

In this equation u_{i-2} is greater, by hypothesis, than u_{i-1}, because $q_{i-2}<q_{i-1}$; *à fortiori* $\frac{u_{i-1}}{q_{i-2}}$ is greater than $\frac{u_{i-1}}{q_{i-1}}$. But u_{i-1} is of the same sign as ϵ; therefore, u_{i-1} and u_{i-2} must have contrary signs. Consequently the quotient

$$\phi_i=-\frac{u_{i-2}}{u_{i-1}}=-\frac{p_{i-2}-q_{i-2}\theta}{p_{i-1}-q_{i-1}\theta}$$

is positive, and greater than unity; and if $\frac{p_{i-1}}{q_{i-1}}$ be developed in a continued fraction having $\frac{p_{i-2}}{q_{i-2}}$ for its last convergent (which is always possible), we obtain

$$\theta=\frac{p_{i-2}+\phi_i p_{i-1}}{q_{i-2}+\phi_i q_{i-1}}=\mu_1+\frac{1}{\mu_2+}\,\frac{1}{\mu_3+}\,\cdots\,\frac{1}{\mu_{i-1}+}\,\frac{1}{\phi_i};$$

i.e. $\frac{p_{i-2}}{q_{i-2}}$ and $\frac{p_{i-1}}{q_{i-1}}=\frac{b}{a}$ are successive convergents to θ.

13. Combining the two theorems of Lagrange, we see that if we have ascertained, by observation, that $p_{i-1}-q_{i-1}\theta$ is less than any difference $y-x\theta$ in which x is less than q_{i-1}, we can at once infer that $p_{i-1}-q_{i-1}\theta$ is also less than any difference $y-x\theta$ in which $q_{i-1}<x<q_i$.

14. The two theorems of Lagrange serve to define the successive minima of the expression $y-x\theta$. The theory of the successive minima of the expression $\frac{y}{x}-\theta$ is perhaps less complete. Thus we have the elementary theorem, that the difference $\frac{p_{i-1}}{q_{i-1}}-\theta$ is less than any difference $\frac{y}{x}-\theta$ in which x does not surpass q_{i-1}, and is also less than any difference of the same sign with itself, in which x does not surpass q_i; but there may be differences of a contrary sign to $\frac{p_{i-1}}{q_{i-1}}-\theta$, in which x does not surpass q_i, and which are less in absolute magnitude than $\frac{p_{i-1}}{q_{i-1}}-\theta$. And again, if $\frac{b}{a}-\theta$ be a minimum difference (*i.e.* if $\frac{b}{a}-\theta$ be less in absolute magnitude than any difference $\frac{y}{x}-\theta$, in which x is less than a), we cannot in general infer that $\frac{b}{a}$ is a convergent to θ. We shall attempt, in what follows, to define accurately the successive minima of the expression $\frac{y}{x}-\theta$, and thus

to give a greater amount of precision to this part of the theory of continued fractions.

15. We still consider a rational or irrational quantity θ, of which the development is

$$\theta = \frac{\phi_i p_{i-1} + p_{i-2}}{\phi_i q_{i-1} + q_{i-2}} = \mu_1 + \frac{1}{\mu_2 +} \cdots \frac{1}{\mu_{i-1} +} \frac{1}{\phi_i};$$

and, adopting the designation of Lagrange, we term the fractions $\frac{P_k}{Q_k} = \frac{kp_{i-1} + p_{i-2}}{kq_{i-1} + q_{i-2}}$, where $0 > k < \mu_i$, *intermediate* fractions. These fractions are evidently intermediate between $\frac{p_{i-2}}{q_{i-2}}$ and $\frac{p_i}{q_i}$; hence θ lies between any one of them and $\frac{p_{i-1}}{q_{i-1}}$. If $\frac{b}{a} - \theta$ is a minimum difference, we can, by reasoning as in Art. 12, arrive at an equation of the form

$$\theta = \mu_1 + \frac{1}{\mu_2 +} \frac{1}{\mu_3 +} \cdots \frac{1}{\mu_{j-1} +} \frac{1}{\lambda +} \frac{1}{\psi},$$

where

$$\frac{b}{a} = \mu_1 + \frac{1}{\mu_2 +} \cdots \frac{1}{\mu_{j-1} +} \frac{1}{\lambda};$$

and we can prove that in this equation ψ is positive. But we cannot prove that ψ is greater than unity; *i.e.* instead of the equation $\lambda = \mu_j$, we have the inequalities $0 < \lambda \leqq \mu_j$; and thus from the hypothesis that $\frac{b}{a} - \theta$ is a minimum difference, we cannot infer that $\frac{b}{a}$ is a convergent to θ, but only that $\frac{b}{a}$ is either a convergent or an intermediate fraction. But not every intermediate fraction can give a minimum difference; for in order that $\frac{P_k}{Q_k} - \theta$ should be a minimum difference, $\frac{P_k}{Q_k} - \theta$ must (at any rate) be less than $\frac{p_{i-1}}{q_{i-1}} - \theta$, because $q_{i-1} < Q_k$. The absolute value of

$$\frac{p_{i-1}}{q_{i-1}} - \theta \text{ is } \frac{1}{q_{i-1}(\phi_i q_{i-1} + q_{i-2})}$$

and the absolute value of

$$\frac{P_k}{Q_k} - \theta \text{ is } \frac{\phi_i - k}{(kq_{i-1} + q_{i-2})(\phi_i q_{i-1} + q_{i-2})};$$

whence, if $\frac{P_k}{Q_k} - \theta$ is a minimum difference, we must have

$$\left.\begin{aligned} &\phi_i < 2k + \frac{q_{i-2}}{q_{i-1}}, \\ \text{or} \quad &\mu_i + \frac{1}{\mu_{i+1}} + \ldots < 2k + \frac{1}{\mu_{i-1}} \frac{1}{\mu_{i-2}} + \ldots \end{aligned}\right\} \quad \ldots\ldots \quad (A)$$

And this necessary condition is also sufficient. For, since θ lies between $\frac{P_k}{Q_k}$ and $\frac{p_{i-1}}{q_{i-1}}$, and since (if the condition (A) be satisfied) θ also lies nearer to $\frac{P_k}{Q_k}$ than to $\frac{p_{i-1}}{q_{i-1}}$, any fraction which is nearer to θ than $\frac{P_k}{Q_k}$ must lie between $\frac{P_k}{Q_k}$ and $\frac{p_{i-1}}{q_{i-1}}$, and must therefore have a denominator greater than Q_k, because

$$\frac{P_k}{Q_k} - \frac{p_{i-1}}{q_{i-1}} = \frac{(-1)^i}{q_{i-1} Q_k}.$$

We are thus led to divide the fractions, intermediate between $\frac{p_{i-2}}{q_{i-2}}$ and $\frac{p_i}{q_i}$ into two sets, according as they do or do not satisfy the condition (A). We may call those fractions which do not satisfy that condition the *inferior*, and those which do satisfy it the *superior* intermediate fractions. We then have the theorem:

'The complete series of successive minima of the expression $\frac{y}{x} - \theta$ is obtained by taking in succession for $\frac{y}{x}$ the convergents, and the superior intermediate fractions in their natural order.'

16. If $k > \frac{1}{2}\mu_i$, the condition (A) is satisfied; if $k = \frac{1}{2}\mu_i$, the condition is satisfied if $\mu_{i-1} < \mu_{i+1}$; if $k = \frac{1}{2}\mu_i$, $\mu_{i-1} = \mu_{i+1}$, the condition is satisfied if $\mu_{i-2} > \mu_{i+2}$, and so on continually. If the continued fraction be finite, symmetrical, and of an uneven number of quotients, $\mu_i = 2k$ being the middle quotient, we have a singular case in which the errors of $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{P_k}{Q_k}$ are exactly equal; we may in this case regard $\frac{P_k}{Q_k}$ as an inferior fraction. It will be seen that, as nearly as possible, one-half of the fractions intermediate between $\frac{p_{i-2}}{q_{i-2}}$ and $\frac{p_i}{q_i}$ are superior. Thus, if $\mu_i = 2h+1$ is uneven, there are h inferior and h superior intermediate fractions; if $\mu_i = 2h$ is even, there are certainly $h-1$ inferior and $h-1$ superior intermediate fractions; but whether $\frac{P_h}{Q_h}$ is inferior or

superior, can only be decided (as we have just seen) by comparing the quotients which precede μ_i with those which follow it.

17. The difference $\frac{p_{i-1}}{q_{i-1}} - \theta$ is of the sign $(-1)^{i-1}$; the differences $\frac{P}{Q} - \theta$, which, in forming the complete series of minima of $\frac{y}{x} - \theta$, we have to intercalate between $\frac{p_{i-1}}{q_{i-1}} - \theta$, and $\frac{p_i}{q_i} - \theta$ are of the same sign as the latter of these differences, *i.e.* they are of the sign $(-1)^i$. Thus, after every convergent there is a change of sign in the series of minimum differences, and the minimum differences formed with convergents are distinguished by this criterion from the minimum differences formed with superior intermediate fractions.

18. Again, if $\frac{b}{a} - \theta$ be any minimum difference, and if $q_{i-1} \leqq a < q_i$, the only differences $\frac{y}{x} - \theta$, which are less than $\frac{b}{a} - \theta$, and which have denominators x less than q_i, are the minimum differences which lie between $\frac{b}{a} - \theta$ and $\frac{p_i}{q_i} - \theta$. It is sufficient to prove this for the case in which $a = q_{i-1}$, $b = p_{i-1}$. Let $\frac{P_{\lambda-1}}{Q_{\lambda-1}}$ be the last of the inferior fractions, intermediate between $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{p_i}{q_i}$; then θ, which lies between $\frac{P_{\lambda-1}}{Q_{\lambda-1}}$ and $\frac{p_{i-1}}{q_{i-1}}$, is nearer to the latter than to the former of those fractions. If then $\frac{y}{x}$ be nearer to θ than $\frac{p_{i-1}}{q_{i-1}}$ is, $\frac{y}{x}$ must itself lie between $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{P_{\lambda-1}}{Q_{\lambda-1}}$. But, if $x < q_i$, $\frac{y}{x}$ cannot lie between $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{p_i}{q_i}$; hence, $\frac{y}{x}$ must lie between $\frac{p_i}{q_i}$ and $\frac{P_{\lambda-1}}{Q_{\lambda-1}}$. But the only fractions between these limits, which have denominators less than q_i, and lie nearer to θ than $\frac{p_{i-1}}{q_{i-1}}$, are the superior fractions intermediate between $\frac{p_{i-2}}{q_{i-2}}$ and $\frac{p_i}{q_i}$. For all such fractions are of the type $\frac{\sigma p_{i-1} + \tau p_{i-2}}{\sigma q_{i-1} + \tau q_{i-2}}$, the relatively prime numbers σ and τ satisfying the inequalities

$$\lambda - 1 < \frac{\sigma}{\tau} < \mu_i, \quad \ldots \ldots \ldots \quad (1)$$

$$\sigma q_{i-1} + \tau q_{i-1} < q_i, \text{ whence } \sigma < \mu_i, \quad \ldots \ldots \quad (2)$$

$$\phi_i < 2\frac{\sigma}{\tau} + \frac{q_{i-2}}{q_{i-1}}. \quad \ldots \ldots \ldots \quad (3)$$

Now if $\mu_i = 2h+1$, we have $\lambda - 1 = h$, and the inequalities $h < \frac{\sigma}{\tau}$, and $\sigma < 2h+1$ (from which equality is excluded), show that unity is the only admissible value for τ. Again, if $\mu_i = 2h$ and $\frac{P_h}{Q_h}$ is an inferior fraction, we have $h < \frac{\sigma}{\tau}$, $\sigma < 2h$, and unity is the only admissible value for τ. In both these cases, therefore, the only fractions having denominators less than q_i, which lie between $\frac{P_{\lambda-1}}{Q_{\lambda-1}}$ and $\frac{p_i}{q_i}$, are the superior intermediate fractions. If, however, $\mu_i = 2h$, and $\frac{P_h}{Q_h}$ is a superior fraction, the inequalities (1) and (2) are satisfied by the values $\sigma = 2h-1$, $\tau = 2$, so that, besides the superior intermediate fractions, the fraction $\frac{(2h-1)\,p_{i-1} + 2\,p_{i-2}}{(2h-1)\,q_{i-1} + 2\,q_{i-2}}$ lies between the limits $\frac{P_{h-1}}{Q_{h-1}}$ and $\frac{p_i}{q_i}$. But this fraction is more remote from θ than $\frac{p_{i-1}}{q_{i-1}}$ is, because the equation $\frac{\sigma}{\tau} = \frac{2h-1}{2}$ is inconsistent with the inequality (3).

19. The inferior intermediate fractions $\frac{P_k}{Q_k}$, $k \leqq \lambda - 1$, do not give minimum differences, because $q_{i-1} < Q_k$, and $\frac{p_{i-1}}{q_{i-1}} - \theta$ is less in absolute magnitude than $\frac{P_k}{Q_k} - \theta$. But, with the single exception of $\frac{p_{i-1}}{q_{i-1}} - \theta$, all other differences $\frac{y}{x} - \theta$, in which x is less than Q_k, are greater than $\frac{P_k}{Q_k} - \theta$. For, if $\frac{y}{x}$ lie between $\frac{P_k}{Q_k}$ and $\frac{p_{i-1}}{q_{i-1}}$, x must be greater than Q_k; if $\frac{P_k}{Q_k}$ lie between $\frac{y}{x}$ and $\frac{p_{i-1}}{q_{i-1}}$, the difference $\frac{y}{x} - \theta$ is certainly greater than $\frac{P_k}{Q_k} - \theta$, because θ lies between $\frac{P_k}{Q_k}$ and $\frac{p_{i-1}}{q_{i-1}}$; lastly, if $\frac{p_{i-1}}{q_{i-1}}$ lies between $\frac{y}{x}$ and $\frac{P_k}{Q_k}$, we find (taking the case in which i is uneven)

$$\frac{y}{x} - \theta > \frac{y}{x} - \frac{p_{i-1}}{q_{i-1}} > \frac{1}{x q_{i-1}} > \frac{1}{Q_k q_{i-1}} = \frac{p_{i-1}}{q_{i-1}} - \frac{P_k}{Q_k} > \theta - \frac{P_k}{Q_k}.$$

20. The theorem of Lagrange admits of an important geometrical interpretation. If with a pair of rectangular axes in a plane we construct a system of unit points (*i.e.* a system of points of which the coordinates are integral numbers), and draw the line $y = \theta x$, we learn from that theorem that if (x, y) be an unit point lying nearer to that line than any other unit point having a less abscissa

(or, which comes to the same thing, lying at a less distance from the origin), $\frac{y}{x}$ is a convergent to θ; and, *vice versâ*, if $\frac{y}{x}$ is a convergent, (x, y) is one of the 'nearest points.' Thus the 'nearest points' lie alternately on opposite sides of the line, and the double area of the triangle, formed by the origin and any two consecutive 'nearest points,' is unity.

In particular, if $\theta = \frac{p}{q}$, p and q being relatively prime integers, the coordinates of the two 'nearest points' above and below the finite line joining the origin to the unit point (q, p) satisfy respectively the equations $px - qy = 1$, and $px - qy = -1$. We thus obtain a simple geometrical method of finding the least solution in integral numbers of either of those indeterminate equations.

XXIX.

NOTE ON THE THEORY OF THE PELLIAN EQUATION, AND OF BINARY QUADRATIC FORMS OF A POSITIVE DETERMINANT.*

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1. LET $\theta_0 = \mu_0 + \dfrac{1}{\mu_1 +}\,\dfrac{1}{\mu_2 + \ldots}$ be any continued fraction, of which $\theta_1, \theta_2, \ldots$ are the complete quotients; $\dfrac{p_0}{q_0}, \dfrac{p_1}{q_1}, \ldots$ the successive convergents; so that

$$p_0 = \mu_0, \quad q_0 = 1, \ldots, \quad p_i = \mu_i p_{i-1} + p_{i-2}, \quad q_i = \mu_i q_{i-1} + q_{i-2},$$

$$\theta = \frac{\theta_i p_{i-1} + p_{i-2}}{\theta_i q_{i-1} + q_{i-2}}, \quad \theta_i = -\frac{p_{i-2} - \theta q_{i-2}}{p_{i-1} - \theta q_{i-1}}.$$

Also, let $\quad p_{i-1} - \theta q_{i-1} = (-1)^i \epsilon_{i-1}$, so that $\theta_i = \dfrac{\epsilon_{i-2}}{\epsilon_{i-1}}$;

we have $\quad \theta_1 = \dfrac{1}{\epsilon_0}$, and hence $\epsilon_{i-1} = \dfrac{1}{\theta_1 \theta_2 \ldots \theta_i}$.

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* The following summary of the contents of this Note may be of use to the reader:—

Art. 1.—The relation, in a continued fraction, being the quantities ϵ and θ.

Art. 2.—The theorem that $T + U\sqrt{D}$ is equal to the product of the complete quotients in the development of $\sqrt{D}$.

Art. 3.—The same theorem for the period of complete quotients in the development of any quadratic surd.

Art. 4.—Theorems as to the number of different periods of complete quotients; viz., equations (1)–(4).

Art. 5.—Theorems as to the number of non-equivalent classes of quadratic forms; viz., equations (5) and (6).

Art. 6.—Equations arising from a comparison of the formulæ (5) and (6) with those of Dirichlet.

Arts. 7–13.—Discussion of the nature of the periods in the more important special cases.

Art. 14.—On the symmetry of any periodic series.

Art. 15.—On the arithmetical conditions under which the various special cases present themselves.

(It would be difficult to say that anything in the Addition (Arts. 7–15) is new: the discussion there attempted has never been given completely (see Art. 7); but this may have been because no one has thought it worth giving.)

This expression, which supplies a measure of the rate of decrease of the difference ϵ_{i-1}, admits of an interesting application to the theory of the Pellian equation, and of binary quadratic forms of a positive determinant.

2. Let D be any positive integer, not a perfect square. In the development of $\sqrt{D}$ in a continued fraction, let

$$\frac{\sqrt{D}+Q_1}{P_1}, \quad \frac{\sqrt{D}+Q_2}{P_2}, \ \ldots, \ \frac{\sqrt{D}+Q_i}{P_i}$$

be the period of complete quotients; so that, if a is the integral number next inferior to $\sqrt{D}$,

$$\frac{\sqrt{D}+Q_1}{P_1} = \frac{\sqrt{D}+a}{D-a^2}, \quad \frac{\sqrt{D}+Q_i}{P_i} = \sqrt{D}+a.$$

Let T and U be the least integral numbers satisfying the equation

$$T^2 - DU^2 = (-1)^i,$$

and let
$$\sqrt{D} = a + \frac{1}{\mu_1+} \frac{1}{\mu_2+} \ldots \frac{1}{\mu_{i-1}+} \frac{1}{2a+} \ldots ;$$

we have
$$T = p_{i-1}, \quad U = q_{i-1}, \quad (-1)^i \epsilon_{i-1} = T - U\sqrt{D};$$

whence, by the preceding theorem,

$$T + U\sqrt{D} = \Pi_{s=1}^{s=i} \frac{\sqrt{D}+Q_s}{P_s}.$$

Example.—The continued fraction equivalent to $\sqrt{13}$ is

$$\sqrt{13} = 3 + \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \frac{1}{6+} \ldots,$$

and the period of complete quotients is

$$\frac{\sqrt{13}+3}{4}, \quad \frac{\sqrt{13}+1}{3}, \quad \frac{\sqrt{13}+2}{3}, \quad \frac{\sqrt{13}+1}{4}, \quad \sqrt{13}+3,$$

giving
$$(\sqrt{13}+3)^2 (\sqrt{13}+1)^2 (\sqrt{13}+2) \times \frac{1}{144} = 18 + 5\sqrt{13},$$

and
$$18^2 - 25 \times 13 = -1.$$

3. Again, let
$$\Omega = a + 2bu + cu^2 = 0$$

represent any properly primitive equation of determinant D (*i.e.* any quadratic equation whatever, in which a, b, c are integral numbers satisfying the equation $b^2 - ac = D$, and a, $2b$, c have no common divisor). If

$$\frac{\sqrt{D}+Q_1}{P_1}, \quad \frac{\sqrt{D}+Q_2}{P_2}, \ \ldots, \ \frac{\sqrt{D}+Q_i}{P_i}$$

is the period of complete quotients obtained by the development of either root of Ω, we shall have, as before,

$$T+U\sqrt{D}=\Pi_{s=1}^{s=i}\frac{\sqrt{D}+Q_s}{P_s}.$$

For the equations of the period are of the type

$$\left.\begin{aligned} a_0+2\beta_0 u_0-a_1 u_0^2&=0,\\ -a_1-2\beta_1 u_1+a_2 u_1^2&=0,\\ a_2+2\beta_2 u_2-a_2 u_2^2&=0,\\ &\cdots\cdots\cdots\\ (-1)^{i-1}a_{i-1}+(-1)^{i-1}2\beta_{i-1}u_{i-1}+(-1)^i a_0 u_{i-1}^2&=0,\\ (-1)^i[a_0+2\beta_0 u_0-a_1 u_0^2]&=0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}\right\}\quad\ldots\ldots\quad(\Omega)$$

so that, if $\mu_0, \mu_1, \mu_2, \ldots, \mu_{i-1}$, is the period of integral quotients, we have

$$u_0=\mu_0+\frac{1}{\mu_1+\cdots}\,\frac{1}{\mu_{i-1}+}\,\frac{1}{u_0},$$

where
$$u_0=\frac{\beta_0+\sqrt{D}}{a_1}.$$

Hence
$$\begin{aligned}\frac{1}{\epsilon_{i-1}}&=\frac{(-1)^i}{p_{i-1}-q_{i-1}u_0},\\ &=u_0u_1u_2\ldots u_{i-1},\\ &=\Pi_{s=1}^{s=i}\frac{\sqrt{D}+Q_s}{P_s}.\end{aligned}$$

But, by a known theorem (see the 'Report on the Theory of Numbers,' in the 'Report of the British Association for 1861,' Art. 96 (i.), p. 315*), we have

$$\left.\begin{aligned} T&=\tfrac{1}{2}(p_{i-1}+q_{i-2}),\\ U&=\frac{p_{i-2}}{a_0}=\frac{p_{i-1}-q_{i-2}}{2\beta_0}=\frac{q_{i-1}}{a_1};\end{aligned}\right\}\quad\ldots\ldots\quad(Q)$$

whence
$$p_{i-1}-q_{i-1}u_0=p_{i-1}-q_{i-1}\frac{\beta_0+\sqrt{D}}{a_1}=T-U\sqrt{D},$$

or
$$T+U\sqrt{D}=\frac{(-1)^i}{T-U\sqrt{D}}=\Pi_{s=1}^{s=i}\frac{\sqrt{D}+Q_s}{P_s}.$$

If the given equation (Ω) is improperly primitive (*i.e.*, if the numbers a, $2b$, c have 2 for their greatest common divisor), we have to replace the

* Vol. i. p. 195.

numbers T and U in the equations (Q), by $\frac{1}{2}T_1$ and $\frac{1}{2}U_1$, where T_1 and U_1 are the least numbers which satisfy the equation $T_1^2 - DU_1^2 = (-1)^i 4$; and we find

$$\tfrac{1}{2}(T_1 + U_1\sqrt{D}) = \Pi_{s=1}^{s=i} \frac{\sqrt{D} + Q_s}{P_s}.$$

4. Every primitive quadratic equation of determinant D, of which one root is positive and greater than unity, and the other negative and less in absolute magnitude than unity, occurs in one, and only in one, of the periods of equations of determinant D (see the Report cited, p. 309, Art. 93*). Hence, every expression of the form $\frac{\sqrt{D} + Q}{P}$, in which, P and Q being positive, Q is less than $\sqrt{D}$, P is a divisor of $D - Q^2$ intermediate between $\sqrt{D} - Q$ and $\sqrt{D} + Q$, and the three numbers P, Q, $\frac{D - Q^2}{P}$ are relatively prime, is the root of an equation contained in a period of equations of determinant D; and, for any given determinant, the number of periods of complete quotients is equal to the number of periods of quadratic equations, if we regard two quadratic equations such as

$$a + 2b\theta + c\theta^2 = 0,$$
$$-a - 2b\theta - c\theta^2 = 0,$$

which differ only in sign, as identical with one another.

Let $P' = \frac{D - Q^2}{P}$, and let k be the number of periods of properly primitive complete quotients of determinant D; we have evidently

$$(T + U\sqrt{D})^k = \Pi \frac{\sqrt{D} + Q}{P}, \quad \ldots\ldots\ldots \quad (1)$$

or

$$k = \frac{1}{\log(T + U\sqrt{D})} \Sigma \log \frac{\sqrt{D} + Q}{P},$$

the sign of multiplication Π, and the sign of summation Σ, extending to all positive numbers Q which do not surpass $\sqrt{D}$, and to all divisors P of $D - Q^2$, which are intermediate between $\sqrt{D} + Q$ and $\sqrt{D} - Q$, and are such that the three numbers P, $2Q$, P' admit of no common divisor other than unity. Let $\psi(Q)$ be the number of such divisors of $D - Q^2$; observing that, if $\frac{\sqrt{D} + Q}{P}$ is a complete quotient in a period, $\frac{\sqrt{D} + Q}{P'} = \frac{P}{\sqrt{D} - Q}$ is also a complete quotient in the

* Vol. i. p. 186.

same or in a different period, we may write

$$(T+U\sqrt{D})^k=\Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\frac{1}{2}\psi(Q)}; \quad \ldots\ldots\ldots \quad (2)$$

where it will be noticed that, if $P=P'$, $D=P^2+Q^2$, the two identical complete quotients $\dfrac{\sqrt{D}+Q}{P}$ and $\dfrac{P}{\sqrt{D}-Q}$ are each of them equal to $\left[\dfrac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\frac{1}{2}}$.

When $D\equiv 1$, mod 4, let k_1 be the number of periods of improperly primitive complete quotients of determinant D, we find as before

$$(\tfrac{1}{2}T_1+\tfrac{1}{2}U_1\sqrt{D})^{k_1}=\Pi\frac{\sqrt{D}+Q}{P}, \quad \ldots\ldots\ldots \quad (3)$$

or

$$k_1=\frac{1}{\log\frac{1}{2}(T_1+U_1\sqrt{D})}\Sigma\log\frac{\sqrt{D}+Q}{P},$$

the symbols Π and Σ extending to all positive uneven numbers Q which are less than $\sqrt{D}$, and to all divisors P of $D-Q^2$ which are intermediate between $\sqrt{D}+Q$ and $\sqrt{D}-Q$, and are such that the three numbers P, $2Q$, P' have 2 for their greatest common divisor. If $\psi_1(Q)$ be the number of such divisors of $D-Q^2$, we may also write

$$(\tfrac{1}{2}T_1+\tfrac{1}{2}U_1\sqrt{D})^{k_1}=\Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\frac{1}{2}\psi_1(Q)} \quad \ldots\ldots\ldots \quad (4)$$

It will be observed that, if Q is even, we have always $\psi_1(Q)=0$.

5. Let h and h_1 respectively denote the numbers of properly and improperly primitive classes of quadratic forms of determinant D, and let $[\tau, \upsilon]$, $[\tau_1, \upsilon_1]$ be the least numbers which satisfy the equations $\tau^2-D\upsilon^2=+1$, $\tau_1^2-D\upsilon_1^2=+4$; so that, when the equations

$$x^2-Dy^2=-1, \quad x^2-Dy^2=-4, \quad \ldots\ldots\ldots \quad (T)$$

are resoluble [they are either both resoluble or both irresoluble],

$$\tau+\upsilon\sqrt{D}=(T+U\sqrt{D})^2, \quad \tfrac{1}{2}\tau_1+\tfrac{1}{2}\upsilon_1\sqrt{D}=(\tfrac{1}{2}T_1+\tfrac{1}{2}U_1\sqrt{D})^2;$$

and when the equations (T) are irresoluble,

$$\tau+\upsilon\sqrt{D}=T+U\sqrt{D}, \quad \tau_1+\upsilon_1\sqrt{D}=T_1+U_1\sqrt{D}.$$

We can now establish the equations

$$(\tau+\upsilon\sqrt{D})^h=\Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\psi(Q)}, \quad \ldots\ldots\ldots \quad (5)$$

$$(\tfrac{1}{2}\tau_1+\tfrac{1}{2}\upsilon_1\sqrt{D})^{h_1}=\Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\psi_1(Q)} \quad \ldots\ldots\ldots \quad (6)$$

For this purpose it is only necessary to show that, when the equations (T) are not resoluble, we have

$$h=2k, \quad h_1=2k_1; \quad \ldots\ldots\ldots\ldots \quad (\alpha)$$

but when these equations are resoluble, we have, instead,

$$h=k, \quad h_1=k_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

To the single period of complete quotients

$$\frac{\sqrt{D}+\beta_0}{a_1}, \quad \frac{\sqrt{D}+\beta_1}{a_2}, \quad \frac{\sqrt{D}+\beta_2}{a_3}, \ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

there correspond two periods of reduced quadratic forms, viz.,

$$(a_0, \beta_0, -a_1), \quad (-a_1, \beta_1, a_2), \quad (a_2, \beta_2, -a_3), \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

and

$$(-a_0, \beta_0, a_1), \quad (a_1, \beta_1, -a_2), \quad (-a_2, \beta_2, a_3), \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

For the complete quotients (7) are the positive roots of the equations (Ω); they are also the positive roots of the same equations with their signs changed; and the period (9) is related to the period $(-\Omega)$ exactly as the period (8) is related to the period (Ω); viz., the coefficients of the forms are the same as the coefficients of the equations, except that in the periods of equations the middle coefficients are alternately positive and negative, whereas in the periods of forms these coefficients are all positive. The two periods of forms are, in general, but not always, distinct; and we shall now prove (what is indeed well known) that these two periods are, or are not, identical, according as the equations (T) are, or are not, resoluble. We may observe that the form $(a, -b, c)$ is termed the *opposite* of the form (a, b, c), and $(-a, -b, -c)$ the *negative* of (a, b, c): thus $(-a, b, -c)$ is the negative of the opposite of (a, b, c).

(i.) If the equations (T) are resoluble, any form (a, b, c) of determinant D is properly equivalent to the negative of its opposite; viz., (a, b, c) is transformed into $(-a, b, -c)$ by

$$\begin{vmatrix} bU-T, & -cU \\ -aU, & bU+T \end{vmatrix}.$$

Hence the reduced forms $(a_0, \beta_0, -a_1)$, $(-a_0, \beta_0, a_1)$ are properly equivalent; either of them is therefore contained in the period of the other; *i.e.* the two periods are identical.

(ii.) If the two periods (8) and (9) are identical, the form $(-a_0, \beta_0, a_1)$ must occur in the period of $(a_0, \beta_0, -a_1)$; and because its first coefficient is negative, it must occupy an even place in that period. Hence the period of complete quotients (7) consists of an uneven number of terms; and we infer from the formulæ (Q) of Art. 3 that the equations (T) are resoluble..

6. If $D \equiv 1$, mod 8, we have $h=h_1$; if $D \equiv 5$, we have $h=3h_1$ when τ_1 and

v_1 are even, but $h = h_1$ when τ_1 and v_1 are uneven, in which case

$$(\tfrac{1}{2}\tau_1 + v_1\sqrt{D})^3 = \tau + v\sqrt{D}.$$

We thus find, in both cases alike,

$$\Pi\Big[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\Big]^{\psi(Q)} = \Pi\Big[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\Big]^{\sigma\psi_1(Q)}, \quad \ldots\ldots \quad (12)$$

where $\sigma = 1$, or $= 3$, according as $D \equiv 1$, or $\equiv 5$, mod 8.

Again, since, by the formulæ of Lejeune Dirichlet, we have (see the Report cited, Art. 101*)

$$h \log[\tau + v\sqrt{D}] = 2\sqrt{D}\,\Sigma\Big(\frac{D}{n}\Big)\frac{1}{n},$$

where the sign of summation extends to all numbers prime to $2D$, and $\left(\frac{D}{n}\right)$ is the generalised symbol of quadratic reciprocity, we obtain

$$2\sqrt{D}\,\Sigma\Big(\frac{D}{n}\Big)\frac{1}{n} = \Sigma\psi(Q)\log\frac{\sqrt{D}+Q}{\sqrt{D}-Q},$$

or

$$\Sigma\Big(\frac{D}{n}\Big)\frac{1}{n} = \Sigma\psi(Q)\Big[\frac{Q}{D} + \frac{Q^3}{3D^2} + \frac{Q^5}{5D^3} + \ldots\Big] \quad \ldots\ldots \quad (13)$$

Similarly, from the formulæ (see *ibid.*)

$$h_1 \log[\tfrac{1}{2}\tau_1 + \tfrac{1}{2}v_1\sqrt{D}] = \frac{2}{\sigma}\sqrt{D}\,\Sigma\Big(\frac{D}{n}\Big)\frac{1}{n},$$

where σ has the same meaning as before, we infer

$$\Sigma\Big(\frac{D}{n}\Big)\frac{1}{n} = \sigma\Sigma\psi_1(Q)\Big[\frac{Q}{D} + \frac{Q^3}{3D^2} + \frac{Q^5}{5D^3} + \ldots\Big]. \quad \ldots\ldots \quad (14)$$

It is probable that a direct demonstration of the equations (12), (13), (14), of which any two involve the third, would offer considerable difficulties.

Addition to the preceding Note.

7. As the preceding determination (equations 5 and 6) of the number of non-equivalent classes for a positive determinant depends on the equations (a) and (b), which assign the relation between the number of periods of complete quotients and the number of periods of reduced forms, it is worth while, for the sake of distinctness, to describe fully the characteristic appearances presented by these periods in certain special cases which are of some importance.

Every form, or class of forms, is, of course, properly equivalent to itself, and improperly equivalent to its opposite. But a form, or class of forms, may be—

(i.) Properly equivalent to its opposite, and improperly equivalent to itself (in this case the class is ambiguous);

* Vol. i. p. 217.

(ii.) Properly equivalent to its negative, and improperly equivalent to the negative of its opposite;

(iii.) Properly equivalent to the negative of its opposite, and improperly equivalent to its negative.

Since, if any two of these specialities coexist, they necessarily involve the third, there are four cases to be considered, viz., the cases (i.), (ii.), (iii.), in which the specialities (i.), (ii.), (iii.) exist singly, and the case (iv.) in which they all exist simultaneously. We shall briefly refer to each of these cases in succession. We may observe, however, that the case (i.), which is that of an ambiguous class, has been fully considered by Gauss ('Disq. Arith.', Art. 187, Obs. 6, 7, 8); of the rest, the case (ii.) has, perhaps, attracted less attention than might have been expected.

8. If the period of reduced forms equivalent to (a, b, c) is

$$(a_0, \beta_0, -a_1), \quad (-a_1, \beta_1, a_2), \ \ldots, \ (-a_{2k-1}, \beta_{2k-1}, a_0), \ . \ . \ . \ . \quad (8)$$

the periods of reduced forms equivalent to $(a, -b, c)$, $(-a, -b, -c)$, $(-a, b, -c)$ are respectively

$$(a_0, \beta_{2k-1}, -a_{2k-1}), \quad (-a_{2k-1}, \beta_{2k-2}, a_{2k-2}), \ \ldots, \ (-a_1, \beta_0, a_0), \ . \ . \quad (10)$$

$$(-a_0, \beta_{2k-1}, a_{2k-1}), \quad (a_{2k-1}, \beta_{2k-2}, -a_{2k-2}), \ \ldots, \ (a_1, \beta_0, -a_0), \ . \ . \quad (11)$$

$$(-a_0, \beta_0, a_1), \quad (a_1, \beta_1, -a_2), \quad \ldots, \ (a_{2k-1}, \beta_{2k-1}, -a_0). \quad (9)$$

As in Art. 3, we designate the period (7) of complete quotients, or, which is the same thing, the period formed by the positive roots of the equation (Ω), by

$$u_0, \ u_1, \ \ldots, \ u_{2k-1}.$$

The negative roots of the same equations we represent by

$$-\frac{1}{v_0}, \quad -\frac{1}{v_1}, \ \ldots, \ -\frac{1}{v_{2k-1}}:$$

so that, if $\mu_0, \mu_1, \ldots, \mu_{2k-1}$ is the period of integral quotients, we have, by a well-known theorem,

$$\mu_s = Iu_s = Iv_{s+1},$$

the symbol Ix denoting the greatest integral number not surpassing x. The four periods of forms (8), (10), (11), (9) we represent for brevity by the symbols

$$\begin{array}{llll} \phi_0, & \phi_1, & \phi_2, & \ldots, \ \phi_{2k-1}, \ . \ . \ . \ . \ . \ . \ . \ . \ . \ \Phi \\ \psi_{2k-1}, & \psi_{2k-2}, & \psi_{2k-3}, & \ldots, \ \psi_0, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ \Psi \\ \bar{\psi}_{2k-1}, & \bar{\psi}_{2k-2}, & \bar{\psi}_{2k-3}, & \ldots, \ \bar{\psi}_0, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ \bar{\Psi} \\ \bar{\phi}_0, & \bar{\phi}_1, & \bar{\phi}_2, & \ldots, \ \bar{\phi}_{2k-1}, \ . \ . \ . \ . \ . \ . \ . \ . \ . \ \bar{\Phi} \end{array}$$

Two such forms as (a, b, c), (c, b, a) are said to be *associated*; thus, ϕ_s and ψ_s, or again $\bar{\phi}_s$ and $\bar{\psi}_s$, are associated forms. The periods Φ and Ψ, and again the periods $\bar{\Phi}$ and $\bar{\Psi}$, are themselves termed associated periods.

The period of complete quotients corresponding to the periods Φ and $\bar{\Phi}$ is (Art. 5)

$$u_0, u_1, \ldots, u_{2k-1};$$

and similarly the period of complete quotients corresponding to the periods Ψ and $\bar{\Psi}$ is

$$v_{2k-1}, v_{2k-2}, \ldots, v_1, v_0.$$

9. Case (i.)—If (a, b, c) is properly equivalent to $(a, -b, c)$, the periods Φ and Ψ must coincide; *i.e.* we must have, for some value of σ, $\phi_0 = \psi_{2\sigma+1}$, the suffix $2\sigma+1$ being uneven, because the extreme coefficients of $\psi_{2\sigma+1}$ must have the same signs as the extreme coefficients of ϕ_0. Hence $\phi_1 = \psi_{2\sigma}$, $\phi_2 = \psi_{2\sigma-1}$, $\ldots$; and finally $\phi_\sigma = \psi_{\sigma+1}$, $\phi_{\sigma+1} = \psi_\sigma$, or there occur in the period two consecutive forms of which each is the associate of the other. As we may begin the period with any form we please, we may suppose that ϕ_0 and ϕ_1 are these two consecutive forms, so that $\beta_1 = \beta_0$, $a_2 = a_0$, $2\beta_0 \equiv 0$, mod a_1. It will be seen that, if $\phi_0 = \psi_{2\sigma+1}$, we have not only $\phi_\sigma = \psi_{\sigma+1}$, $\phi_{\sigma+1} = \psi_\sigma$, but also

$$\phi_{\sigma+k} = \psi_{\sigma+1-k} = \psi_{\sigma+k+1}, \quad \phi_{\sigma+k+1} = \psi_{\sigma+k}.$$

Thus a sequence of two associated forms occurs twice in the period; and, assuming (as we have done) that $\sigma = 0$, the period of forms is of the type

$$\psi_1, \phi_1; \quad \phi_2, \phi_3, \ldots, \phi_{k-1}; \quad \phi_k, \psi_k; \quad \psi_{k-1}, \psi_{k-2}, \ldots, \psi_2,$$

where $\phi_{s+1} = \psi_{2k-s}$ for every value of s. The period of integral quotients is of the type

$$\lambda_0, \mu_1, \mu_2, \ldots, \mu_{k-1}; \quad \lambda_k, \mu_{k-1}, \mu_{k-2}, \ldots, \mu_1,$$

where

$$\lambda_0 = \frac{2\beta_0}{a_1}, \quad \lambda_k = \frac{2\beta_k}{a_{k+1}}.$$

The period of complete quotients is of the type

$$v_1, u_1; \quad u_1, u_3, \ldots, u_{k-1}; \quad u_k, v_k; \quad v_{k-1}, v_{k-2}, \ldots, v_2,$$

where $u_{s+1} = v_{2k-s}$ for every value of s. The periods of the coefficients a and β are respectively of the types

$$a_1; \quad a_2, \ldots, a_k; \quad a_{k+1}; \quad a_k, \ldots, a_2;$$
$$\beta_1, \beta_1; \quad \beta_2, \ldots, \beta_{k-1}; \quad \beta_k, \beta_k; \quad \beta_{k-1}, \ldots, \beta_2.$$

The two *ambiguous* forms are ϕ_1 and ψ_k.

10. Case (ii.)—If (a, b, c) is properly equivalent to $(-a, -b, -c)$, the periods Φ and $\bar{\Psi}$ must coincide. Hence we must have (for some even value of the suffix 2σ) $\phi_0 = \bar{\psi}_{2\sigma}$; whence $\phi_\sigma = \bar{\psi}_\sigma$, and also $\phi_{\sigma+k} = \bar{\psi}_{\sigma-k} = \bar{\psi}_{\sigma+k}$. The

equation $\phi_\sigma = \bar{\psi}_\sigma$ is equivalent to the equation $a_\sigma = a_{\sigma+1}$; we thus see that the period contains two forms, in each of which the extreme coefficients are equal in absolute magnitude; so that (supposing, as before, that $\sigma = 0$), we have

$$D = \beta_0^2 + a_0^2, \quad D = \beta_k^2 + a_k^2.$$

The period of reduced forms is of the type

$$\bar{\psi}_0 = \phi_0; \quad \phi_1, \phi_2, \ldots, \phi_{k-1}; \quad \phi_k = \bar{\psi}_k; \quad \bar{\psi}_{k-1}, \bar{\psi}_{k-2}, \ldots, \psi_1,$$

where $\phi_s = \bar{\psi}_{2k-s}$; the period of complete quotients is of the type

$$u_0 = v_0; \quad u_1, u_2, \ldots, u_{k-1}; \quad u_k = v_k; \quad v_{k-1}, v_{k-2}, \ldots, v_1,$$

where $u_s = v_{2k-s}$.

Lastly, the periods of integral quotients and of the coefficients α and β are of the types

$$\mu_0, \mu_1, \ldots, \mu_{k-1}; \quad \mu_{k-1}, \mu_{k-1}, \ldots, \mu_1, \mu_0;$$

$$a_1, a_2, \ldots, a_{k-1}, a_k, a_k, a_{k-1}, \ldots, a_2, a_1;$$

$$\beta_0; \quad \beta_1, \beta_2, \ldots, \beta_{k-1}; \quad \beta_k; \quad \beta_{k-1}, \beta_{k-2}, \ldots, \beta_1.$$

11. Case (iii.)—If (a, b, c) is properly equivalent to $(-a, b, -c)$, the periods Φ and $\bar{\Phi}$ coincide, and we must have $\phi_0 = \bar{\phi}_{2\sigma+1}$, $\bar{\phi}_0 = \phi_{2\sigma+1}$ where $2\sigma+1$ is less than $2k$. From these equations we infer $\bar{\phi}_0 = \bar{\phi}_{2(2\sigma+1)}$, or $2\sigma+1 = k$, since if $\bar{\phi}_0 = \bar{\phi}_m$, m is a multiple of $2k$. The period of forms is therefore of the type

$$\phi_0, \phi_1, \ldots, \phi_{k-1}, \bar{\phi}_0, \bar{\phi}_1, \ldots, \bar{\phi}_{k-1},$$

k being an uneven number; the periods of complete quotients, of integral quotients, and of the coefficients α and β, consist each of a period of k terms, twice repeated. The period of equations (Ω) in like manner consists of a period of k equations, twice repeated; but each equation appears in the second half of the period with its sign changed.

12. Case (iv.)—If (a, b, c) is properly equivalent to any two of the forms $(a, -b, c)$, $(-a, -b, -c)$, $(-a, b, c)$, and therefore to all three of them, the nature of the periods is most readily ascertained by considering the series of integral quotients. Since the conditions characteristic of the cases (i.) and (iii.) must be united, the semi-period

$$\lambda_0, \mu_1, \mu_2, \ldots, \mu_{k-1}$$

must be term for term identical with the semi-period

$$\lambda_k, \mu_{k-1}, \mu_2, \ldots, \mu_1,$$

k being an uneven number $2i+1$. Hence $\lambda_0 = \lambda_k$, and the period is of the type

$$\lambda, \mu_1, \mu_2, \ldots, \mu_i, \mu_i, \ldots, \mu_2, \mu_1,$$

twice repeated; which combines the characters of the periods of integral quotients in the cases (i.), (ii.), (iii.).

The period of forms is of the type

$$\psi_1, \phi_1;\ \phi_2, \phi_3, \ldots, \phi_{i+1}, \bar{\psi}_i, \bar{\psi}_{i-1}, \ldots, \bar{\psi}_2;$$
$$\bar{\psi}_1, \bar{\phi}_1;\ \bar{\phi}_2, \bar{\phi}_3, \ldots, \bar{\phi}_{i+1}, \psi_i, \psi_{i-1}, \ldots, \psi_2;$$

where $$\phi_s = \bar{\psi}_{2i+2-s} = \bar{\phi}_{s-2i-1} = \psi_{4i+3-s};$$

the period of complete quotients is of the type

$$v_1, u_1;\ u_2, u_3, \ldots, u_{i+1};\ v_i, v_{i+1}, \ldots, v_2;$$

twice repeated, where $u_s = v_{2i+2-s}$; and the periods of the coefficients α and β are respectively

$$a_1;\ a_2, a_3, \ldots, a_{i+1}, a_{i+1}, a_i, \ldots, a_2;$$
$$\beta_1, \beta_1;\ \beta_2, \ldots, \beta_i, \beta_{i+1}, \beta_i, \ldots, \beta_2;$$

each of them twice repeated.

13. If therefore we develope the two roots of a given primitive quadratic equation, we obtain, in the general case, two distinct associated periods of complete quotients and four distinct periods of reduced forms. In the special cases (i.) and (ii.), we have but one period of complete quotients and two periods of reduced forms; in case (i.) the two associated periods of each pair combine; in case (ii.) each period of reduced forms becomes identical with the negative of the opposite of its associated period. In case (iii.) we have two distinct periods of complete quotients; but only two distinct periods of reduced forms; the period of any form being identical with the period of the opposite of its negative, and consisting of an uneven number of forms followed by the opposites of the negatives of the same forms; the period of complete quotients contains only half as many terms as the period of reduced forms. Lastly, in case (iv.) we have but one period of complete quotients, and but one period of reduced forms, the four periods, which in the general case are distinct, being all identical with one another.

We may observe that, if the equation

$$\nu a - 2\mu b + \lambda c = 0 \quad \ldots\ldots\ldots\ldots \quad (p)$$

can be satisfied by three numbers λ, μ, ν, which also satisfy the condition

$$\mu^2 - \lambda\nu = 1, \quad \ldots\ldots\ldots\ldots \quad (q)$$

the form (a, b, c) is transformed into $(a, -b, c)$ by the substitution $\begin{vmatrix} \mu, & -\nu \\ -\lambda, & \mu \end{vmatrix}$, and consequently has a period of the type (i.). If, instead of the condition (q), the condition

$$\mu^2 - \lambda\nu = -1 \quad \ldots\ldots\ldots\ldots \quad (r)$$

is satisfied by the three numbers (λ, μ, ν), (a, b, c) is transformed into $(-a, -b, -c)$ by $\begin{vmatrix} \mu, & \nu \\ -\lambda, & -\mu \end{vmatrix}$ and the period of (a, b, c) is of the type (ii.). Lastly, if the equation (p) can be satisfied by two different sets of numbers, of which one set satisfies the condition (q) and the other the condition (r), the period of (a, b, c) is of the type (iv.).

14. The periods which we have to consider in the cases (i), (ii.) and (iv.) afford examples of each of the three kinds of symmetry which can exist in a periodic series. Let $\ldots c_0, c_1, c_2, \ldots c_{n-1}, \ldots$ be a period of n terms repeated indefinitely in both directions; it will be found that the series thus formed may be symmetrical (*i.e.* may be the same whether we follow it forwards or backwards) in three, and only in three, different ways.

(i.) Let n be even; and let the series continued from c_0 forwards coincide with the series continued from c_{2k+1} backwards, so that $c_0 = c_{2k+1}$, $c_1 = c_{2k}$, $\ldots$: the period then is

$$c_0, c_1, \ldots, c_k;\ c_k, \ldots, c_0, c_{2k+2}, \ldots, c_{n+1};$$

or, if $n = 2\nu$,

$$c_{-\nu+k+1}, c_{-\nu+k+2}, \ldots, c_0, c_1, \ldots, c_k,\ c_k, \ldots, c_0, c_{-1}, \ldots, c_{-\nu+k+1};\ \ldots \quad (A)$$

where there are two centres of symmetry, one falling between the two terms c_k, the other falling between the two terms $c_{-\nu+k+1}$.

Of this type is the period of the coefficients β in case (i.); and the periods of the integral quotients and of the coefficients α in case (ii.).

(ii.) Let n still be even, but let the series continued forwards from c_0 coincide with the series continued backwards from c_{2k}; the period is

$$c_0, c_1, \ldots, c_{k-1},\ c_k, c_{k-1}, \ldots, c_0,\ c_{2k+1}, \ldots, c_{n-1},$$

or, if $n = 2\nu$,

$$c_{-\nu+k};\ c_{-\nu+k+1}, \ldots, c_0, c_1, \ldots, c_{k-1};\ c_k;\ c_{k-1}, \ldots, c_0, \ldots, c_{-\nu+k+1}, \quad \ldots \quad (B)$$

where there are two centres of symmetry falling on the terms $c_{-\nu+k}$ and c_k respectively. The symmetry of the integral quotients, and of the coefficients α in case (i.), and of the coefficients β in case (ii.) is of this type.

(iii.) Let $n = 2\nu + 1$ be uneven; and let the series continued forward from c_0 coincide with the series continued backward from c_{2k+1}. The period is of the type

$$c_0, c_1, \ldots, c_k,\ c_k, \ldots, c_0,\ c_{2k+2}, \ldots, c_{n-1},$$

or

$$c_{-\nu+k};\ c_{-\nu+k+1}, c_{-\nu+k+2}, \ldots, c_0, c_1, \ldots, c_k, c_k, \ldots, c_0, c_{-1}, \ldots, c_{-\nu+k+1},$$

where again there are two centres of symmetry, one falling on the term $c_{-\nu+k}$, the other between the two terms c_k. If we had supposed $c_0 = c_{2k}$ we should have obtained a period of the same form. It is evident that, if this period be doubled,

it combines the symmetries of the periods (A) and (B). Of this type are the periods of integral quotients and of the coefficients α and β in case (iv.) In case (iii.) there is no symmetry; but an unsymmetrical uneven period is twice repeated.

15. Every determinant has ambiguous classes; and every ambiguous class has a period of the type (i.). But developments of the types (ii.), (iii), (iv.) can only present themselves in the case of determinants of the form P or $2P$, where P is a product of uneven prime numbers of the form $4n+1$. For in case (ii.) D must, as we have seen, be the sum of two square numbers prime to one another, and in case (iii.) the equation $T^2 - DU^2 = -1$ must be resoluble, whence again D is the sum of two squares prime to one another.

If μ is the number of different primes dividing P, the number of ways in which P can be decomposed into the sum of an even and uneven square prime to one another is $2^{\mu-1}$. Let $D=P$, and let $D=A^2+B^2$ be one of these decompositions, A being uneven and B even; the forms $(-A, B, A)$, $(A, B, -A)$, $(-B, A, B)$, $(B, A, -B)$ are all reduced, and the first two are properly, the last two improperly, primitive. We thus have $2^{\mu-1}$ properly primitive periods of reduced forms of the type (ii.); and as many improperly primitive periods of the same type; *i.e.* since there are $2^{\mu-1}$ properly and as many improperly primitive ambiguous classes, there are as many classes having periods of the type (ii.) as there are classes having periods of the type (i.).

If $D=2P$, we have a similar result; viz., there are $2^{\mu-1}$ equations of the form $D=2P=A^2+B^2$, in which A and B are both uneven. We thus obtain 2^{μ} properly primitive periods of reduced forms of the type (ii.); *i.e.* as many as there are of type (i.) There are, of course, no improperly primitive classes of a determinant of the form $2P$.

When the equations (T) are not resoluble, but the determinant is of either of the forms P or $2P$, the developments of the type (i.) and those of the type (ii.) are entirely distinct from one another. On the other hand, when the equations (T) are resoluble, the developments of the types (i.) and (ii.) coincide, giving rise to developments of the type (iv.), and all the remaining developments are of the type (iii.).

It is known that, when D is an uneven power of an uneven prime of the form $4n+1$, the equations (T) are always resoluble. But when D has any other value of either of the forms P or $2P$, there is no known criterion for deciding whether these equations are or are not resoluble.

XXX.

ON THE VALUE OF A CERTAIN ARITHMETICAL DETERMINANT.

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LET (m, n) denote the greatest common divisor of the integral numbers m and n; and let $\psi(m)$ be the number of numbers not surpassing m and prime to m; the symmetrical determinant

$$\Delta_m = \Sigma \pm (1, 1)(2, 2) \dots (m, m)$$

is equal to

$$\psi(1) \times \psi(2) \times \dots \times \psi(m).$$

This theorem may be established as follows. Let p_1, p_2, ... be all the different primes dividing m, and consider the columns (P) of which the indices are

$$m, \quad \frac{m}{p_1}, \quad \frac{m}{p_2}, \quad \dots, \quad \frac{m}{p_1 p_2}, \quad \dots, \quad \frac{m}{p_1 p_2 p_3}, \dots.$$

Take these columns with the signs of the corresponding terms in the product

$$\psi(m) = m \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots;$$

and, attending to these signs, replace the terms of the last column of Δ_m by the sum of the corresponding terms in the columns (P). The value of Δ_m is not changed: the term (m, m) is evidently replaced by $\psi(m)$; and we shall now show that every other term (m, k) in the last column is replaced by zero; *i.e.* that $\Delta_m = \psi(m) \times \Delta_{m-1}$, which is the theorem to be proved.

First, let k be prime to m; then

$$(m, k) = 1, \quad \left(\frac{m}{p}, k\right) = 1, \quad \&c.;$$

and (m, k) has to be replaced by a sum of units, of which as many are negative as positive; *i.e.* by zero.

Secondly, let k be a divisor of m, other than unity or m itself; and let us separate the primes p into two classes, q and r, in such a manner that k does not divide any quotient of the form $\frac{m}{q}$, but does divide every quotient of the form $\frac{m}{r}$. There may or may not be any primes q, but there must be at least one prime r, or we should have $k = m$: we further observe that

$$(m, k) = \left(\frac{m}{r}, k\right) = \left(\frac{m}{r_1 r_2}, k\right) = \ldots = k;$$

$$\left(\frac{m}{q}, k\right) = \frac{k}{q}, \quad \left(\frac{m}{q_1 q_2}, k\right) = \frac{k}{q_1 q_2}, \ldots.$$

Thus, if we were to attend only to those columns of which the indices are $m, \frac{m}{q}, \ldots, \frac{m}{q_1 q_2}, \ldots$, we should have to replace (m, k), or k, by $k\,\Pi\left(1 - \frac{1}{q}\right)$, just as before we replaced m by $\psi(m)$. But we have to attend to the complete series of columns (P); and thus we have to replace (m, k), not by $k\Pi\left(1 - \frac{1}{q}\right)$ taken once, but by $k\Pi\left(1 - \frac{1}{q}\right)$ taken as often as there are terms in the product $\Pi\left(1 - \frac{1}{r}\right)$, and taken each time with the sign proper to the corresponding term of that product; *i. e.* (m, k) is replaced by zero.

Lastly, let $k = h\delta$, δ being the greatest common divisor of k and m; so that $(m, k) = (m, \delta) = \delta$. If d is any divisor of m, we have the elementary theorem $\left(\frac{m}{d}, \delta\right) = \left(\frac{m}{d}, h\delta\right)$. For, if $\left(\frac{m}{d}, \delta\right) = \delta'$, we have also $(m, d\delta) = d\delta'$; and hence δ, which is a common divisor of m and $d\delta'$, divides $d\delta'$, which is the greatest common divisor of those two numbers. But h is prime to $\frac{m}{\delta}$; therefore, *à fortiori*, h is prime to $\frac{m}{d\delta'}$; *i. e.* $\left(\frac{m}{d\delta'}, h\frac{\delta}{\delta'}\right) = 1$, for $\frac{m}{d\delta'}$ is prime to $\frac{\delta}{\delta'}$ as well as to h, or, which is the same thing, $\left(\frac{m}{d}, h\delta\right) = \delta' = \left(\frac{m}{d}, \delta\right)$. It appears from this that in the columns (P) the terms which lie in the row of which the index is $h\delta$, are precisely the same as the terms which lie in the row of which the index is δ; and hence $(m, h\delta)$ is replaced by terms of which the sum is zero, because (m, δ) is replaced by terms of which the sum is zero.

The following remarks are suggested by the preceding theorem, or by its demonstration :

(1) If we denote by Ia the greatest integer not surpassing the positive quantity a, the theorem may be expressed by the equation

$$\frac{\Delta_m}{1 \cdot 2 \cdot 3 \ldots m} = \Pi \left(1 - \frac{1}{p}\right)^{I\frac{m}{p}}, \quad \ldots\ldots\ldots \quad (A)$$

the sign of multiplication extending to all primes not surpassing m.

(2) Instead of the greatest common divisors themselves, we may consider their powers of exponent s ; writing $\Psi_s(m) = m^s \Pi \left(1 - \frac{1}{p^s}\right)$, and following the same course of demonstration, we obtain the theorem

$$\begin{aligned} \Delta_{m,s} &= \Sigma \pm (1, 1)^s (2, 2)^s \ldots (m, m)^s \\ &= \Psi_s(1) \times \Psi_s(2) \times \Psi_s(3) \times \ldots \times \Psi_s(m), \quad \ldots\ldots \quad (B) \end{aligned}$$

or

$$\frac{\Delta_{m,s}}{(1 \cdot 2 \cdot 3 \ldots m)^s} = \Pi \left(1 - \frac{1}{p}\right)^{I\frac{m}{p}};$$

from which we infer, as a particular result,

$$\frac{\Delta_{m,2}}{\Delta_{m,1}} = 1 \cdot 2 \cdot 3 \ldots m \times \Pi \left(1 + \frac{1}{p}\right)^{I\frac{m}{p}}.$$

(3) The equation (B) is an identity with respect to the exponent s, which may have any value whatever : the case in which $s = -1$ is especially interesting. Let $[m, n]$ be the least common multiple of m and n, so that $[m, n] = \frac{m \times n}{(m, n)}$; we find

$$\begin{aligned} \nabla_m &= \Sigma \pm [1, 1][2, 2] \ldots [m, m] \\ &= 1 \cdot 2 \cdot 3 \ldots m \cdot \Pi (1 - p)^{I\frac{m}{p}}, \end{aligned}$$

whence

$$\frac{\nabla_m}{\Delta_m} = \pm \Pi \cdot p^{I\frac{m}{p}};$$

and, in general, if

$$\nabla_{m,s} = \Sigma \pm [1, 1]^s \ldots [m, m]^s,$$

$$\frac{\nabla_{m,s}}{\Delta_{m,s}} = \pm (\Pi \cdot p^{I\frac{m}{p}})^s :$$

the sign being that of $(-1)^{\Sigma I\frac{m}{p}}$.

(4) If, for the greatest common divisor δ of m and n, we substitute any function whatever $\phi(\delta)$ of δ, and denote by $\Phi(m)$ the function

$$\phi(m) - \Sigma \phi\left(\frac{m}{p}\right) + \Sigma \phi\left(\frac{m}{p_1 p_2}\right) - \ldots,$$

we arrive at the identity

$$\Sigma \pm \phi(1,1)\,\phi(2,2)\ldots\phi(m,m) = \Phi(1)\times\Phi(2)\times\Phi(3)\times\ldots\times\Phi(m).$$

Two particular cases are worth attention.

(α) Let s be an integral number, and let

$$\phi_s(m) = 1^s + 2^s + 3^s + \ldots + (m-1)^s + m^s,$$

so that, when $s > 1$,

$$\phi_s(m) = \frac{m^{s+1}}{s+1} + \tfrac{1}{2}m^s + B_1\frac{s}{1\,.\,2}m^{s-1} - B_3\frac{s(s-1)(s-2)}{1\,.\,2\,.\,3\,.\,4}m^{s-3} + \ldots,$$

$B_1, B_3, \ldots$ being the fractions of Bernoulli, and the last term being

$$(-1)^{\frac{s-3}{2}}B_{s-2}\frac{s}{2}m^2, \text{ or } (-1)^{\frac{s-2}{2}}B_{s-1}m,$$

according as s is uneven or even. Let also $\psi_s(m)$ be the sum of the powers s of the numbers prime to m and not surpassing m; we shall have

$$\Sigma \pm \phi_s(1,1)\,\phi_s(2,2)\ldots\phi_s(m,m) = \psi_s(1)\times\psi_s(2)\times\psi(3)\times\ldots\times\psi_s(m,m).$$

The forms of the functions $\psi_s(m)$ are deducible from the expression for $\phi_s(m)$ (see a paper by Mr. Thacker in 'Crelle,' Vol. XL., p. 89); we thus find

$$\psi_1(m) = \tfrac{1}{2}m^2\Pi\left(1-\frac{1}{p}\right),\quad \psi_2(m) = \tfrac{1}{3}m^3\Pi\left(1-\frac{1}{p}\right) + \tfrac{1}{6}m\Pi(1-p),$$

the general rule being that, in order to obtain $\psi_s(m)$, we are to substitute $m^k\Pi(1-p^{n-k})$ for m^k in the expression for $\phi_s(m)$.

(β) Let $\sigma_s(m)$ be the sum of the powers s of the divisors of m; it will be found that

$$\sigma_s(m) - \Sigma\sigma_s\left(\frac{m}{p}\right) + \Sigma\sigma_s\left(\frac{m}{p_1p_2}\right) - \ldots = m^s.$$

For, if $m = p_1^{\alpha_1}p_2^{\alpha_2}\ldots$, and if we put

$$P = (1 + p^s + \ldots + p^{\alpha s}),$$

$$P' = (1 + p^s + \ldots + p^{(\alpha-1)s}),$$

we have

$$\sigma_s(m) = P_1\times P_2\times\ldots,$$

$$\sigma_s\left(\frac{m}{p_1}\right) = P_1'\times P_2\times\ldots,$$

$$\sigma_s\left(\frac{m}{p_1p_2}\right) = P_1'\times P_2'\times P_3\ldots,$$

whence

$$\sigma_s(m) - \Sigma\sigma_s\left(\frac{m}{p}\right) + \Sigma\sigma_s\left(\frac{m}{p_1p^2}\right) - \ldots = (P_1 - P_1')(P_2 - P_2')\ldots = m^s.$$

We thus obtain the equation

$$\Sigma \pm \sigma_s(1,1)\,\sigma_s(2,2)\,\sigma_s(3,3)\ldots\sigma_s(m,m) = (1\,.\,2\,.\,3\ldots m)^s,$$

in which s may be any quantity whatever: the cases in which $s=0$, $s=-1$, $s=+1$, are equally remarkable.

(5) Returning to the equation

$$\begin{aligned}\Delta_m &= \Sigma \pm (1,1)(2,2)\dots(m,m)\\ &= \psi(1)\times\psi(2)\times\dots\times\psi(m),\end{aligned}$$

we may observe that it is by no means necessary that the numbers 1, 2, 3, ..., m should be the natural series of numbers. We may, in fact, take any different numbers $\mu_1, \mu_2, \dots, \mu_m$ that we please, subject only to the condition that, if μ be any one of these numbers, every divisor of μ must also appear among them, a condition which implies that unity is always one of the numbers μ. Subject to this condition, we have always

$$\Sigma \pm (\mu_1,\mu_1)(\mu_2,\mu_2)\dots(\mu_m,\mu_m) = \psi(\mu_1)\times\psi(\mu_2)\dots\psi(\mu_m),$$

or, more generally (see 4 *supra*),

$$\Sigma \pm \phi(\mu_1,\mu_1)\,\phi(\mu_2,\mu_2)\dots\phi(\mu_m,\mu_m) = \Phi(\mu_1,\mu_1)\Phi(\mu_2,\mu_2)\dots\Phi(\mu_m,\mu_m).$$

The most obvious cases are—(α) when we reject the multiples of given primes; *e.g.*, when the numbers μ are the uneven numbers in their natural order; (β) when we consider only numbers composed with given primes, *e.g.* when the numbers are all the divisors of one of them μ; (γ) when we consider only *linear* numbers, *i.e.* numbers not divisible by any square. In all these cases the results are immediately obtained by the methods which we have already used, and which it is unnecessary to exemplify further.

(6) Lastly, the symbols μ need not represent integral numbers at all, but may be any quantities which admit of resolution into factors in a definite manner. If, for example, $\alpha^{ij} = x^{i-1}$ or x^{j-1} according as $i<j$, or $j<i$, we have

$$\begin{aligned}\Sigma \pm x_{11}, \alpha_{22}, \dots, \alpha_{mm} &= (x^{m-1}-x^{m-2})(x^{m-2}-x^{m-3})\dots(x-1)\\ &= (x-1)^{m-1}x^{\frac{1}{2}(m-1)(m-2)}.\end{aligned}$$

XXXI.

ON THE PRESENT STATE AND PROSPECTS OF SOME BRANCHES OF PURE MATHEMATICS.

[Proceedings of the London Mathematical Society, vol. viii. pp. 6–29. Read November 9, 1876.]

I HAVE been led to believe that the Society may not be unwilling to allow a certain latitude in the scope of the remarks which they permit their Presidents to address to them upon retiring from the Chair. Relying upon this belief, I propose, on the present occasion, to invite your attention to some considerations relating to the present state of Mathematical Science, with especial reference to its cultivation in this country, and to our own position as representing a great number of those who are interested in its advancement. The subject is so extensive that I am sure you will excuse me if I endeavour to limit it in every way I can. I propose, therefore, to exclude from what I have to say all that relates to Applied Mathematics, and to ask you to confine your attention to questions of Pure Mathematics only. I am well aware how much by this exclusion I restrict the field before me; but the restriction is forced upon me, not only by the limit of time, but by the far narrower limits of my own knowledge. And I cannot help adding that I shall regard it as a fortunate circumstance, if the attention of my successor, when he in his turn is looking round him for a subject for his own Presidential Address, should be attracted by a domain, upon which I must myself decline to enter, but of which he, better perhaps than anyone among us, is fitted to take a clear and comprehensive view.

The restriction which I have mentioned is far from being the only one which I must impose upom myself. I can only presume to offer fragmentary remarks upon great subjects, in the hope that even such casual and hasty notices may not be without their use, if they serve to remind us of the vastness of our science, and yet of its unity; of its unceasing development, rapid at the present

time, and promising to be no less rapid in the immediate future; of its marvellous power of assimilating to itself the accessions which each year brings to our knowledge of external nature, while yet it derives strength and vitality from roots which strike far back into the past, so that the organic continuity of its gigantic growth has been preserved throughout.

In every science there is a time and place for general contemplations, as well as for minute investigations. And it is a rule of sound philosophy that neither should be neglected in its proper season. 'Itaque alternandæ sunt istæ contemplationes,' says Lord Bacon, 'et vicissim sumendæ, ut intellectus reddatur simul penetrans et capax.'* Perhaps it is the besetting sin of mathematicians to concentrate the mental vision upon as narrow and definite a field as possible. And there is much to be said in excuse for our indulgence of this tendency. If we are to find anything worth finding in the mines of mathematical research, we must dig deep; and if we want to dig deep, we must, if we are not gifted with Herculean force, confine our efforts to a narrow superficial area. But the tendency is not without its peril. The illustrious mathematician under whose auspices this Society was founded, felt it right in his opening address to warn us against the danger. 'Our subject,' said Augustus De Morgan, on the 15th of January, 1865, with a characteristic irony of expression, 'Our subject is really rather a wide one. But there are mathematical publications in which it is contracted; and it is often treated as a narrow subject.' He cautioned us against falling into 'a line which may be useful, but which is still confined and partial'; and, while exhorting us to do our part in the additions to the more rapidly developing 'branches of the science,' he bid us at the same time take care 'not to let any one particular branch overgrow us.' It would not have been Augustus de Morgan if he had not added some pointed criticisms upon examinations in general, and on Cambridge examinations in particular, and if he had not cautioned us against any excessive admiration for that part of mathematical ingenuity which devotes itself to the narrowest of all the narrow fields ever chosen by a mathematician, the invention and solution of 'ten-minute conundrums.'

It is now nearly twelve years since these warnings were given to the infant Society by its first President; and perhaps the time may have arrived when we might put to ourselves the question whether its subsequent history has shown that we have profited by the lessons of that eminent and large-minded teacher. It would ill become me to attempt to answer such a question. I

* Novum Organon, Lib i., Aph.

would only venture to express, and that with great diffidence, the double opinion—that, on the one hand, the mathematical world will wholly acquit the Society of having devoted its energies to little or trivial subjects; but that, on the other hand, while it would be universally conceded that the volumes of our Proceedings contain memorable additions to mathematical knowledge, it might be alleged by an 'advocatus diaboli' (if such a character should be assumed by some severe critic) that we, in this respect resembling the other mathematicians of our country, have shown, and still continue to show, a certain partiality in favour of one or two great branches of the science, to the comparative neglect and possible disparagement of others. Perhaps it would be well to begin our reply by denying the charge; but, having done so, if we should be advised to urge a second and somewhat contradictory plea, we might with great plausibility rejoin that ours is not a blameable partiality, but a well-grounded preference. So great (we might contend) have been the triumphs achieved in recent times by that combination of the newer algebra with the direct contemplation of space which constitutes the modern geometry—so large has been the portion of these triumphs which is due to the genius of a few great English mathematicians—so vast and so inviting has been the field thus thrown open to research,—that we do well to spend our time and our labour upon a country which has, we might say, been 'prospected' for us, and in which we know beforehand that we cannot fail to obtain results which will repay our trouble, rather than adventure ourselves into regions where, soon after the first step, we should have no beaten tracks to guide us to the lucky spots, and where the daily earnings of the searcher for mathematical treasure are (at the best) but small, and do not always make a great show even after long years of work. Such regions, however, there are in the domain of pure mathematics, and it cannot be for the interest of science that they should be altogether neglected by the rising generation of English mathematicians.

I propose therefore, in the first instance, to direct your attention to some few of these by us comparatively neglected regions; and foremost among them I must name the Theory of Numbers. Of all branches of mathematical enquiry this is the most remote from practical applications; and yet, more perhaps than any other, it has kindled an extraordinary enthusiasm in the minds of some of the greatest mathematicians. We have the examples of Fermat, Euler, Lagrange, Legendre, and Gauss, of Cauchy, Jacobi, Lejeune Dirichlet, and Eisenstein, without mentioning the names of others who have passed away, and of a few who are still living. But, somehow, the practical genius of the English mathe-

matician has in general given a different direction to his pursuits; and it would sometimes seem as if we in England measured the importance of the subject by what we find of it in our text-books of Algebra, or as if we regarded its enquiries as problems of mere curiosity, without a wider scope, and without direct bearing on other branches of mathematics. I might endeavour to remove this impression—if indeed it exists in the minds of any of those who hear me—by enumerating instances in which the advancement of Algebra and of the Integral Calculus appears to depend on the progress of the arithmetic of whole numbers. But, instead of wearying you with the details which would be necessary to make such an enumeration intelligible, I would rather ask you to listen to what is recorded of the most eminent master of this branch of science. 'Gauss,' we are told by his biographer, 'held Mathematics to be the Queen of the Sciences, and Arithmetic to be the Queen of Mathematics.' 'She sometimes condescends'—so spoke the great Astronomer and Physicist—'to render services to Astronomy and the other natural sciences, but under all circumstances the first place is her due.'* In a more serious mood he wrote, 'The higher arithmetic presents us with an inexhaustible storehouse of interesting truths—of truths, too, which are not isolated, but stand in the closest relation to one another, and between which, with each successive advance of the science, we continually discover new and sometimes wholly unexpected points of contact. A great part of the theories of Arithmetic derive an additional charm from the peculiarity that we easily arrive by induction at important propositions, which have the stamp of simplicity upon them, but the demonstration of which lies so deep as not to be discovered until after many fruitless efforts; and even then it is obtained by some tedious and artificial process, while the simpler methods of proof long remain hidden from us.'† Or, again, let the young mathematician, who feels an instinctive liking for arithmetical enquiry, be encouraged by the observation which has been put on record by Jacobi in his brief notice of the life of Göpel, that many of those who have a natural turn for mathematical speculation find themselves in the first instance attracted by the Theory of Numbers.‡

There are three great departments of arithmetic, not, it must be admitted, wholly separable from one another, which seem to me at the present time to offer a very inviting field to the researches of the mathematician. Of these

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* 'Gauss. Zum Gedächtniss.' Von W. Sartorius v. Waltershausen. Leipzig, 1856.

† Preface to Eisenstein's 'Mathematische Abhandlungen.' Berlin, 1849.

‡ Notiz über A. Göpel, 'Crelle's Journal,' vol. xxxv. p. 313.

I will name first the arithmetical theory of homogeneous forms, or quantics, as we in England have now learned to call them. It is worthy of remembrance that some of the most fruitful conceptions of modern algebra had their origin in arithmetic, and not in geometry or even in the theory of equations. The characteristic properties of an invariant, and of a contravariant, appear with distinctness for the first time in the 'Disquisitiones Arithmeticæ';* and in that treatise also the attention of mathematicians was for the first time directed to the study of quantics of any order and of any number of indeterminates.† Again, Eisenstein in the course of his researches on the arithmetical theory of binary cubic forms was led to the discovery of the first covariant ever considered in analysis—the Hessian of the cubic form.‡ But the progress of modern algebra and of modern geometry has far outstripped the progress of arithmetic, and one great problem which arithmeticians have before them at the present time is to endeavour to turn to account for their own science the great results which have been obtained in the sister sciences. How difficult this problem may prove is perhaps best attested by the little progress that has been made towards its complete solution. One or two instances may serve to illustrate the actual position of the enquiry. The algebraical problem of the automorphics of a quadratic form, containing any number of indeterminates, may be regarded as completely solved by a formula due to M. Hermite and Professor Cayley.§ The arithmetical formula

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* 'Disq. Arith.,' Arts. 157, 267, 268, where binary and ternary quadratic forms are considered.

† 'Sed manifesto hoc argumentum' [the theory of binary quadratic forms] 'tanquam sectionem maxime particularem disquisitionis generalissimæ de functionibus algebraicis rationalibus integris homogeneis plurium indeterminatarum et plurium dimensionum considerare . . . possumus.' 'Sufficiat hunc campum vastissimum geometrarum attentioni commendavisse, in quo materiem ingentem vires suas exercendi, arithmeticamque sublimiorem egregiis incrementis augendi invenient.' ('Disq. Arith.,' Art. 266.)

‡ See 'Crelle's Journal,' vol. xxvii. p. 89. It is remarkable that the cubic covariant does not explicitly appear in this paper (December, 1843) or in the note (p. 105) which immediately follows it. This omission is supplied in a subsequent note, dated March 3, 1844 (*ibid.* p. 319). The earliest papers of Boole, who approached the study of linear transformations from a geometrical point of view, belong to the years 1841 and 1843 (Cambridge Mathematical Journal, vol. ii. p. 64 and vol. iii. p. 1 and p. 106). In these papers (of which perhaps only the first should be cited here) covariants do not appear, but the first general theorem of invariance ever enunciated, the theorem of the invariance of the discriminant of any quartic, is distinctly stated and proved.

§ M. Hermite, in 'Crelle's Journal,' vol. xlvii. p. 309, appears to have considered forms of three indeterminates only; his solution was subsequently generalised by Professor Cayley (*ibid.* vol. i. p. 288). See also a later memoir by Professor Cayley 'On the Automorphic Linear Transformation of a Bipartite Quadric Function,' in the 'Philosophical Transactions' for 1858.

which gives the automorphics of a binary quadratic form has long formed a part of the elements of the Theory of Numbers; and the corresponding investigation for an indefinite ternary quadratic form may be now regarded as completed by the memoirs in which M. Paul Bachmann has followed up the earlier researches of M. Hermite.* Again, the problem of the equivalence of two positive or definite ternary quadratic forms was completely solved by Seeber; and the problem of the arithmetical automorphics of such forms, by Eisenstein.† The corresponding but far more difficult problem of equivalence for indefinite ternary forms has received its first solution only in very recent times from M. Eduard Selling;‡ and perhaps it is not too much to hope that these profound researches may receive some further development from their distinguished author, and may be brought into closer relation with other parts of arithmetical and algebraical theory. So far, then, as binary and ternary quadratic forms are concerned, we have not much reason to complain of the slowness of the advances made by arithmetic. But if we pass to quadratic forms of four or more indeterminates, we shall find that the limits within which our arithmetical knowledge is confined are indeed restricted. The fundamental theorem of M. Hermite, that the number of non-equivalent classes of quadratic forms having integral coefficients and a given discriminant is finite, and the recent researches of M. Zolotareff and Korkine on the minima of positive quadratic forms, mark the extremest limit to which enquiry has been pressed in this direction.§

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* The solution of the problem is made to depend on the solution in integral numbers of the indeterminate equation $p^2+F(q_1, q_2, q_3)=1$, where F is the contravariant of the given ternary form. No general method, however, of obtaining the solutions of this equation has as yet been given. (See M. Hermite in the memoir already cited, 'Crelle,' vol. xlvii. p. 307 *sqq.*; M. Bachmann, 'Borchardt,' vol. lxxi. p. 296, and vol. lxxvi. p. 331, together with the note by M. Hermite completing his former solution, *ibid.* vol. lxxviii. p. 325.)

† See L. Seeber, 'Untersuchungen ueber die Eigenschaften der positiven ternären quadratischen Formen,' Freiburg, 1831; and, in connection with this work, the review of it by Gauss (in the Göttingen 'Gelehrte Anzeige' for 1831; or in 'Crelle,' vol. xx. p. 312; or in the collected edition of Gauss' Works, vol. ii. p. 188), and the subsequent and simpler investigations of Dirichlet ('Crelle,' vol. xl. p. 209), and M. Hermite (*ibid.* p. 173). The theory of the automorphics of positive ternary quadratic forms is given by Eisenstein in the Appendix to his 'Table of Reduced Positive Ternary Quadratic Forms.' ('Crelle,' vol. xli. p. 227.) He observes (see the note at p. 230) that Seeber, without actually solving the problem, had come extremely near to its solution.

‡ 'Borchardt's Journal,' vol. lxxvii. p. 143.

§ See the letters of M. Hermite to Jacobi ('Crelle,' vol. xl. p. 261, *sqq.*), and the papers of M. Zolotareff and Korkine ('Clebsch,' vol. v. p. 581, and vol. vi. p. 366). I may perhaps also be allowed to refer to my own papers 'On the Orders and Genera of Quadratic Forms containing more than three Indeterminates,' in the Proceedings of the Royal Society, vol. xiii. p. 199, and vol. xv. p. 387.

As a second and much simpler instance of the difficulties which remain for arithmetic after the work of algebra is done, let us consider the system of two binary quadratic forms. The first question that we naturally ask, is, what is the arithmetical meaning of the evanescence of their joint invariant? I gave myself an answer to this question some years ago in the following theorem, which for brevity I express in the proper technical language.

'If the joint invariant of two properly primitive forms vanishes, the determinant of either of them is represented primitively by the duplicate of the other.'*

This theorem is very far from exhausting the subject to which it refers. But it may serve as a fair illustration of the class of enquiries which I wish to propose to the attention of the Society as likely to be not unfruitful. The geometrical interpretation of the invariantive character to which the theorem relates is (as we all know) that the two pairs of elements, represented by the two quantics, are harmonically conjugate; and I think it especially deserving of notice that the same invariantive character has an important meaning both in arithmetic and in geometry, but that neither of the two interpretations seems in the least likely to suggest the other. If we pass on to the case of two ternary quadratic forms, the geometrical signification of the evanescence of either of their joint invariants is now embodied in well-known elementary theorems; but I do not think that any answer has been given to the corresponding arithmetical question, nor indeed do I know that anyone has occupied himself with it. I would, however, venture to hazard a conjecture that the arithmetical interpretation of these invariantive conditions may have an important bearing on the researches of M. Selling, to which I have already referred.

I do not wish to weary the Society with too many particular examples; but I will venture to point to one more instance from which it would appear that modern geometrical and analytical conceptions may help us a little, if only a little, on our way in the trackless wilds of arithmetic. Let us take a question which has some relation to the familiar notions of 'unicursality' and 'one-to-one correspondence.' It is an old theorem, that if the homogeneous indeterminate equation of the second degree containing three variables admits of one solution, it admits of an infinite number; and there is a Memoir of Cauchy† showing how

* Report on the Theory of Numbers in the Reports of the British Association for 1863, p. 783, Art. 123 [vol. i. p. 284].

† 'Exercices de Mathématiques,' vol. i. p. 233. Cauchy considers the ternary cubic as well as the ternary quadratic equation in this memoir.

from one given solution all the solutions are to be derived. But here two things deserve our notice—(1) that no geometry (so far as I am aware) helps us in any way to decide whether the given equation does or does not admit of solution. The criterion turns on the definition (first given by Eisenstein) of the generic characters of ternary quadratic forms—a definition which itself depends on a simple arithmetical inference from the algebra of such forms.* But (2), in strong contrast to what I have just stated, when once we have a single solution, the rest is a matter of intuitive geometry. Our equation represents a conic; the conic is real, because we have a single rational point on it; every rational line drawn through this point meets the conic again in a rational point; and thus the unicursality of the conic enables us to deduce from any one solution all the solutions which exist, and, what is more, to obtain them all in a natural sequence. To pass to a question of a somewhat higher order of difficulty, there is no known criterion (so far as I am aware) for deciding whether a ternary homogeneous cubic equation does or does not admit of solution in integral numbers. Such a criterion would be of great interest, and ought not, one would suppose, to lie beyond the present scope of analysis.† But here again geometry shows us at once, that if we have one solution, we have, in general, an infinite number. For the tangential of a rational point on a rational cubic curve is itself a rational point, and the line joining two given rational points meets the cubic in a third rational point. (There are, of course, cases of exception to this mode of derivation of one integral solution from another, but I need not advert to them here.) Advancing a little further, we have not to look very far into the connexion which modern algebra has established between ternary cubic and binary quadratic forms in order to satisfy ourselves that the Diophantine problem of rendering a biquadratic expression a perfect square (a problem which has been the subject of numerous researches ever since the time of Euler) is the same as the problem of finding rational points upon a cubic curve of which the equation is rational; and that, in particular, the *tangential* method to which I have just referred enables us in general to deduce an infinite number of solutions of the Diophantine problem from any given solution.

A second department of Arithmetic which, as it seems to me, has in quite recent times received less attention than it deserves, is the Theory of Congru-

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* See Eisenstein, in 'Crelle,' vol. xxxv. p. 117; and a note of my own in the Proceedings of the Royal Society for 1864, vol. xiii. p. 110 [No. XIII. vol. i. p. 410].

† Special cases of the equation here considered have attracted much attention. It will be sufficient to mention Fermat's theorem of the impossibility of the equation $x^3+y^3+z^3=0$.

ences. Some time before the notation of congruences had been introduced into the Theory of Numbers by Gauss, Lagrange, to whom the conception of a congruence (apart from any special notation) was perfectly familiar, had established the elementary theorem :*

'If an expression of the form $\frac{f(x)}{\phi(x)}$, where $f(x)$ and $\phi(x)$ are rational functions of x having integral coefficients, acquires an integral value for any given integral value of x, the value of $\phi(x)$ must be a divisor of the resultant of $f(x)$ and $\phi(x)$.'

This theorem naturally suggests another which was subsequently given by Cauchy :†

'If two congruences which have the same modulus admit of a common solution, the modulus is a divisor of their resultant.'

These propositions suggest the possibility of transferring to the Theory of Numbers some at least of the results which have been obtained by modern researches in the theory of algebraical elimination. For example, we are led to consider the problem :

'Given two congruences, to find the number of their common roots when we take in succession for modulus each divisor of their resultant.'

In all such inquiries we shall find that the considerations which suffice for the solution of the algebraical problem enter as indispensable elements into the arithmetical investigation, but that this investigation compels us to take notice of other elements also, with which, in algebra, we were not concerned. Thus, in the solution of the problem which I have mentioned, and which I hope at some future day to bring more fully under the notice of the Society, we should have not only to consider in a general manner the system of the divisors of the resultant itself, but we should also have to distinguish, in that system of numbers, those which are at the same time common divisors of certain systems of minors in the dialytic matrix of which the resultant is the determinant.

If I may be allowed to regard the subject of complex numbers as belonging to the theory of congruences, I must also be allowed to modify to a certain

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* The 'Disquisitiones Arithmeticæ' were published in 1801. The Memoir of Lagrange, entitled 'Nouvelle méthode pour résoudre les Problèmes indéterminés,' appeared in the Transactions of the Academy of Berlin for 1768. See also paragraph 4 (p. 528) of his Additions to the French translation of the Algebra of Euler (Lyons, 1774, and often reprinted since); the reference here is to the edition of the 'an iii de l'ère républicaine.'

† 'Exercices de Mathématiques,' vol. i. p. 164.

extent what I have said as to the indifference with which that theory has been looked upon in very recent times; for there is no reason to complain that complex numbers have received insufficient attention, at least from the mathematicians of the Continent. Without referring here to the results obtained by Lejeune Dirichlet, or to the splendid series of researches upon the complex numbers formed with roots of unity which we owe to M. Kummer, we may notice that the general theory has attracted the attention of M. Kronecker, whose investigations relating to it have unfortunately not as yet been published, and are only known from the application which he has made of them to the equations which present themselves in the Theory of Elliptic Functions.* In a Supplement, added to the second edition of Lejeune Dirichlet's Lectures on the Theory of Numbers, M. Dedekind has given the outlines of a complete and very original theory of complex numbers, in which he has to a certain extent deviated from the course pursued by M. Kummer, and has avoided the introduction (at least in a formal manner) of ideal numbers.† I may remind my hearers that the inapplicability, in general, of Euclid's theory of the greatest common divisor to complex numbers formed with the roots of equations having integral coefficients renders it impossible to define the prime factors of such numbers in the same way in which we can define the prime factors of common integers, or of complex numbers of the form $a+b\sqrt{-1}$; and that, in the effort to overcome the difficulty thus arising, M. Kummer was led to introduce into arithmetic an entirely new and very important conception—that of ideal numbers. I shall ask leave to mention a very recent and very interesting application of this conception which has been made by M. Zolotareff to the solution of a problem of the Integral Calculus, which was first attempted by Abel, and has since attracted the attention of MM. Tchébychef and Weierstrass. We owe to Abel the remarkable theorem that the differential expression

$$du = \frac{(x+\lambda)\,dx}{\sqrt{(x^4+ax^3+bx^2+cx+d)}}$$

can or cannot be integrated by logarithms, for some value of the parameter λ, according as the radical can or cannot be developed in a periodic continued

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* See the 'Monatsberichte' of the Academy of Berlin for June 26, 1862, p. 370. M. Dedekind also refers to the investigations of M. Kronecker at p. viii. of his Preface to the second edition of Dirichlet's Lectures, presently to be noticed.

† M. Dedekind has also commenced the publication of a *résumé* of his theory in the 'Bulletin des Sciences Mathématiques et Astronomiques' for December, 1876.

fraction. But Abel gave no criterion for deciding whether the development is or is not periodic—a question which obviously cannot be decided by mere trial. The problem was first solved by M. Tchébychef for the case in which the coefficients a, b, c, d are rational; and the complete solution has at last been obtained by M. Zolotareff, with the help of a new theory of complex ideal numbers.*

The last part of arithmetical theory to which I would wish to direct the attention of some of the younger mathematicians of this country is the determination of the mean or asymptotic values of arithmetical functions. This is a field of enquiry which presents great difficulties of its own; and it is certainly one in which the investigator will not find himself incommoded by the number of his fellow-workers. 'Vix ullus reperietur geometra,' wrote Euler† in the last century, 'qui non, ordinem numerorum primorum investigando, haud parum temporis inutiliter consumpserit.' But I do not think that, as a rule, the

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* The Memoir of Abel, 'Sur l'intégration de la formule différentielle $\frac{\rho dx}{\sqrt{R}}$, R et ρ étant des fonctions entières,' will be found in 'Crelle,' vol. i. p. 185, or 'Œuvres Complètes,' vol. i. p. 33. In this memoir Abel demonstrates the general theorem that, if R be a rational and integral function of x of any order, such that $\sqrt{R}$ can be expressed by a symmetrical and periodic continued fraction of the form

$$\sqrt{R} = r + \frac{1}{2\mu +} \frac{1}{2\mu_1 +} \cdots\cdots \frac{1}{2\mu_1 +} \frac{1}{2\mu +} \frac{1}{2r +} \cdots\cdots, \quad \ldots\ldots\ldots \quad (1)$$

it is always possible to find a rational and integral function ρ of x such that

$$\int \frac{\rho dx}{\sqrt{R}} = \log\left(\frac{y+\sqrt{R}}{y-\sqrt{R}}\right) + C, \quad \ldots\ldots\ldots\ldots \quad (2)$$

y being a rational function of x; and that, conversely, if the equation (2) can be satisfied, the development of $\sqrt{R}$ is of the form (1). Abel further shows that y is always the convergent $[r, 2\mu, 2\mu_1, \ldots, 2\mu_1, 2\mu]$; and he gives the application of his theorem to the case in which R is a biquadratic function. The memoir of M. Tchébychef was first published in the 'Bulletin de l'Académie de St. Pétersbourg' for 1860 (tom. iii. pp. 1–12), and has been reprinted in the 'Mélanges Mathématiques et Astronomiques de St. Pétersbourg' (tom. iii. pp. 182–192), and also in 'Liouville's Journal' (Second Series, 1864), vol. ix. p. 225. M. Tchébychef has given his method without demonstration; this was first supplied by M. Zolotareff in a paper, 'Sur la méthode d'intégration de M. Tchébychef' ('Mathematische Annalen,' vol. v. p. 560). The complete solution of the problem is contained in a work by M. Zolotareff ('Théorie des Nombres entiers complexes,' St. Pétersbourg,) which I have not yet seen. The result obtained by M. Weierstrass (however remarkable in itself), that the integrability by logarithms of the differential du depends on the possibility of expressing a certain constant in the form $aK + i\beta K'$, where a and β are rational numbers, and K and K' are certain complete elliptic integrals, does not supply an available criterion. (See the 'Fortschritte der Mathematik,' vol. vi. p. 118.)

† Dec. 1, 1760, 'Novi Commentarii Petropolitani,' vol. ix. p. 99; or, 'Commentationes Arithmeticæ Collectæ' ('Petropoli, 1849), vol. i. p. 356.

mathematicians of the present day have any reason to reproach themselves on this score, or stand in any need of the apology which Euler proceeds to deliver. Nevertheless, much has been done in this direction since the days of Euler; enough, certainly, to give abundant encouragement to further inquiry. The first asymptotic results that were obtained are due to Gauss, and are given without demonstration in the 'Disquisitiones Arithmeticæ';* they relate to the average number of classes of binary quadratic forms of a positive or negative determinant. The general principles on which such inquiries depend were laid down by Lejeune Dirichlet, forty-eight years later, in a memoir entitled 'Ueber die Bestimmung der mittleren Werthe in der Zahlentheorie,' and inserted in the Transactions of the Berlin Academy for 1849. The subject has recently been resumed by M. Mertens of Cracow, who, in an interesting memoir ('Borchardt's Journal,' vol. lxxvii. p. 289), has determined the asymptotic values of several numerical functions. In particular, he has demonstrated the expression given by Gauss for the mean value of the number of quadratic forms of a negative determinant; this mean value being, in the vicinity of n, when n is a considerable number, the quotient obtained by dividing $\frac{2}{7}\sqrt{n}$ by the sum of the cubes of the reciprocals of the natural numbers.

As to our knowledge of the series of the prime numbers themselves, the advance since the time of Euler has been great, if we think of the difficulty of the problem; but very small if we compare what has been done with what still remains to do. We may mention, in the first place, the undemonstrated, and indeed conjectural, theorems of Gauss and Legendre as to the asymptotic value of the number of primes inferior to a given limit x (the former theorem assigns for this value the integral logarithm $\mathrm{Li}(x) = \int_0^x \frac{dx}{\log x}$, the latter an expression of the form $\frac{x}{\log x - a}$).† But these theorems are only approximately

* See Arts. 302, 304, and the Additamenta to Art. 306, X; also various passages in the posthumous fragments, 'De nexu inter multitudinem classium,' &c. (Gauss, 'Werke,' vol. ii. pp. 269–303.)

† See Legendre, 'Théorie des Nombres' (Paris, 1830), vol. ii. p. 65: Gauss to Encke (Dec. 24, 1849), in Gauss's Collected Works, vol. ii. p. 444. Legendre assigns to a the conjectural value 1·08366. Gauss compares his own formula and that of Legendre with the results of actual enumerations of the prime numbers, and finds that, as far as the end of the third million, the comparison is in favour of the formula of Legendre, but that the error of that formula shows a tendency to increase more rapidly than the error of his own. He also observes that the mean value of the constant a tends to decrease, and that its true limit may possibly be 1, or may differ from 1 by a quantity of the order $\frac{1}{\log x}$. Encke

consistent with one another, and are perhaps still less approximately true. The memoir of Bernhard Riemann, 'Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse,' contains (so far as I am aware) the only investigation of the asymptotic frequency of the primes which can be regarded as rigorous.* He shows that, if $F(x)$ be the number of primes inferior to x, there exists an analytical expression for the series

$$F(x) + \tfrac{1}{2} F(x^{\frac{1}{2}}) + \tfrac{1}{3} F(x^{\frac{1}{3}}) + \ldots,$$

which consists (1) of a term which does not increase without limit with x; (2) of a non-periodic term $\mathrm{Li}(x)$; (3) of an infinite series of periodic terms of the type $\mathrm{Li}(x^{\frac{1}{2}-\alpha i}) - \mathrm{Li}(x^{\frac{1}{2}+\alpha i})$, the constants α being the roots (infinite in number) of a certain transcendental equation. It follows that the non-periodic part of the expression for $F(x)$ is of the type

$$\mathrm{Li}(x) - \tfrac{1}{2}\mathrm{Li}(x^{\frac{1}{2}}) - \tfrac{1}{3}\mathrm{Li}(x^{\frac{1}{3}}) - \tfrac{1}{5}\mathrm{Li}(x^{\frac{1}{5}}) + \tfrac{1}{6}\mathrm{Li}(x^{\frac{1}{6}}) - \tfrac{1}{7}\mathrm{Li}(x^{\frac{1}{7}}) + \ldots;$$

and thus the equation of Gauss, $F(x) = \mathrm{Li}(x)$, can, roughly speaking, be correct only as far as quantities of the order $x^{\frac{1}{2}}$.

No less important than the investigation of Riemann, but approaching the problem of the asymptotic law of the series of primes from a different side, is the celebrated memoir, 'Sur les Nombres Premiers,' by M. Tchébychef,† in which he has established the existence of limits within which the sum of the logarithms of the primes P inferior to a given number x must be comprised. The limits assigned by M. Tchébychef are not very close, not even close enough to determine the asymptotic value of the quotient $\dfrac{\Sigma \log P}{x}$, to which the value 1 has been conjecturally assigned. But the theorem of M. Tchébychef may be, perhaps, said to mark the furthest point to which our knowledge of the series of prime numbers has yet been carried; and while it is truly remarkable that, in a matter of so much difficulty, a process so apparently simple as that which he has employed

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had communicated to Gauss a formula of his own, $\dfrac{x}{\log x} \times 10^{\frac{1}{2\log x}}$, which, as Gauss observes, may, for very great values of x, be regarded as coinciding with $\dfrac{x}{\log x - \frac{1}{2}\log 10}$. (See the letter cited, and also the note of M. Schering, *ibid.* p. 521.)

* See the 'Monatsberichte' of the Academy of Berlin for November, 1859; or Riemann's 'Mathematische Werke,' p. 137. In the 'Annali di Matematica,' tom. iii. p. 52 (1860), M. Genocchi has given a very interesting account of the method of Riemann, and has arrived at a result differing in one respect (see p. 58) from that of Riemann.

† 'Liouville,' First Series, vol. xvii. p. 366. The memoir was presented to the Academy of St. Petersburg in 1850.

should be capable of leading to a result of so much interest and importance, it is somewhat disappointing to find that this method, even in the hands of its eminent inventor, should seem incapable of being pursued further, and unlikely to furnish any nearer approximation to the truth. The method of M. Tchébychef, profound and inimitable as it is, is in point of fact of a very elementary character, and in this respect contrasts strongly with that of Riemann, which depends throughout on very abstruse theorems of the Integral Calculus.

I do not know that the great achievements of such men as Tchébychef and Riemann can fairly be cited to encourage less highly gifted investigators; but at least they may serve to show two things—first, that nature has placed no insuperable barrier against the further advance of mathematical science in this direction; and, secondly, that the boundaries of our present knowledge lie so close at hand that the inquirer has no very long journey to take before he finds himself in the unknown land. It is this peculiarity, perhaps, which gives such perpetual freshness to the higher arithmetic. It is one of the oldest branches, perhaps the very oldest branch, of human knowledge; and yet some of its most abstruse secrets lie close to its tritest truths. I do not know that a more striking example of this could be found than that which is furnished by the theorem of M. Tchébychef. To understand his demonstration requires only such algebra and arithmetic as are at the command of many a schoolboy; and the method itself might have been invented by a schoolboy, if there were again a schoolboy with such an early maturity of genius as characterised Pascal, Gauss, or Evariste Galois.*

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* In addition to the memoirs to which reference has been made in the text, we may also mention the following:—(1) M. Tchébychef, 'Sur la totalité des nombres premiers inférieurs à une limite donnée' ('Liouville,' 1st Series, vol. xvii. p. 341). In this paper (which was presented to the Academy of St. Petersburg in 1848) M. Tchébychef proves (among other things) that, if the expression $\frac{x}{F(x)} - \log x$ has a limit at all, the value of that limit must be -1. This result shows that in the approximate formula of Legendre, $F(x) = \frac{x}{\log x - a}$, we ought to take $a = +1$. (2) A paper by the late Judge Hargreave, in the 'Philosophical Magazine' for 1849 (vol. xxxv. p. 36), in which it is shown (but not by any very rigorous demonstration) that the average interval between two consecutive primes in the vicinity of any very great number x is $\log x$; a result at which Gauss had arrived while still a boy, as may be inferred from his letter to Encke, quoted above. (3) A paper in the 'Mathematische Annalen,' vol. ii. p. 636, in which the author, M. Meissel, by the aid of a method suggested by those employed by Legendre ('Théorie des Nombres,' vol. ii. p. 86), obtains a formula which greatly facilitates the determination of the number of prime numbers contained between given limits; he has thus found that the number of primes in the first ten millions is 664579: and in a later note ('Mathematische Annalen,' vol. iii. p. 523) that the number of primes in the first hundred millions is 5,761,460. (4) A

I pass on to speak of some other branches of analysis which appear to me at the present moment to promise much in the immediate future.

I will first refer to one or two points to which the transition from the arithmetic of whole numbers is easy and natural.

We owe to Jacobi the first suggestion of a method of approximation which forms a natural extension of the theory of continued fractions, but which still remains in an incomplete condition. In the memoir 'De functionibus duorum variabilium quadrupliciter periodicis,' ('Crelle,' vol. xiii. p. 55,) Jacobi demonstrated the theorem that an uniform function of a single variable can at most be doubly periodic; and that, if it be doubly periodic, the ratio of the two periods is necessarily imaginary. He effected this by proving that, if $\alpha\alpha'\alpha''$, $bb'b''$ are independent irrational quantities, it is always possible to find integral numbers $mm'm''$ such that the value of each of the two expressions

$$m\alpha + m'\alpha' + m''\alpha'',$$
$$mb + m'b' + m''b'',$$

shall be less than any quantity that can be assigned.

This idea of Jacobi was subsequently further developed by M. Hermite, who showed its connexion with the theory of the reduction of quadratic forms (see his letters to Jacobi in 'Crelle's Journal,' vol. xl. p. 261 *sqq.*). The same conception lies at the basis of Lejeune-Dirichlet's researches on complex units, and led him to his celebrated generalisation of the theory of the Pellian Equation.*

Since the death of Jacobi, a memoir of his (apparently left incomplete) has been published in 'Borchardt's Journal' (vol. lxix. p. 29), in which he examines the relations between the successive sets of integral numbers x_0, x_1, x_2, which render an expression such as

$$x_0 + x_1\omega_1 + x_2\omega_2$$

(where ω_1 and ω_2 are irrational quantities) approximately equal to zero. He

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note by Mr. J. W. L. Glaisher in the Report of the British Association for 1872 ('Transactions of the Sections,' p. 19), in which the results of some enumerations of the primes are given, and are compared with Mr. Hargreave's theorem as to their average frequency. (5) A paper by M. Mertens ('Borchardt's Journal,' vol. lxxviii. p. 46), in which he determines the asymptotic values of the functions $\Sigma\frac{1}{P}$, $\Pi\left(1-\frac{1}{P}\right)$. These functions had been already considered by Legendre ('Théorie des Nombres,' vol. ii. p. 67), and by M. Tchébychef in the memoir already cited in this note: but M. Mertens obtains a more precise result by more rigorous reasoning. (6) A preliminary note on an enumeration of primes, by Mr. Glaisher, in the Proceedings of the Cambridge Philosophical Society, Dec. 4, 1876.

* See the 'Monatsberichte' of the Academy of Berlin for 1842, p. 95, and for 1846, p. 105.

applies the theory to the examples $\omega_1 = 2^{\frac{1}{3}}, = 3^{\frac{1}{3}}, = 5^{\frac{1}{3}}$; $\omega_2 = 2^{\frac{2}{3}}, = 3^{\frac{2}{3}}, = 5^{\frac{2}{3}}$, and finds that in each case the development is periodic; but he appears not to have obtained any demonstration of the general theorem that the corresponding development in the case of the root of any cubic equation having integral coefficients is always periodic.* These unfinished researches of Jacobi, to which M. Borchardt has called the special attention of mathematicians (in the preface to the 68th volume of his Journal) have been resumed by M. Bachmann,† and still more recently, though from a somewhat different point of view, by M. Fürstenau. The latter of these mathematicians defines a continued fraction of the second order to be a continued fraction in which each element is itself a continued fraction; and, availing himself of this definition, he has succeeded in showing that we can always approximate to the real root of an equation of the order n, having integral coefficients, by means of a periodic continued fraction of the order $n-1$.‡ It is evident that the discovery of such a mode of approximation to the root of an equation may lead to theoretical considerations of great interest, though it is hardly likely that the method itself will be found practically useful.

Closely allied to the investigation of new methods of approximation is the problem of determining the arithmetical or transcendental character of irrational quantities. It was first shown by M. Liouville§ that irrational quantities exist which cannot be the roots of any equation whatever having integral coefficients; a proposition which certainly required the proof which it has received from him, although it might easily seem incredible *à priori* that such irrational quantities should not exist. We may, perhaps, be allowed to designate by the terms arithmetical and transcendental the two classes of irrational quantities between which the theorem of M. Liouville has taught us to distinguish; and it becomes a problem of great interest to decide to which of these two classes we are to assign the irrational numbers, such as e and π, which have acquired a fundamental importance in analysis. To Lambert, the eminent Berlin mathematician of the last century, the first great step in this direction is due. He showed that neither π nor π^2 is rational; with regard to e he was even more successful, for he was able

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* This theorem had been already obtained by M. Hermite. See the Letters already cited, 'Crelle,' vol. xl. pp. 286–289.

† 'Borchardt's Journal,' vol. lxxv. p. 25.

‡ E. Fürstenau, 'Ueber Kettenbrücke höherer Ordnung,' Wiesbaden. I regret to say that I only know this work from the notice in the 'Jahrbuch über die Fortschritte der Mathematik' for 1874.

§ 'Comptes Rendus,' vol. xviii. (1844), p. 883, and p. 910; reproduced with additions in 'Liouville's Journal' (1st series), vol. xvi. p. 133.

to prove that no power of e, of which the exponent is rational, can itself be rational.* There (with one slight exception) the question remained for more than a century; and it was reserved for M. Hermite, in the year 1873, to complete by a singularly profound and beautiful analysis, the exponential theorem of Lambert, and to prove that the base of the Napierian logarithms is a transcendental irrational.† But, in the letter to M. Borchardt already cited, M. Hermite declines to enter on a similar research with regard to the number π. 'Je ne me hasarderai point,' he says, 'à la recherche d'une démonstration de la transcendance du nombre π. Que d'autres tentent l'entreprise; nul ne sera plus heureux que moi de leur succès; mais croyez m'en, mon cher ami, il ne laissera pas que de leur en coûter quelques efforts.' It is a little mortifying to the pride which mathematicians naturally feel in the advance of their science to find that no progress should have been made for one hundred years and more toward answer-

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* 'Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,' in the 'Mémoires de l'Académie des Sciences de Berlin' for 1761, p. 265; the demonstration depends on the continued fraction

$$\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{x}{1+}\,\frac{x^2}{3+}\,\frac{x^2}{5+}\,\frac{x^2}{7+\ldots}.$$

A different method of proving the incommensurability of e (depending on the exponential series) has found its way into many elementary treatises; it would seem that this simple method cannot be applied to prove the more general proposition that $e^{\frac{m}{n}}$ is incommensurable; but it has been successfully employed by M. Liouville ('Liouville's Journal,' 1st series, vol. v. p. 192) to show that neither e nor e^2 can be the root of a rational quadratic equation. This result forms the only extension which the theorem of Lambert had received up to the date of the memoir of M. Hermite, 'Sur la fonction exponentielle,' to which we shall presently refer. The only elementary work in which (so far as I know) the incommensurability of $e^{\frac{m}{n}}$ is demonstrated, is Mr. Todhunter's 'Algebra,' ed. 5, p. 530. The theorems as to the incommensurability of π and π^2 are excluded from English text-books; the only exception that occurs to me being Sir David Brewster's English edition (Edinburgh, 1824) of the Geometry of Legendre, where Legendre's reproduction of the demonstration of Lambert is given in Note iv. p. 239. The exclusion of these theorems is a matter of regret; for they constitute the only 'short method with the circle-squarers'; and perhaps the extraordinary prevalence within the United Kingdom of the form of delusion known as circle-squaring may partly arise from the appearance of an 'ipsi dixerunt' on the part of the mathematicians, which is certainly suggested by the omission in elementary works of any rigorous demonstration of the irrationality of π. M. Hermite has given a demonstration of the irrationality of π and π^2, which is very beautiful and entirely different from that of Lambert (Letter to M. Borchardt in 'Borchardt's Journal,' vol. lxxvi. p. 342); with this single exception, the theory of the quadrature of the circle rests to-day where Lambert left it in the year 1761.

* See the Memoir 'Sur la fonction exponentielle,' already cited in the preceding note, 'Comptes Rendus,' vol. lxxvii. pp. 18 etc.; and also published separately by Gauthier-Villars, 1874.

ing the last question which still remains to be answered with regard to the quadrature and rectification of the circle. But mathematical discovery is like electricity; it follows the lines of least resistance; and an adherence to the rule which this analogy suggests is certainly conducive to the comfort of the individual mathematician, and is probably also, in the long run, conducive to the progress of mathematics themselves. It has often happened in mathematical history that a difficulty, which had for ages resisted all direct attempts to overcome it, has yielded at last to the gradual advance of science; just as in the operations of strategy a strong position, which cannot be carried by a front attack, may nevertheless be turned and taken in the rear by an enemy who has possessed himself of the country round it.*

I may, perhaps, mention yet one more class of questions lying on the border land of arithmetic and algebraic analysis; I mean the questions which relate to the transcendental or algebraic character of developments in the form of infinite series, infinite products, or infinite continued fractions. The theorems of Eisenstein and M. Heine, of which a simple and beautiful demonstration has lately been laid before us by our colleague M. Hermite, are amply sufficient to awaken the expectation of great future discoveries in this almost unexplored field of enquiry.†

I have detained you so long over arithmetical and quasi-arithmetical subjects that I can only venture to glance hastily at some topics on which I could have wished to have dwelt much longer. I am afraid that I have only given you an additional instance of that one-sidedness against which, as I have reminded you, we were cautioned by our first President. In the hope of convincing you that I have not wholly forgotten the claims of other parts of our science, I will now hazard the assertion, that (after all) the advancement of the Integral Calculus is at once the most arduous and the most important task to which a mathematician can address himself. In the applications of mathematics to physics

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* I find that I am here closely following (haud passibus æquis) some observations of the late Dr. Hermann Hankel in his inaugural address, 'Die Entwickelung der Mathematik in der letzten Jahrhunderten,' Tübingen, 1869. This discourse, by one who was at once a learned scholar and an original investigator, contains much which deserves the attention of all who are interested in the progress of mathematical science, or who wish to see a higher spirit infused into the mathematical teaching given in the schools and universities of this country.

† Eisenstein, 'Monatsberichte' of the Berlin Academy for 1852, p. 441; M. Heine in 'Crelle's Journal,' vol. xlv. p. 285, and vol. xlviii. p. 267. The note of Eisenstein is reproduced in the first of these memoirs.

the Integral Calculus is confessedly of ever increasing importance; and it is especially interesting to observe that some of the most recent developments which it has received have had their origin in considerations of pure analysis, and yet have come just in time to furnish us with the most appropriate instruments for dealing with the problems which at the present moment are the most prominent in physical enquiries. But I must not dwell on the prospects of great future extension which are thus opened up for the various branches of mathematical physics; I can only advert (and that very hastily) to some of those parts of the integral calculus which, even from the point of view of the pure mathematician, seem to promise an abundant and immediate harvest.

Let me first mention the theory of ordinary differential equations. This is a subject which ought to have a special interest for ourselves, as one of the latest advances that have been made in it—the introduction of symbolical methods—is due in great measure to English mathematicians, and above all others to George Boole. Nor have Englishmen been behindhand in the cultivation of another branch of the subject (intimately connected with the use of symbolical methods) —the representation of the solutions of differential equations by means of definite integrals. But, simultaneously with these investigations, a line of research has been pursued on the Continent to which we in England have not paid equal attention. I refer to the endeavours which have been made to determine the nature of the function defined by a differential equation, from the differential equation itself, and not from any analytical expression of the function, obtained by first 'solving' the differential equation. The generality and importance of such an enquiry (whatever be its difficulty) cannot be overrated; for, just as we long since learned to regard integration in a finite form (or, more properly, integration by means of algebraic or exponential and logarithmic functions) as only a very small part of the problem which the Integral Calculus has to solve with regard to differential expressions containing a single variable, so also, when we come to differential equations, we are forced to remember that the variety and complexity of the functional relations expressed by them may altogether transcend any other means of expression at our disposal. Perhaps we may regard as the fundamental theorem in the whole subject the proposition of Cauchy, that every differential equation admits (in the vicinity of any non-singular point) of an integral which is synectic within a certain circle of convergence, and which is consequently (within that circle) developable by the series of Taylor. Various applications of this theorem (together with a demonstration somewhat simpler than that given by Cauchy) will be found in the classical treatise of MM. Briot

and Bouquet.* Closely allied to the point of view indicated by the theorem of Cauchy is that adopted by Riemann, who regards a function of a single variable as defined by the position and nature of its singularities, and who has applied this conception to the linear differential equation of the second order which is satisfied by the hyper-geometrical series. In the memoir, 'Beiträge zur Theorie der durch die Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$ darstellbaren Functionem,'† Riemann sets out with the conception of a function which possesses three discriminantal points (I venture to propose this word as the most natural English rendering of 'Verzweigungs-punkte'), and which is further characterised by the property that any three of the values which it admits at any point are connected by a linear and homogeneous equation with constant coefficients. Such a function Riemann shows, by reasoning of great beauty and originality, necessarily satisfies the linear differential equation of the hypergeometric series; and thus the nature and mode of existence of the functions defined by that equation are put before us with a precision and clearness which could not, perhaps, have been attained by any application of the ordinary methods of analysis to the discussion or integration of the equation.

The collected works of Riemann include another, but unfortunately unfinished memoir ('Zwei allgemeine Lehrsätze ueber lineäre Differential-gleichungen mit algebraischen Coefficienten'), relating to the case in which the number of independent functional values is any whatever instead of only three. And the fertility of the conceptions of Cauchy and of Riemann is further attested by the researches to which they have given rise, and are still giving rise, in Germany—researches among which I must especially mention those of L. Fuchs, whose papers on linear differential equations, in the 66th and subsequent volumes of 'Borchardt's Journal,' must form, it seems to me, the basis of all future inquiries on this part of the subject.

There is one celebrated problem connected with differential equations which, after all that has been written and said about it, remains a problem still; I mean the problem of Singular Solutions. If it were not for the papers of M. Darboux ('Bulletin des Sciences Mathématiques et Astronomiques,' tom. iv. p. 158), and

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* 'Théorie des Fonctions Elliptiques,' ed. 2, Paris, 1875, p. 325. See also a memoir by the same authors in the 'Journal de l'Ecole Polytechnique,' cahier 36, p. 137. For an enumeration of Cauchy's Memoirs on Differential Equations, see his Life by M. Valson, Paris, 1868, vol. ii. cap. 9, pp. 104–117.

† 'Transactions of the Academy of Göttingen' for 1875, vol. vii., or in Riemann's Collected Works (Leipzig, 1876).

of Professor Cayley ('Messenger of Mathematics,' vol. ii. p. 6, and vol. vi. p. 23), I do not know where I should advise a student to turn to acquire any distinctness of insight into this important question. These papers have at any rate rendered one great service; they clearly show that there is a difficulty, and a difficulty not yet surmounted. The point of the difficulty I presume to be, that whereas a singular solution, from the point of view of the integrated equation, ought to be a phenomenon of universal, or at least of general, occurrence, it is, on the other hand, a very special and exceptional phenomenon from the point of view of the differential equation. The explanation suggested by M. Darboux is (to say the least) deserving of very careful consideration. He says, at p. 167 of the memoir just cited, 'Since differential equations are formed by the elimination of constants from an equation in finite terms and its derived equations, writers have supposed (and it would seem erroneously) that, when we are given a differential equation of the first order (for example), it always possesses an integral of the first order expressible in the form $f(x, y, c) = 0$, where f is a function having in the whole extent of the plane the properties generally recognised in analytical functions. This function f might be more or less difficult to find, but it was conceived of in every case as existing. Now this is just the disputable point, and we think that recent researches on the theory of functions ought to lead us to adopt a different view.' It is evident that, if the observations of M. Darboux are well founded, an important series of questions arises as to the nature of the integral equation answering to a given differential equation; and, further, that some of the elementary considerations with which it is usual to introduce the subject of differential equations must be abandoned as untenable. The rules that can be given in aid of mathematical discovery are, I suppose, very few, and I have already ventured to call your attention to one of them—the rule that bids us follow up any opening that may present itself, rather than try to force a way against obstacles which may prove insurmountable. In the case before us, I think we come upon an illustration of another rule, which is of less general application, but nevertheless often useful. The rule is, that an apparent contradiction (as distinct from a mere misunderstanding) is always to be regarded as an indication of some undiscovered truth. Yet it is remarkable what a tendency there is in the minds of men to ignore or soften down such apparent contradictions, instead of looking for the reality which lies at the bottom of them. The equation $\int_1^x \frac{dx}{x} = \log x$ must have been familiar to mathematicians for a century at least before they set themselves seriously to examine the apparent contra-

diction presented by the equation of a single-valued to a multiple-valued expression. And yet what a flood of light was thrown on the whole theory of functions by the researches to which Cauchy and others were led when the endeavour was at last made to account for this and similar apparently inexplicable phenomena. If the clue offered by such familiar instances as the equations

$$\int_1^x \frac{dx}{x} = \log x, \quad \int_0^x \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x,$$

had been seized and followed up, it is difficult to believe that the main outlines of the Theory of Elliptic Functions would not have been discovered much sooner than they actually were. If we look back on the history of the past, the discovery of the 'principle of double periodicity,' and, with it, of the essential characteristic of an elliptic function, cannot but appear to us as one of the most extraordinary efforts of mathematical genius. The integration of the differential equation of elliptic integrals by Euler—an integration obtained by a sort of divination, which has deservedly remained celebrated in the history of science; the systematisation of the calculus of elliptic integrals by Legendre; the simultaneous discovery by Jacobi and Abel of the double periodicity latent in the equation of Euler,—these were the successive steps—and each one a gigantic step—by which those great mathematicians arrived at the theory of elliptic functions in the form in which we now possess it. But if we compare the actual history of the discovery with the outlines of the theory, as we find them, for example, in the work of MM. Briot and Bouquet, it is impossible not to be struck with the contrast. Each step in the theory, as exhibited in that work, appears to follow from those that precede it in such a natural and necessary order that we are inclined to wonder why those who discovered the great results themselves should have failed to find the easiest path of access to them.

If I had had the honour of addressing the Mathematical Society ten years ago, I think I should have had to complain of the neglect in England of the study of elliptic functions. But I cannot do so now. The University of Cambridge has given this subject a place in its Mathematical Tripos; the University of London in its examination for the Doctorate of Science. The British Association has supplied the funds requisite to defray the cost of printing Tables of the Theta function—Tables of which the mathematicians of this country may justly be proud, and which will form an enduring memorial of the great ability and indefatigable industry of our colleague, Mr. Glaisher. We further owe to Professor Cayley an introductory treatise on elliptic functions, the first which has appeared

in our language. I consider that the service which he has thus rendered to students is an important one, and one for which we ought to be very grateful. I am convinced that nothing so hinders the progress of mathematical science in England as the want of advanced treatises on mathematical subjects. We yield the palm to no European nation for the number and excellence of our text-books of the second grade—I mean, such text-books as are intended to guide the studies of the undergraduate within the courses prescribed by our University examinations in honours. But we want works adapted to the requirements of the student when his examinations are over—works which will carry him to the frontiers of knowledge in various directions, which will direct him to the problems which he ought to select as the objects of his own researches, and which will free his mind from the narrow views he is too apt to contract while 'getting up' subjects with a view to passing an examination, or, a little later in his life, while preparing others for examination. Can we doubt that much of the preference for geometrical and algebraical speculation which we notice among our younger mathematicians is due to the admirable works of Dr. Salmon; and can we also doubt that, if other parts of mathematical science had been equally fortunate in finding an expositor, we should observe a wider interest in, and a juster appreciation of, the progress which has been achieved? There are, of course, other treatises besides those of Professor Cayley and Dr. Salmon to which I might refer; there is, for example, the work of Boole on Differential Equations; and there are the great historical treatises of Mr. Todhunter, so suggestive of research and so full of its spirit; we have also a recent work by the same author on the functions of Laplace, Lamé, and Bessel. But the field is not nearly covered, though, indeed, my enumeration is not complete; and, even without leaving the domain of the Integral Calculus, I might point out that there are at least three treatises which we greatly need—one on Definite Integrals, one on the Theory of Functions in the sense in which that phrase is understood by the school of Cauchy and of Riemann, and one (though he should be a bold man who would undertake the task) on the Hyperelliptic and Abelian Integrals. I fear that our colleague, Professor Clifford, would hardly listen to us if we were to appeal to him to undertake this task; but at least we may express the hope that he may be able to continue the profound researches which he has commenced on this great branch of analysis.

I feel I must now bring these somewhat desultory remarks to a conclusion; though, if your time and patience were unlimited, there are many things I could wish to say. Among other matters I should have adverted to the great efforts

which have been made in very recent times, in Germany, in Russia, and in Norway, to advance the theory of partial differential equations; and I should have noted with pleasure that our own Society has received important communications on this subject from Professor Tanner. And again, leaving the field of the Integral Calculus, I think I should have hazarded some references to the Theory of Substitutions, and its applications to the Theory of Equations; and though I should have been relapsing into a region dangerously near to the Theory of Numbers, I should have exhorted the younger mathematicians of our time not to turn away from a subject which, if forbidding at the first aspect, contains so much promise of future development, and lies so near the very centre and fountain of much that is important in Algebra. Lastly, I should have endeavoured to make my peace with Geometry, which all this time I have been treating with such marked neglect, and would have invited your attention, though it were but for a few moments, to some of those questions of geometry which the natural advance of science has brought to the forefront at the present time. Even here, I might have found one example more of a study which we in England too much neglect; and I might perhaps have reminded you of the great hopes which Gauss entertained of the Geometry of Situation, containing, according to him, vast and as yet quite uncultivated regions over which our present analytical methods can pretend to no dominion. I certainly should not have forgotten to congratulate the Society on the part which its members, under the guidance and inspiration of Professor Sylvester, have taken in the development of the great geometrical theory of link-work movements.

'Verum hæc ipse equidem spatiis exclusus iniquis
Prætereo, atque aliis post commemoranda relinquo.'

I will not sit down without again offering my excuses for the fragmentary and disconnected nature of the reflexions I have laid before you this evening. But, indeed, over so wide a field I could only take a wandering course. My object has been to impress upon those who have been indulgent enough to listen to me that the vast increase which this century has witnessed in the extent of the ground already covered by mathematical science has been accompanied by a proportionate increase in the number and variety of the objects of interest to which the mathematician may turn his attention, and by an even more than proportionate increase in the opportunities of discovering new truths which have been brought within his reach. Our border on every side is the unknown; and the further our boundary line is extended, the more multitudinous become the points at which we may hope to penetrate beyond it. In these days when so

much is said of original research and of the advancement of scientific knowledge, I feel that it is the business of our Society to see that, so far as our own country is concerned, mathematical science should still be in the vanguard of progress. I should not wish to use words which may seem to reach too far, but I often find the conviction forced upon me that the increase of mathematical knowledge is a necessary condition for the advancement of science, and, if so, a no less necessary condition for the improvement of mankind. I could not augur well for the enduring intellectual strength of any nation of men, whose education was not based on a solid foundation of mathematical learning, and whose scientific conceptions, or, in other words, whose notions of the world and of the things in it, were not braced and girt together with a strong framework of mathematical reasoning. It is something for men to learn what proof is, and what it is not; and I do not know where this lesson can be better learned than in the schools of a science which has never had to take one footstep backward, which has never asserted without proof, or retracted a proved assertion; a science which, while ever advancing with human civilization, is as unchangeable in its principles as human reason; the same at all times and in all places; so that the work done at Alexandria or Syracuse two thousand years ago (whatever may have been added to it since) is as perfect in its kind, and as direct and unerring in its appeal to our intelligence, as if it had been done yesterday at Berlin or Göttingen by one of our own contemporaries. Perhaps also it might not be impossible to show, and even from instances within our own time, that a decline in the mathematical productiveness of a people implies a decline in intellectual force along the whole line; and it might not be absurd to contend that on this ground the maintenance of a high standard of mathematical attainment among the scientific men of a country is an object of almost national concern. But I need not ask your assent to such wide assertions; I shall be more than satisfied if anything that may have fallen from me may induce any one of us to think more highly than he has hitherto done of the first and greatest of the sciences, and more hopefully of the part which he himself may bear in its advancement.

XXXII.

ON THE CONDITIONS OF PERPENDICULARITY IN A PARALLELEPIPEDAL SYSTEM.

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1. THE conception of a parallelepipedal system (*i.e.*, of a space divided by three systems of equidistant parallel planes into similar and equal parallelepipeds) which forms the basis of the usually received theory of crystallography,* has also led to important researches in a domain belonging partly to arithmetic and partly to geometry.†

It is the object of the present paper to discuss, more completely than perhaps has been done as yet, the conditions for the perpendicularity of lines and planes in a parallelepipedal system. The arithmetical principles required in the discussion are very simple, and are to be found, for the most part, in articles 1–7 of a memoir on 'Linear Systems of Indeterminate Equations and Congruences' in the 'Philosophical Transactions' for 1861‡. The geometrical considerations involved, although well known, are less elementary, and relate to the theory of

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* See Dr. Leonhard Sohncke 'Die unbegrenzten regelmässigen Punktsysteme als Grundlage einer Theorie der Krystallstruktur,' Karlsruhe, 1876, for a different and more general hypothesis as to the ultimate structure of a crystal. See also a memoir by Dr. Sohncke, 'Die regelmässigen ebenen Punktsysteme von unbegrenzter Ausdehnung' in 'Borchardt's Journal,' vol. lxxvii. p. 47, and a memoir by M. Camille Jordan 'Sur les groupes de mouvements' in the 'Annali di Matematica' (Brioschi e Cremona), vol. ii. p. 167.

† See (1) Gauss's Review of Seeber's 'Untersuchungen ueber die Eigenschaften der positiven ternären quadratischen Formen' in the Göttingen 'Gelehrte Anzeige' for 1831, or in 'Crelle's Journal,' vol. xx. p. 312; (2) Lejeune Dirichlet, 'Ueber die Reduction der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen,' 'Crelle's Journal,' vol. xl. p. 209; (3) the 'Études Crystallographiques' of Auguste Bravais, Paris, 1866; (4) A Letter of Eisenstein to M. Charles Hermite ('Journal de Mathématiques,' vol. xvii. p. 473). The researches to which this letter refers are purely arithmetical, but they have an important bearing on a class of questions which are naturally suggested by the present enquiry, and of which the following may be taken as an example, 'In what cases can a parallelepipedal system, possessing a spherical symmetry, be regarded as containing a system in which the parallelepipeds are cubes?'

‡ No. XII. vol. 1. p. 367.

contravariant systems of cones. For the definition and some of the principal properties of such systems, we may refer to a paper 'On some Geometrical Constructions' in the 'Proceedings of the London Mathematical Society,' vol. ii. p. 85*; the most important case, that of two-fold systems, forms an essential part of the theory of cubic cones or curves; the considerations relating to that theory which we shall have occasion to employ, will be found in the 'Introduzione alla Teorica delle Curve Piane' of Professor Cremona, or in the 'Treatise on the Higher Plane Curves' of Dr. Salmon.

The results of the present enquiry (which has been undertaken at the request of Professor N. S. Maskelyne, and owes much to his suggestions) are submitted to the Mathematical Society with great diffidence; because, while they do not seem likely to admit of any direct application to the measurement of crystals, there is also some uncertainty as to their exact relation to other parts of crystallographic theory.

2. It is, perhaps, hardly necessary to explain that by a 'line of the system' we understand a line joining any two points of a given parallelepipedal system; by a 'plane of the system,' a plane containing any three points of the system; the 'points of the system' being the points of intersection of the three sets of equidistant parallel planes by which the system is defined. For our present purpose it will be sufficient to consider exclusively *origin* lines and planes; *i.e.*, lines and planes passing through a fixed point of the system taken as origin.

Whenever a line of the system is perpendicular to a plane of the system, the system has a certain 'symmetry of aspect' with regard to that plane. Let Ω be the plane, and let O be any point of the system lying in it. The planes and lines of the system which pass through O are symmetrically distributed with regard to Ω; but the points of the system are not (in general) symmetrically distributed with regard to Ω; thus, if OP is any line of the system, and OQ is the *reflexion* of OP in the plane Ω, OQ is a line of the system, but the points of the system which lie on OQ are not, in general, the reflexions of the points which lie on OP. Hence, while the points of the system are not themselves symmetrically distributed with regard to Ω, the directions in which they would be viewed by an eye situated at O, are symmetrically distributed; and this we may express by saying that the system has a 'symmetry of aspect' with regard to Ω. It will be seen—(1) that Ω is, or is not, a plane of absolute symmetry (*i.e.*, a plane of symmetry with regard to the points of the system),

* No. XIX. vol. i. p. 524.

according as the point of the system, that lies the nearest to O upon the normal at O, does, or does not, lie upon the nearest plane of the system parallel to Ω; (2) that two parallelepipedal systems may have the same aspect without coinciding. Suppose, for example, that the given parallelepipedal system has a symmetry of aspect with regard to three rectangular planes intersecting in O (this is the case of an 'ellipsoidal symmetry' to which we shall presently refer). Let OA, OB, OC be the three lines of intersection of these planes, A, B, C being the points of the system nearest to O on these lines respectively. The four points O, A, B, C determine a new parallelepipedal system $(OABC)$ composed of rectangular parallelepipeds. This system is *contained* in the given system, because every point of $(OABC)$ is a point of the given system; but the two systems are not *equivalent*, because every point of the given system is not a point of $(OABC)$. In general, a certain number of points of the given system lie in each rectangular parallelepiped of $(OABC)$, and are similarly distributed in each of them, though not symmetrically in any one of them. But the 'aspect' of the two systems from the point O (or from any point common to both of them) is the same; and the question whether the given system has an 'ellipsoidal symmetry of aspect' is the same as the question whether it contains a rectangular parallelepipedal system. Similar considerations apply to every case of symmetry of aspect; and indeed, whenever one of two parallelepipedal systems contains the other, the two systems have the same aspect at any one of their common points.*

As we shall have no occasion to consider planes of absolute symmetry, we shall, henceforward, for the sake of brevity, use the word symmetry in the sense of 'symmetry of aspect.' Thus any line, and any plane of the system, which are at right angles to one another, are an axis and a plane of symmetry.

The cases of symmetry, as thus defined, which can present themselves in a parallelepipedal system, are four in number. There is (1) the case of *simple* symmetry, where there is only one axis and one plane of symmetry; and there are three cases of triple symmetry, which may be characterized as (2) the ellipsoidal, (3) the spheroidal, and (4) the spherical. In an ellipsoidal system there are three rectangular planes of the system which are planes of symmetry; in a

* The relation between a contained and a containing parallelepipedal system is the same as the relation between a contained and a containing ternary quadratic form (see Gauss, *loc. cit.*). The problem, 'To find the conditions that a given parallelepipedal system should have an ellipsoidal symmetry of aspect,' may be stated in a purely arithmetical form as follows:—'To find the conditions that a definite ternary quadratic form (of which the coefficients may be rational or irrational) should contain a form of the type $Ax^2 + By^2 + Cz^2$.

spheroidal system there is one equatorial plane of symmetry, but every plane of the system at right angles to this plane is also a plane of symmetry; in a system having spherical symmetry, every plane of the system is a plane of symmetry, and every line of the system an axis of symmetry. Two simple symmetries cannot coexist without forming a triple symmetry, which is ellipsoidal, if the axis of one of the symmetries lies in the plane of the other, but is spheroidal in every other case; three simple symmetries form an ellipsoidal symmetry, if the three axes are at right angles to one another; a spheroidal symmetry, if one of the axes is at right angles to the plane of the other two, which are not at right angles to one another; a spherical symmetry in every other case. These assertions are, in part, simple consequences of the well-known theorem that if a plane contains two pair of lines of the system at right angles to another, every line of the system in that plane is at right angles to a line of the system; in part, however, their demonstration requires a fuller discussion of the conditions of symmetry, to which we now proceed.

3. *General Conditions of Perpendicularity.*

We begin by recording, in the form which is most suitable for our present purpose, the conditions for the perpendicularity of lines and planes in space referred to oblique axes.

The coordinates of any point being x, y, z, and X, Y, Z being the angles between the positive directions of the axes, the formula

$$F(x, y, z) = x^2 + y^2 + z^2 + 2yz\cos X + 2xz\cos Y + 2xy\cos Z,$$

which denotes the square of the distance of the point x, y, z from the origin, supplies a convenient mode of expressing the condition for the perpendicularity of any two lines

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

$$\frac{x}{l'} = \frac{y}{m'} = \frac{z}{n'};$$

viz., this condition is

$$l'\frac{dF}{dl} + m'\frac{dF}{dm} + n'\frac{dF}{dn} = 0, \quad \ldots\ldots\ldots \quad (1)$$

or

$$l\frac{dF'}{dl'} + m\frac{dF'}{dm'} + n\frac{dF'}{dn'} = 0$$

where

$$F = F(l, m, n), \quad F' = F(l', m', n'),$$

Similarly the condition for the perpendicularity of the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

and the plane $\lambda x+\mu y+\nu z=0,$

is
$$\frac{\frac{dF}{dl}}{\lambda}=\frac{\frac{dF}{dm}}{\mu}=\frac{\frac{dF}{dn}}{\nu}. \quad \ldots\ldots\ldots\ldots \quad (2)$$

To obtain, in a form similar to (1), the condition for the perpendicularity of two planes, we introduce the *contravariant* of F, *i.e.*, the function

$$\Phi(\xi,\eta,\zeta)=\xi^2\sin^2 X+\eta^2\sin^2 Y+\zeta^2\sin^2 Z+2\eta\zeta(\cos Y\cos Z-\cos X)$$
$$+2\zeta\xi(\cos Z\cos X-\cos Y)+2\xi\eta(\cos X\cos Y-\cos Z),$$

or, as it may be written, if X_1, Y_1, Z_1 are the angles between the normals to the coordinate planes,

$$\Phi(\xi,\eta,\zeta)=\xi^2\sin^2 X+\eta^2\sin^2 Y+\zeta^2\sin^2 Z+2\eta\zeta\sin Y\sin Z\cos X_1$$
$$+2\zeta\xi\sin Z\sin X\cos Y_1+2\xi\eta\sin X\sin Y\cos Z_1.$$

The condition for the perpendicularity of the two planes

$$\lambda x+\mu y+\nu z=0,$$
$$\lambda' x+\mu' y+\nu' z=0,$$

may now be expressed in either of the two equivalent forms

$$\left.\begin{aligned}\lambda'\frac{d\Phi}{d\lambda}+\mu'\frac{d\Phi}{d\mu}+\nu'\frac{d\Phi}{d\nu}&=0,\\ \lambda\frac{d\Phi'}{d\lambda'}+\mu\frac{d\Phi'}{d\mu'}+\nu\frac{d\Phi'}{d\nu'}&=0;\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3)$$

and the condition (2) for the perpendicularity of a plane and a line may be expressed in a second, and equally convenient form,

$$\frac{\frac{d\Phi}{d\lambda}}{l}=\frac{\frac{d\Phi}{d\mu}}{m}=\frac{\frac{d\Phi}{d\nu}}{n}. \quad \ldots\ldots\ldots\ldots \quad (4)$$

The formulæ (1), (2), (3), (4) are well-known, and are only given here for convenience of reference; their demonstration by the ordinary methods of analytical geometry presents no difficulty. We may, however, call attention to the following points:

(i.) The equation $F(x,y,z)=0$ represents a sphere of evanescent radius having its centre at the origin; or, which is the same thing, a certain imaginary cone having its vertex at the origin, and asymptotic to any sphere concentric with the origin. Such an imaginary sphere-cone possesses the characteristic properties,

that (α) any two lines, (β) any line and plane, (γ) any two planes, which are harmonically conjugate with regard to it, are at right angles to one another. The conditions (1) and (2) are the analytical expression of the properties (α) and (β).

(ii.) Again, the tangential equation of the imaginary sphere-cone, *i. e.*, the condition that the plane $\xi x+\eta y+\zeta z=0$ should be one of its tangent planes, is

$$\Phi(\xi, \eta, \zeta)=0;$$

and the conditions of perpendicularity (3) and (4) are the analytical expressions, by means of the tangential equation Φ, of the properties (γ) and (β).

(iii.) As the formula $F(x, y, z)$ expresses the square of any rectilineal segment in terms of its oblique projections upon the three coordinate axes; so the formula $\Phi(\xi, \eta, \zeta)$ expresses the square of any plane area in terms of its oblique projections $\xi \sin X$, $\eta \sin Y$, $\zeta \sin Z$, upon the three coordinate planes. The imaginary sphere-cone possesses, it will be noticed, the two paradoxical properties, that the square of any segment lying on one of its lines, and the square of any area lying on one of its tangent planes, are each of them equal to zero.

4. *Conditions of Perpendicularity in a Parallelepipedal System.*

Adopting the notation of the classical treatise of Professor Miller, we designate by a, b, c the parameters of a parallelepipedal system; we thus have, for the square of the distance between any two points of the system, the expression

$$\begin{aligned} f(x, y, z) &= F(ax, by, cz) \\ &= a^2x^2+b^2y^2+c^2z^2+2bcyz\cos X+2cazx\cos Y+2abxy\cos Z, \end{aligned}$$

where xyz now denote integral numbers. Again, if

$$\begin{aligned} \phi(\xi, \eta, \zeta)=\Phi(bc\xi, ca\eta, ab\zeta) &= b^2c^2\xi^2\sin^2 X+c^2a^2\eta^2\sin^2 Y+a^2b^2\zeta^2\sin^2 Z \\ &+2a^2bc\eta\zeta\sin Y\sin Z\cos X_1+2b^2ca\zeta\xi\sin Z\sin X\cos Y_1 \\ &+2c^2ab\xi\eta\sin X\sin Y\cos Z_1, \end{aligned}$$

the form ϕ, which is the contravariant of f, characterises (in the manner indicated by Gauss) a new parallelepipedal system (the polar system of Bravais) in which every line is perpendicular to a plane of the given system, and in which the parameter corresponding to any line is the elementary parallelelogram in the plane to which the line is perpendicular.

Writing, for brevity,

$$\begin{aligned} f &= Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy, \\ \phi &= A_1\xi^2+B_1\eta^2+C_1\zeta^2+2A_1'\eta\zeta+2B_1'\zeta\xi+2C_1'\xi\eta, \end{aligned}$$

(so that

$A = a^2, \ldots, A' = bc\cos X, \ldots, A_1 = b^2c^2\sin^2 X, \ldots, A_1' = a^2bc\sin Y\sin Z\cos X_1, \ldots,$)

and denoting the determinant

$$\begin{vmatrix} A, & C', & B' \\ C', & B, & A' \\ B', & A', & C \end{vmatrix} = a^2b^2c^2\left(1-\cos^2 X-\cos^2 Y-\cos^2 Z+2\cos X\cos Y\cos Z\right)$$

by Δ, we have the well known relations

$$A_1 = BC - A'^2, \quad B_1 = CA - B'^2, \quad C_1 = AB - C'^2,$$
$$A_1' = B'C' - AA', \quad B_1' = C'A' - BB', \quad C_1' = A'B' - CC',$$
$$\begin{vmatrix} A_1, & C_1', & B_1' \\ C_1', & B_1, & A_1' \\ B_1', & A_1', & C_1 \end{vmatrix} = \Delta^2,$$
$$\Delta A = B_1C_1 - A_1'^2, \quad \Delta B = C_1A_1 - B_1'^2, \quad \Delta C = A_1B_1 - C_1'^2,$$
$$\Delta A' = B_1'C_1' - A_1A_1', \quad \Delta B' = C_1'A_1' - B_1B_1', \quad \Delta C' = A_1'B_1' - C_1C_1';$$

which serve to show that, if we suppose $\Delta = 1$, or the volume of the elementary parallelepiped of the system f to be unity (a supposition which is admissible, because, if Δ be not unity, we may alter the parameters in the ratio of $\sqrt[6]{\Delta} : 1$), the relation between the two parallelepipedal systems is reciprocal; *i.e.*, the system f is derived from the system ϕ in the same way in which the system ϕ has been derived from the system f.

The five quantities upon which the nature of the parallelepipedal system ultimately depends are unquestionably the angles X, Y, Z, and the ratios of the parameters a, b, c. But the combinations of these quantities which enter into the conditions of perpendicularity and symmetry are precisely the six *covariant* coefficients

$$A, B, C, A', B', C',$$

and the six *contravariant* coefficients

$$A_1, B_1, C_1, A_1', B_1', C_1',$$

Thus, if

$$\left.\begin{aligned} \frac{x}{au} &= \frac{y}{bv} = \frac{z}{cw}, \\ \frac{x}{au_1} &= \frac{y}{bv_1} = \frac{z}{cw_1}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (5)$$

are any two lines of the system, the condition for their perpendicularity is

$$u_1\frac{df}{du} + v_1\frac{df}{dv} + w_1\frac{df}{dw} = 0,$$

or $$u\frac{df_1}{du_1}+v\frac{df_1}{dv_1}+w\frac{df_1}{dw_1}=0,$$

i. e., the condition that the lines (5) should be perpendicular is the same as the condition that the lines

$$\frac{x}{u}=\frac{y}{v}=\frac{z}{w}, \qquad \frac{x}{u_1}=\frac{y}{v_1}=\frac{z}{w_1}, \quad \ldots\ldots\ldots \quad (6)$$

should be harmonically conjugate with regard to the cone $f=0$. Again, the condition that the two planes of the system

$$\left.\begin{aligned}\frac{hx}{a}+\frac{ky}{b}+\frac{lz}{c}&=0,\\ \frac{h_1x}{a}+\frac{k_1y}{b}+\frac{l_1z}{c}&=0,\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7)$$

should be perpendicular to one another is the same as the condition that the planes

$$\left.\begin{aligned}hx+ky+lz&=0,\\ h_1x+k_1y+l_1z&=0,\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8)$$

should be harmonically conjugate with regard to the same cone $f=0$, of which the tangential equation is $\phi=0$; viz., this condition is

$$h_1\frac{d\phi}{dh}+k_1\frac{d\phi}{dk}+l_1\frac{d\phi}{dl}=0,$$

or $$h\frac{d\phi_1}{dh_1}+k\frac{d\phi_1}{dk_1}+l\frac{d\phi_1}{dl_1}=0.$$

And similarly the condition that one of the lines (5) should be perpendicular to one of the planes (7), is the same as the condition that the corresponding line (6) and plane (8) should be a polar line and polar plane with regard to the cone $f=0$ or $\phi=0$.

We shall, in what follows, adopt the mode of expressing the conditions of perpendicularity which has just been explained. By rational lines, planes, or cones, we shall understand lines, planes, or cones, of which the equations have integral numbers for their coefficients. We shall term the lines and planes (6) and (8), the rational lines and planes corresponding to the lines and planes of the system (5) and (7). Except in the case of a spherical symmetry, the cone $f=0$, or $\phi=0$, is not itself a rational cone: and thus (in other cases) the problem, 'to determine all the lines and planes which are perpendicular to one another in a given parallelepipedal system,' is the same as the problem, 'to find all the rational lines and planes which are harmonically conjugate to one another with regard to a given irrational cone.'

5. *Linear Relations connecting the Coefficients.*

Let the parallelepipedal system contain a pair of perpendicular lines (5); the condition of perpendicularity gives immediately the relation

$$Auu_1+Bvv_1+Cww_1+A'(vw_1+wv_1)+B'(wu_1+uw_1)+C'(uv_1+u_1v)=0. \quad . \quad (9)$$

Unless therefore the six covariant coefficients are connected by a linear homogeneous equation having integral coefficients, no two lines of the system can be perpendicular to one another; and, correlatively, unless the six contravariant coefficients are connected by a similar relation, no two planes of the system can be perpendicular to one another. But the existence of such a relation connecting the six coefficients (or the six contravariant coefficients), though a necessary condition, is not a sufficient condition for the existence in the system of a pair of perpendicular lines (or planes). We proceed, therefore, to examine more closely the circumstances which present themselves, when the coefficients are connected by one, two, three, four, or five relations. By a relation connecting the coefficients we understand a linear homogeneous equation of the type

$$pA+qB+rC+2p'A'+2q'B'+2r'C'=0, \quad (10)$$

in which p, q, r, p', q', r' are integral numbers; in connection with such a relation we shall have to consider the quadratic form

$$\Psi=p\xi^2+q\eta^2+r\zeta^2+2p'\eta\zeta+2q'\zeta\xi+2r'\xi\eta,$$

and its reciprocal form

$$\Psi=(p'^2-qr)x^2+(q'^2-rp)y^2+(r'^2-pq)z^2 \\ +2(pp'-q'r')yz+2(qq'-r'p')zx+2(rr'-p'q')xy;$$

these we shall term the quadratic form and the reciprocal quadratic form appertaining to the given relation.

For brevity, we shall attend only to the cases in which given relations exist between the six covariant coefficients $A, B, C, \ldots$; the cases, in which given relations exist between the six contravariant coefficients, are simply the correlatives of these; conditions for the perpendicularity of two planes answering to conditions for the perpendicularity of two lines, and every condition of symmetry admitting of a two-fold expression according as we employ the covariant, or the contravariant coefficients.

It is remarkable that, in every case, the conditions of perpendicularity and symmetry depend solely on the coefficients of the linear relations connecting the crystallographic coefficients; so that two parallelepipedal systems in which the crystallographic coefficients have different ratios, but satisfy the same linear

relations, would resemble one another exactly in respect of symmetry and perpendicularity.

6. *Case of one Linear Relation between the Coefficients.*

Here we have the theorem:

'The system contains a single pair of perpendicular lines, or contains no such pair whatever, according as the reciprocal form appertaining to the given relation is, or is not, a perfect square.'

For (i.) if the system contain a pair of perpendicular lines, the coefficients are connected by a relation of the type (9); and the reciprocal form appertaining to this relation is the perfect square

$$\tfrac{1}{4} \times \begin{vmatrix} x, & y, & z \\ u, & v, & w \\ u_1, & v_1, & w_1 \end{vmatrix}^2.$$

Conversely, (ii.) if the coefficients are connected by a relation of the type (10), which satisfies the condition $\Psi = \square$,

the system contains a single pair of perpendicular lines. Let

$$\Psi = (\alpha x + \beta y + \gamma z)^2,$$

and consider the matrix

$$\begin{vmatrix} p, & r' - \gamma, & q' + \beta \\ r' + \gamma, & q, & p' - \alpha \\ q' - \beta, & p' + \alpha, & r \end{vmatrix},$$

in which it is easily proved that every first minor is equal to zero, and in which we may suppose that the nine elements are freed by division from any common divisor which they may possess. If in this matrix we represent by u, v, w the greatest common divisors of the elements in the first, second, and third rows respectively, we find immediately

$$p = uu_1, \quad q = vv_1, \quad r = ww_1,$$
$$2p' = vw_1 + wv_1, \quad 2q' = wu_1 + uw_1, \quad 2r' = uv_1 + vu_1;$$

whence it follows that the equation (10) assumes the form of the equation (9) and that the two lines

$$\frac{x}{au} = \frac{y}{bv} = \frac{z}{cw},$$
$$\frac{x}{au_1} = \frac{y}{bv_1} = \frac{z}{cw_1},$$

are perpendicular to one another.

For the condition that Ψ must be a perfect square, we may, if we please, substitute the condition that the form ψ appertaining to the given relation should resolve itself into two linear factors having integral coefficients. If these factors are

$$u\xi + v\eta + w\zeta,$$
$$u_1\xi + v_1\eta + w_1\zeta,$$

the lines (5) are two perpendicular lines.

It is important to observe that the reciprocal form Ψ cannot be identically equal to zero. If it were so, the form ψ would itself be a perfect square, and the cone $f=0$, which is certainly imaginary, would contain a real line. Again, when we say that Ψ is to be a perfect square, we understand that Ψ itself is to be the square of a linear function of x, y, z (and not merely a multiple of such a square by a number which is not itself a square). Thus the condition $\Psi = \square$ may be replaced by the two conditions, (1) that the discriminant of ψ is to be zero, (2) that the greatest common divisor of the first minors of this discriminant is to be a perfect square; and of these the second may, if we please, be replaced by the condition that, of the three principal minors, one, which is not zero, is to be a perfect square. (These three minors cannot all be zero, and if one of them is a perfect square different from zero, the other two are either zero or perfect squares.)

7. Since the form f is positive and definite, the coefficients A, B, C, A', B', C' are subject to certain inequalities. But, subject to these, they may be any quantities whatever, rational or irrational. Hence no relation of the form (10) can in general subsist between them; or, which is the same thing, a parallelepipedal system cannot, in the most general case, contain a single right angle. The problem, 'Given any number of irrational quantities to determine the rational linear relations (if any) which subsist between them,' presents great difficulties, and, in the present state of indeterminate analysis, is perhaps insoluble. We observe, however, that if $m+n$ quantities $Y_1, Y_2, \ldots, Y_{m+n}$ are connected by m independent equations of the form

$$g_{i,1}Y_1 + g_{i,2}Y_2 + \ldots + g_{i,m+n}Y_{m+n} = 0,$$

where $i = 1, 2, 3, \ldots, m$, and the numbers g are integral, we can always (and indeed, unless $n=1$, in an infinite number of ways) assign n quantities Z_1, Z_2, Z_n such that the $m+n$ quantities Y can be expressed in terms of these by means of a system of equations of the type

$$Y_j = h_{j,1} Z_1 + h_{j,2} Z_2 + \ldots + h_{j,n} Z_n,$$

where the numbers h are integral, and $j = 1, 2, 3, \ldots, m+n$.*

It follows from this theorem that, if the coefficients are connected by one, and only one, relation, there is an equation of the type

$$f(x, y, z) = \omega_1 \Sigma_1 + \omega_2 \Sigma_2 + \ldots + \omega_5 \Sigma_5,$$

where $\Sigma_1, \Sigma_2, \ldots, \Sigma_5$ are quadratic forms in x, y, z having integral coefficients, and $\omega_1, \omega_2, \ldots, \omega_5$ are irrational quantities between which no linear rational homogeneous equation can exist. In general, if $i+j=6$, and if there are i relations between the coefficients, we have an equation of the type

$$f = \omega_1 \Sigma_1 + \ldots + \omega_j \Sigma_j.$$

The rational cones of which the equations are

$$\Sigma_1 = 0, \ \Sigma_2 = 0, \ \ldots, \ \Sigma_j = 0,$$

and again the rational cones of which the tangential equations are

$$\psi_1(\xi, \eta, \zeta) = 0, \quad \psi_2(\xi, \eta, \zeta) = 0, \ \ldots, \ \psi_i(\xi, \eta, \zeta) = 0,$$

determine two systems of contravariant cones; every cone of the system (Σ) circumscribing harmonically every cone of the system (ψ), and, *vice versâ*, every cone of the system (ψ) being harmonically inscribed in every cone of the system (Σ).†

We have seen that the existence of a pair of perpendicular lines in a parallelepipedal system implies the existence of a relation between the coefficients; but that not every such relation implies the existence of a pair of perpendicular lines. When the reciprocal form Ψ appertaining to a given relation is not a perfect square, we can only assert that the cone f harmonically circumscribes the cone ψ. But this statement seems to have no crystallographic meaning.

8. *Case of two linear relations between the coefficients.*

Let ψ_1, ψ_2; Ψ_1, Ψ_2 be the forms appertaining to the two given linear relations; and let us at first suppose that

$$\Psi_1 = \square, \quad \Psi_2 = \square \ ;$$

so that the system contains two pairs of perpendicular lines.

Three particular cases require especial notice.

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* See the memoir to which we have already referred, 'Phil. Trans.,' 1861, p. 299, *sqq.* [vol. i. p. 375].

† 'Proceedings of the London Mathematical Society,' vol. ii. pp. 90, 91 [vol. i. p. 528].

(1) If the squares Ψ_1 and Ψ_2 are identical, or differ only by a numerical factor, so that, retaining the notation of Art. 6, we have the equations,

$$\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2} = \frac{\gamma_1}{\gamma_2},$$

the planes of the two right angles coincide; and every line of the system that lies in this plane has a line of the system at right angles to it in the same plane.

(2) The plane of one of the right angles may contain one of the rays of the other right angle.

(3) The two right angles may have one ray in common; their common ray is then at right angles to the plane of the other two rays, and this plane becomes a plane of simple symmetry. The condition that this should happen is that ψ_1 and ψ_2 should have a linear factor in common; viz., if these two forms have a common linear factor, this factor is of necessity rational, the two reciprocal forms are perfect squares, and the common factor is

$$\begin{vmatrix} \xi, & \eta, & \zeta \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix}.$$

Still retaining the hypothesis that Ψ_1 and Ψ_2 are perfect squares, but excluding the particular cases to which we have just referred, we observe that, besides the two pairs of perpendicular lines which we obtain immediately from the two given relations, the system contains one, and only one, other pair of perpendicular lines. This is a consequence of the elementary theorem that if AOB, COD are two right angles in different planes, the intersection of the planes AOD, BOC is at right angles to the intersection of the planes AOC, BOD; a theorem which is itself a particular case of a general property of two pairs of conjugate lines with regard to any cone (or conjugate points with regard to any conic), first given by Otto Hesse.

We now quit the hypothesis that Ψ_1 and Ψ_2 are squares. When only one linear relation subsists between the coefficients, we are certain that, if the reciprocal form appertaining to it is not a perfect square, the parallelepipedal system cannot contain a single pair of perpendicular lines. But, when we have two such relations, we have still to inquire whether, by combining them linearly with one another, we cannot obtain a new relation which shall satisfy the condition

$$\Psi = \square.$$

Let θ, θ_1, θ_2 be the three roots of the discriminantal cubic of $\psi_1 + \theta\psi_2$. If

these roots are irrational, the system contains not a single pair of perpendicular lines. If one of them, for example θ, is rational, we have still to examine whether the factors of $\psi_1+\theta\psi_2$ are rational; if they are, we have a pair of perpendicular lines. If all the three roots are rational, we have to examine the factors of the three forms $\psi_1+\theta\psi_2$, $\psi_1+\theta_1\psi_2$, $\psi_1+\theta_2\psi_2$; according as these factors are or are not rational (if the factors of two of the forms are rational, the factors of the third are also rational), we obtain one or three pairs of perpendicular lines, or no pair at all of such lines.

If two of the roots of the discriminating cubic are equal, the cones ψ_1 and ψ_2 touch one another; if the three roots are all equal, these cones osculate (the cases of double contact and of super-osculation cannot present themselves, as they would imply the existence of a real line on the cone $f=0$). When two of the roots are equal, we have either the special case (2) considered above, or we have one, and only one, pair of perpendicular lines; when the three roots are all equal, we have a single pair of perpendicular lines.

Lastly, the coefficients of the discriminating cubic may all vanish. If this happens, either (α) Ψ_1 and Ψ_2 differ (if at all) only by a numerical factor, or (β) ψ_1 and ψ_2 have a common linear factor. The case (α) is the particular case (1) to which we have already referred. For each of the equations $\psi_1=0$, $\psi_2=0$ represents a pair of lines in the plane $\surd\Psi_1=0$ or $\surd\Psi_2=0$; these two pairs of lines may themselves be irrational, but, even if they are, every rational line lying in their plane has a rational conjugate in the involution determined by them, and the pairs of this involution are conjugate pairs with regard to the cone $f=0$.

The case (β), when ψ_1 and ψ_2 have a common linear factor, is the case of simple symmetry (the particular case (3) already mentioned).

We may therefore enunciate the theorem:

'The conditions that a parallelepipedal system should possess a simple symmetry are

'(a) That the coefficients should be connected by two linear relations;

'(b) That the two quadratic forms appertaining to these relations should have a linear factor in common.'

For the condition (b) we may substitute the following:—

'The four invariants of ψ_1 and ψ_2 must vanish; the first minors of the discriminants of ψ_1 and ψ_2 not being proportional to one another.'

This is only an explicit statement of the conditions which are necessary and sufficient to ensure the presence of a common factor in ψ_1 and ψ_2.

9. *Case of three linear relations between the coefficients.*

If the parallelepipedal system includes three pairs of lines at right angles to one another (these pairs being *asyzygetic*, *i.e.*, such that the existence of one of them is not a necessary consequence, in the manner already explained, of the existence of the other two), the six coefficients are connected by three independent linear relations of the type (10), and the reciprocal forms appertaining to these relations are perfect squares. Conversely, when these conditions are satisfied, the system contains not merely three pairs of lines at right angles to one another, but (as we shall presently see) an infinite number of such pairs; indeed, this will happen even if only one of the three reciprocal forms is a perfect square. And, even when none of these forms is a perfect square, the system *may* contain rectangular pairs of lines, because (as in the last case) it may be possible to combine linearly the given relations in such a manner as to obtain a new relation satisfying the condition

$$\Psi = \square.$$

Such a combination, as we now shall prove, is, or is not, possible, according as the intermediate equation

$$C = \begin{vmatrix} \frac{d\psi_1}{d\xi}, & \frac{d\psi_1}{d\eta}, & \frac{d\psi_1}{d\zeta} \\ \frac{d\psi_2}{d\xi}, & \frac{d\psi_2}{d\eta}, & \frac{d\psi_2}{d\zeta} \\ \frac{d\psi_3}{d\xi}, & \frac{d\psi_3}{d\eta}, & \frac{d\psi_3}{d\zeta} \end{vmatrix} = 0$$

does, or does not, admit of resolution in integral numbers.

In accordance with the theorem of Art. 7, we represent f by an equation of the type

$$f = \omega_1\Sigma_1 + \omega_2\Sigma_2 + \omega_3\Sigma_3;$$

and since no linear relation can exist between ω_1, ω_2, ω_3, we infer that any pair of rational lines which are conjugate with regard to f are also conjugate with regard to Σ_1, Σ_2, and Σ_3; the problem before us, therefore, is, 'To determine all the rational lines which are conjugate with regard to Σ_1, Σ_2, Σ_3.'

With reference to this problem, we consider the system of contravariant cones ψ_1, ψ_2, ψ_3, and Σ_1, Σ_2, Σ_3; the equations $\psi = 0$ being equations in plane coordinates, and the equations $\Sigma = 0$ being equations in point coordinates. The Cayleyan cone of the system is the cone of the third class $C = 0$, of which the equation has been already given. This cone is the Jocobian of the cones

ψ_1, ψ_2, ψ_3; the Jacobian of the cones Σ_1, Σ_2, Σ_3 is the Hessian cone of the system; its equation is

$$H = \begin{vmatrix} \frac{d\Sigma_1}{dx}, & \frac{d\Sigma_1}{dy}, & \frac{d\Sigma_1}{dz} \\ \frac{d\Sigma_2}{dx}, & \frac{d\Sigma_2}{dy}, & \frac{d\Sigma_2}{dz} \\ \frac{d\Sigma_3}{dx}, & \frac{d\Sigma_3}{dy}, & \frac{d\Sigma_3}{dz} \end{vmatrix} = 0.$$

The Cayleyan C is a combinantive covariant of the three forms ψ_1, ψ_2, ψ_3; the Hessian H is a combinantive contravariant of the same three forms; the developed expressions of C and H have been given by Dr. Salmon,* and it is unnecessary to repeat them here.

All the pairs of lines harmonically conjugate with regard to the system of cones (Σ) lie on the cubic cone $H = 0$; and *vice versâ*, every line lying on this cubic cone has a corresponding line lying on the same cone which is its conjugate with regard to all the cones of (Σ); it only remains, therefore, to ascertain whether any of these pairs of conjugate lines are rational. The problem, 'To determine whether a homogeneous cubic equation, containing three indeterminates, and having integral coefficients, does, or does not, admit of solution in integral numbers,' is one of which, in its general form, no complete discussion has as yet been given. It is obvious, however, that, if the equation $H = 0$ admits of a single solution in integral numbers, it admits of an infinite number of such solutions. In general this is true for any homogeneous cubic equation containing three indeterminates, and having integral coefficients. For we may regard the cubic equation as defining a rational cubic cone; if there is a rational line appertaining to this cone, the successive *tangentials* of this rational line are themselves rational lines; and the plane meeting the cone in any two rational lines meets it in a third rational line. This mode of construction is subject to certain exceptions; if, for example, the given rational line is a line of inflexion, it is its own tangential, and gives rise to no second rational line. But in the case before us (when the cubic cone appears as the Hessian of a rational system of cones) the line conjugate to any given rational line of the Hessian cone is always rational, and distinct from the given line; and the plane joining the two lines meets the Hessian cone in a third line distinct from either of them, of which we may again take the conjugate line, and so on continually. The line

* 'Conic Sections,' Ed. 5, pp. 346, 347.

conjugate to a given line of the Hessian cone is most readily found by taking the polar planes of the given line with regard to the cones (Σ); these polar planes are all rational, and intersect in a rational line, which is the line required. We may also note—(1) That if AA', BB' are two pairs of rational conjugate lines, the lines $(AB', A'B)$, $(AB, A'B')$ are themselves a pair of rational conjugate lines; (2) that if C is any rational line of the Hessian, the planes CA and CA' meet the Hessian again in a pair of rational conjugate lines; (3) that A, A' have the same rational tangential D, which is the conjugate of the line in which the plane AA' meets the Hessian again.

From what precedes, it appears that, if the equation $H=0$ admits of a single solution in integral numbers, the parallelepipedal system contains an infinite number of pairs of perpendicular lines all lying on the cubic cone $H(xa, yb, zc)=0$. And the same thing is true (as has been already said) if the equation $C=0$ admits of a single solution in integral numbers; for the two equations $C=0$ and $H=0$ are simultaneously resoluble or irresoluble; or, which is the same thing, if the Cayleyan cone $C=0$ has rational tangent planes, the Hessian cone $H=0$ has rational lines, and *vice versâ;* this, indeed, is evident, because the plane containing any two conjugate lines of the Hessian is a tangent plane of the Cayleyan; and the intersection of any two conjugate planes tangent to the Cayleyan is a line of the Hessian.

10. *Three Relations between the Coefficients—Conditions of Symmetry.*

The system may have a simple symmetry, or an ellipsoidal symmetry, or none at all. But it cannot have a spheroidal or a spherical symmetry.

If there is a simple symmetry, the cones (Σ) must have in common a rational polar line and polar plane; but must not have in common a rational self-conjugate system of diametral planes. On the other hand, when there is an ellipsoidal symmetry, the cones (Σ) must have such a system of diametral planes.

(A) *Conditions that the cones of the system* (Σ) *should have in common a single polar line and polar plane* (the line and plane are necessarily rational).

We take for the axis of z the polar line, the polar plane for the plane of xy, and for the axes of x and y the lines in which this polar plane is intersected by the tangent planes drawn from the axis of z to touch any one of the cones (ψ). (Except in a special case, which we shall notice presently, these cones all touch the same pair of planes intersecting in the axis of z.) We may then take for the representatives of the system (Σ) the three loci

$$x^2=0,\quad y^2=0,\quad z^2-2\lambda xy=0\,;$$

and for the representatives of the system (ψ) the three envelopes

$$\zeta\eta=0,\quad \zeta\xi=0,\quad \lambda\zeta^2+2\xi\eta=0\,;$$

λ being a constant different from zero. Thus the Hessian and the Cayleyan respectively become

$$xyz \quad \text{and} \quad \zeta(2\xi\eta-\lambda\zeta^2),$$

the former resolving itself into a product of three linear factors, the latter into a product of a linear by a quadratic factor. The axis of z is conjugate (with regard to the cones Σ) to every line in the plane xy; every line in either of the planes xz or yz is conjugate to a line in the other of those two planes; and the planes containing these pairs of conjugate lines envelope the Cayleyan cone $2\xi\eta-\lambda\zeta^2=0$.

The conditions that a ternary cubic should resolve itself into three linear factors is that the Hessian of the form should coincide with the form itself; and the condition that a ternary cubic should resolve itself into a linear and a quadratic factor is that the evectants of its two principal invariants should coincide.*.

The conditions, therefore, for a simple symmetry are that the Hessian of H should coincide with H, and that the two evectants of C should coincide with one another; but that the Hessian of C should not coincide with C.

The rational line corresponding to the axis of symmetry is the line, and the only line, common to all the quadratic cones represented by the first minors of the Hessian; the linear factor of C represents it tangentially. The rational plane corresponding to the plane of symmetry is the polar plane of this rational line with regard to the quadratic Cayleyan cone. One of the linear factors of the Hessian represents this polar plane, and is certainly rational; the other two factors (x and y) may be irrational, or even imaginary. When they are rational, there exist in the parallelepipedal system two planes, not at right angles to one another, intersecting in the axis of symmetry, and each containing an infinite number of lines of the system at right angles to one another.

These two planes may coincide; if they do, the plane of coincidence is certainly rational. The special case at which we thus arrive has been excluded from the preceding discussion; for we have supposed that the cones ψ all touch two distinct tangent planes intersecting in the common polar line of the cones Σ.

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* Dr. Salmon's 'Higher Plane Curves,' pp. 190 and 202, *sqq.*

But the cones ψ may instead touch two coincident planes; *i. e.*, they may all pass through the polar line, and touch a fixed plane along that line. If we take this plane for the plane of yz, and any other plane through the polar line for the plane of xz, we may represent the systems (Σ) and (ψ) by the loci

$$x^2=0, \quad xy=0, \qquad y^2+\lambda z^2=0,$$

and the envelopes

$$\eta\zeta=0, \quad \xi\zeta=0, \quad -\lambda\eta^2+\zeta^2=0.$$

The Hessian and the Cayleyan are respectively

$$x^2z \text{ and } \zeta(\zeta^2+\lambda\eta^2),$$

the former being a product of a linear by a square factor; the latter being a product of three linear factors, of which two are certainly imaginary, because, if λ were negative, the cones Σ would have two real lines in common, which would consequently appertain to the cone f. The conditions for this special case of simple symmetry, which is characterised by the presence of a single plane passing through the axis of symmetry, and containing an infinite number of pairs of perpendicular lines, are that the Hessian of H should vanish identically, and that the Hessian of C should coincide with C.

(B.) *Conditions that the system* (Σ) *should have in common a self-conjugate system of diametral planes, of which one at least is rational.*

The cones (ψ) must have three tangent planes in common (viz., the three self-conjugate planes of the system Σ). Thus the Cayleyan cone resolves itself into three lines (the three lines of intersection of the diametral planes), and the Hessian resolves itself into those three diametral planes themselves.

For this the necessary and sufficient conditions are that the Hessian of the Hessian should coincide with the Hessian, and the Hessian of the Cayleyan with the Cayleyan.

The three linear factors of the Cayleyan (or the Hessian) may, however, be all irrational, or one of them only may be rational, or they may all three be rational; and accordingly there may be either no symmetry at all, or a simple symmetry, or an ellipsoidal symmetry.

To distinguish between these three cases it is sufficient to examine the discriminantal cubic of any two of the cones Σ, or of any two of the cones ψ; for it is not difficult to verify that the linear factors of the Cayleyan, or Hessian, are rational or irrational according as the roots of any one of these equations are rational or irrational.

11. *Three Relations between the Coefficients—Special Cases.*

Some special cases, in which there is no symmetry, have (nevertheless) a certain interest. Of these we may briefly mention two.

(1) The cones ψ have one, and only one, tangent plane A in common. The Hessian in this case consists of A, and of a quadratic cone which is the *reciprocal* cone of A with regard to the cones of Σ. Thus, in the parallelepipedal system, there is a cone and plane such that every line lying in the plane has a line at right angles to it lying in the cone, and *vice versâ*; the lines of intersection of the plane and cone are at right angles, and *may* be lines of the system.

(2) The cones ψ have a polar line and polar plane in common.

The result is as follows :—The parallelepipedal system has an infinite number of pairs of lines at right angles to one another, all lying in the same plane ; and it *may* have a second set of such pairs lying on the surface of a quadratic cone, the plane of each such pair passing through the polar line of the first-named plane with regard to the cone.

12. *Case of Four Linear Relations between the Coefficients.*

In this case the form f may be expressed by an equation of the type

$$f = \omega_1 \Sigma_1 + \omega_2 \Sigma_2,$$

where the ratio of ω_1 and ω_2 is irrational, and where the cones Σ_1, Σ_2 cannot have any real line in common.

Every rational line has a rational line conjugate to it with regard to each of the cones $\Sigma_1 = 0$, $\Sigma_2 = 0$, and consequently with regard to the cone $f = 0$. Hence every line of the parallelepipedal system has a line at right angles to it; and this distribution of pairs of perpendicular lines may exist without the presence of any symmetry whatever.

Conditions of Symmetry.

The symmetry (if any) may be simple, or ellipsoidal, or spheroidal, but cannot be spherical.

There is a simple symmetry when the discriminantal cubic of the system (Σ_1, Σ_2) has one rational root, and an ellipsoidal symmetry when this cubic has three rational and unequal roots. When two of the roots are equal, the cones Σ_1 and Σ_2 have an imaginary double contact, and the axis and plane of the double contact are the axis and equatorial plane of a spheroidal symmetry.

If we represent by S the harmonic covariant of Σ_1 and Σ_2, the equation

$$J = \begin{vmatrix} \frac{dS}{dx}, & \frac{dS}{dy}, & \frac{dS}{dz} \\ \frac{d\Sigma_1}{dx}, & \frac{d\Sigma_1}{dy}, & \frac{d\Sigma_1}{dz} \\ \frac{d\Sigma_2}{dx}, & \frac{d\Sigma_2}{dy}, & \frac{d\Sigma_2}{dz} \end{vmatrix} = 0$$

represents the three diametral planes which are self-conjugate with regard to the cones Σ_1 and Σ_2; and its left-hand member J vanishes identically when Σ_1 and Σ_2 have double contact.

The Jacobian J is a combinantive covariant of Σ_1 and Σ_2; its coefficients therefore involve only the determinants (of the second order) which can be formed with the coefficients of Σ_1 and Σ_2. But these determinants* are proportional to the reciprocal determinants (of the fourth order) formed with the coefficients of $\psi_1, \psi_2, \psi_3, \psi_4$; *i.e.*, the coefficients of the Jacobian can be expressed in terms of these reciprocal determinants. To avoid introducing a new notation, we abstain from giving the developed expression of J, which indeed is not required here. It will be observed, however, that, in the case which we are now considering, the conditions for the various kinds of symmetry, and the determination of the planes of symmetry, depend on a single ternary cubic form (resoluble into three linear factors) which is a combinantive contravariant of the four forms ψ. (See the concluding remark in Art. 5.)

13. *Case of Five Linear Relations between the Coefficients.*

In this case the ratios of the coefficients are themselves evidently rational; and the parallelepipedal system has a spherical symmetry.

For, the coefficients of f being rational (after division, if necessary, by a common irrational factor), every rational line has a rational polar plane; *i.e.*, every line of the parallelepipedal system has a plane of the system at right angles to it.

And, conversely, if a parallelepipedal system has a spherical symmetry, the ratios of the coefficients are rational; for, if we consider any three lines of the system at right angles to one another, we obtain three independent relations (see Art. 5) between the coefficients; and if we consider any line of the system

* See 'Philosophical Transactions,' 1861, p. 301 [vol. i. p. 376].

(not lying in the same plane with two of the first three), and the plane perpendicular to it, we obtain two more relations between the coefficients, independent of one another and of the former three; *i.e.*, the ratios of the coefficients are rational.*

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* The question of the rationality or irrationality of the ratios of the crystallographic coefficients had already attracted the attention of Gauss, who, as appears from his Life ('Gauss. Zum Gedächtniss;' by W. Sartorius von Waltershausen; Leipzig, 1856), had in the year 1831 devoted himself with great ardour to the study of crystallography. See the concluding paragraphs of the Review of Seeber's 'Untersuchungen' already cited.

XXXIII.

ON THE CONDITIONS OF PERPENDICULARITY IN A PARALLELEPIPEDAL SYSTEM.

[Philosophical Magazine, Ser. v., vol. iv. pp. 18–25. Read before the Crystallological Society, June 14, 1876.

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1. THE conception of a parallelepipedal system (*i.e.* of a space divided by three systems of equidistant parallel planes into similar and equal parallelepipeds) may be regarded as forming the basis of the usually received theory of crystallography. It is the object of the present note to state some of the conditions for the perpendicularity of lines and planes in such a system. The results of this enquiry (which has been undertaken at the request of Professor N. S. Maskelyne, and owes much to his suggestions) are submitted to the Crystallological Society with great diffidence, because they do not seem likely to admit of any direct application to the practical work of the crystallographer. Such interest as they possess belongs to a domain which borders on the one hand on pure arithmetic, and on the other hand on pure geometry.

2. It is perhaps hardly necessary to explain that by a 'line of the system' we understand a line joining any two points of the given parallelepipedal system, by 'a plane of the system' a plane containing three points of the system, the points of the system being the points of intersection of the three sets of equidistant parallel planes by which the system is defined. It will be sufficient to consider *origin*-lines and planes, *i.e.* lines and planes passing through a fixed point of the system taken as origin.

3. Whenever a line of the system is perpendicular to a plane of the system, the system has a certain 'symmetry of aspect' with regard to that plane. Let Ω be the plane, and let O be any point of the system lying in it. The planes and lines of the system which pass through O are symmetrically distributed with regard to Ω; but the points of the system are not (in general) symmetrically

distributed with regard to Ω: thus, if OP is any line of the system not lying in the plane Ω, and if OQ is the *reflection* of Ω with regard to the plane Ω, OQ is a line of the system as well as OP, but the points of the system which lie on OQ are not (in general) the reflections of the points of the system which lie on OP. Hence, while the points of the system are not themselves symmetrically distributed with regard to Ω, the directions in which they would be viewed by an eye situated at O are symmetrically distributed; and this is what we intend to express by saying that the system has a 'symmetry of aspect' with regard to the plane Ω.

As we shall have no occasion in what follows to consider planes of absolute symmetry, we shall for the sake of brevity use the word symmetry in the sense of 'symmetry of aspect.' Thus any line and any plane of the system which are at right angles to one another are an axis and a plane of symmetry.

4. The cases of symmetry, as thus defined, which can present themselves in a parallelepipedal system are four in number. There is (1) the case of *simple* symmetry, when there is only one axis and one plane of symmetry; and there are three cases of triple symmetry, which may be characterised as (2) the *ellipsoidal*, (3) the spheroidal, and (4) the spherical. In an ellipsoidal system there are three mutually rectangular planes, which are planes of symmetry; in a spheroidal system there is one equatorial plane of symmetry, but every plane of the system at right angles to this plane is also a plane of symmetry; in a system having spherical symmetry every plane of the system is a plane of symmetry, and every line of the system an axis of symmetry. Two simple symmetries cannot coexist without forming a triple symmetry, which is ellipsoidal if the axis of one of the symmetries lies in the plane of the other, but is spheroidal in every other case: three simple symmetries form an ellipsoidal symmetry if the three axes are at right angles to one another, a spheroidal symmetry if one of the axes is at right angles to the plane of the other two which are not at right angles to one another, a spherical symmetry in every other case.

5. Adopting the notation of the classical treatise of Professor W. H. Miller, we designate by a, b, c the parameters appertaining to the three lines of the system taken for the coordinate axes; we also denote by X, Y, Z the angles between the coordinate axes, and by X_1, Y_1, Z_1 the angles between the normals to the coordinates planes. We thus have for the square of the distance between any two points of the system the expression

$$f(x, y, z) = a^2x^2 + b^2y^2 + c^2z^2 + 2bcyz\cos X + 2cazx\cos Y + 2abxy\cos Z,$$

where x, y, z denote any positive or negative integral numbers; and this ternary quadratic form may be regarded as characterising the given parallelepipedal system. Again, if

$$\begin{aligned}\phi(\xi, \eta, \zeta) = {} & b^2 c^2 \xi^2 \sin^2 X + c^2 a^2 \eta^2 \sin^2 Y + a^2 b^2 \zeta^2 \sin^2 Z \\ & + 2a^2 bc\, \eta\zeta \sin Y \sin Z \cos X_1 + 2b^2 ca\, \zeta\xi \sin Z \sin X \cos Y_1 \\ & + 2c^2 ab\, \xi\eta \sin X \sin Y \cos Z_1,\end{aligned}$$

the form ϕ, which is the contravariant of f, characterises (in the same way in which f characterises the given system) a new parallelepipedal system (the polar system of Auguste Bravais) in which every line is perpendicular to a plane of the given system, and in which the parameter corresponding to any line is the elementary parallelogram of the given system lying in the plane to which the line is perpendicular.

6. We write for brevity

$$\begin{aligned} f &= Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy, \\ \phi &= A_1\xi^2 + B_1\eta^2 + C_1\zeta^2 + 2A_1'\eta\zeta + 2B_1'\zeta\xi + 2C_1'\xi\eta \end{aligned}$$

(so that

$$A = a^2, \ldots,\ A' = bc\cos X, \ldots,\ A_1 = b^2c^2 \sin X, \ldots,\ A_1' = a^2 bc \sin Y \sin Z \cos X_1, \ldots);$$

and we observe that although the five quantities upon which the nature of the parallelepipedal system ultimately depends are the ratios of the parameters a, b, c, and the three angles X, Y, Z, yet the combinations of these quantities which it is most convenient to consider in discussing the conditions of perpendicularity are precisely the six coefficients

$$A, B, C, A', B', C',$$

and the six contravariant coefficients

$$A_1, B_1, C_1, A_1', B_1', C_1'.$$

Thus the condition that the lines of the system

$$\left.\begin{aligned} \frac{x}{au} &= \frac{y}{bv} = \frac{z}{cw}, \\ \frac{x}{au_1} &= \frac{y}{by_1} = \frac{z}{cw_1}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

should be perpendicular to one another is

$$u_1 \frac{df}{du} + v_1 \frac{df}{dv} + w_1 \frac{df}{dw} = 0,$$

or

$$u \frac{df}{du_1} + v \frac{df}{dv_1} + w \frac{df}{dw_1} = 0;$$

the condition that the planes of the system

$$\left.\begin{aligned} \frac{hx}{a}+\frac{ly}{b}+\frac{kz}{c}&=0,\\ \frac{h_1x}{a}+\frac{l_1y}{b}+\frac{k_1z}{c}&=0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

should be perpendicular to one another is

$$h_1\frac{d\phi}{dh}+k_1\frac{d\phi}{dk}+l_1\frac{d\phi}{dl}=0,$$

or

$$h\frac{d\phi}{dh_1}+k\frac{d\phi}{dk_1}+l\frac{d\phi}{dl_1}=0;$$

and the conditions that the first of the lines (i) should be perpendicular to the first of the planes (ii) may be written in one or other of the equivalent forms

$$\frac{\left(\frac{df}{du}\right)}{h}=\frac{\left(\frac{df}{dv}\right)}{k}=\frac{\left(\frac{df}{dw}\right)}{l},$$

$$\frac{\left(\frac{d\phi}{dh}\right)}{u}=\frac{\left(\frac{d\phi}{dk}\right)}{v}=\frac{\left(\frac{d\phi}{dl}\right)}{w}.$$

7. Let us now suppose that the given parallelepipedal system contains a pair of perpendicular lines (i); the condition of perpendicularity gives immediately

$$Auu_1+Bvv_1+Cww_1+A'(vw_1+wv_1)+B'(wu_1+uw_1)+C'(uv_1+vu_1)=0.$$

Unless, therefore, the six covariant coefficients are connected by a linear homogeneous relation having integral coefficients, no two lines of the system can be perpendicular to one another; and correlatively, unless the six contravariant coefficients are connected by a similar relation, no two planes of the system can be perpendicular to one another. But the existence of such a relation connecting the six covariant coefficients (or the six contravariant coefficients), though a necessary condition, is not a sufficient condition for the existence of a pair of perpendicular lines or planes. We proceed, therefore, very briefly to describe the principal cases which present themselves when the coefficients are connected by one, two, three, four, or five linear relations. By a linear relation connecting the coefficients we understand a linear homogeneous equation of the type $pA+qB+rC+2p'A'+2q'B'+2r'C'=0,$

where p, q, r, p', q', r' are integral numbers which we may suppose free from

any common divisor. In connexion with such a relation we shall have to consider the quadratic form

$$\psi = p\xi^2 + q\eta^2 + r\zeta^2 + 2p'\eta\zeta + 2q'\zeta\xi + 2r'\xi\eta$$

and its contravariant or reciprocal form

$$\Psi = (p'^2 - qr)x^2 + (q'^2 - rp)y^2 + (r'^2 - pq)z^2 + 2(pp' - q'r')yz + 2(qq' - r'p')zx + 2(rr' - p'q')xy.$$

These we shall term the quadratic form and the reciprocal quadratic form appertaining to the given relation. For brevity we shall attend only to the cases in which given relations exist between the six covariant coefficients A, B, C, A', B', C', the cases in which given relations exist between the six contravariant conditions being simply the correlatives of these. It is remarkable that in every case the conditions of perpendicularity and symmetry depend solely on the coefficients of the linear relations connecting the crystallographic coefficients; so that two parallelepipedal systems, in which the crystallographic coefficients have different ratios but satisfy the same linear relations, would resemble one another exactly in respect of symmetry and perpendicularity.

8. *Case of one linear relation between the coefficients.*

Here we have the theorem, 'The system contains a single pair of perpendicular lines, or contains no such pair whatever, according as the reciprocal form appertaining to the given relation is or is not a perfect square.'

For the condition that the reciprocal form Ψ should be a perfect square, we may if we please substitute the condition that the quadratic form ψ appertaining to the given relation should resolve itself into two rational factors. Or, again, we may replace this condition by the two conditions, (1) that the discriminant of ψ is to be zero, (2) that the greatest common divisor of the first minors of this discriminant is to be a perfect square.

9. *Case of two linear relations between the coefficients.*

We represent the quadratic forms and the reciprocal quadratic forms appertaining to these relations by ψ_1, Ψ_1, ψ_2, Ψ_2, and by θ, θ', θ'' the roots of the discriminantal cubic of $\psi_1 + \theta\psi_2$. If these roots are irrational, the system contains not a single pair of perpendicular lines. If one of them, for example θ, is rational, we still have to examine whether the factors of $\psi_1 + \theta\psi_2$ are rational; if they are, we have a pair of perpendicular lines. If all the three roots θ, θ', θ'' are rational, we have to examine the factors of each of the three forms $\psi_1 + \theta\psi_2$, $\psi_1 + \theta'\psi_2$, $\psi_1 + \theta''\psi_2$; according as these factors are or are not rational

(if the factors of two of them are rational the factors of the third are so too), we obtain one or three pairs of perpendicular lines, or no pair at all of such lines.

When two of the roots θ, θ', θ'' are equal, we have either one, and only one, pair of perpendicular lines; or we may have two pairs, the plane of one of the right angles containing one of the rays of the other right angle. When the three roots are all equal we have a single pair of perpendicular lines.

Lastly, the coefficients of the discriminating cubic may all vanish. If this happens, either (α) Ψ_1 and Ψ_2 differ, if at all, by a numerical factor, and every line of the system that lies in a certain plane has a line of the system at right angles to it in the same plane; or (β) ψ_1 and ψ_2 have a common linear factor, and the system possesses a simple symmetry.

We may thus enunciate the theorem:

'The conditions that a parallelepipedal system should possess a simple symmetry are (a) that the coefficients should be connected by two linear relations, (b) that the two quadratic forms appertaining to these relations should have a linear factor in common.'

10. *Case of three linear relations between the coefficients.*

We represent by ψ_1, ψ_2, ψ_3 the quadratic forms appertaining to the given relations, and we obtain the following theorem:

'The system contains no right angle, or an infinite number, according as the indeterminate cubic equation

$$C = \begin{vmatrix} \frac{d\psi_1}{d\xi}, & \frac{d\psi_1}{d\eta}, & \frac{d\psi_1}{d\zeta} \\ \frac{d\psi_2}{d\xi}, & \frac{d\psi_2}{d\eta}, & \frac{d\psi_2}{d\zeta} \\ \frac{d\psi_3}{d\xi}, & \frac{d\psi_3}{d\eta}, & \frac{d\psi_3}{d\zeta} \end{vmatrix} = 0$$

does or does not admit of solution in integral numbers.'

By virtue of the three given relations the characteristic expression $f(x, y, z)$ of Art. 5 assumes the form

$$f(x, y, z) = \omega_1 f_1 + \omega_2 f_2 + \omega_3 f_3,$$

the ratios of the quantities ω_1, ω_2, ω_3 being irrational, but the coefficients of the quadratic forms f_1, f_2, f_3 being integral numbers. If $H(x, y, z)$ denote the Jacobian of these three forms, we have the theorem:

'When the indeterminate equation $C = 0$ admits of solution, the infinite number of right angles which the system contains all lie on the cubic cone

$H(xa, yb, zc) = 0$; viz. an infinite number of lines of the system lie on this cone, and every line of the system which lies on it has a line at right angles to it, also lying on the cone.'

The system may have a simple symmetry or an ellipsoidal symmetry, or none at all; but it cannot have a spheroidal or a spherical symmetry.

The conditions for a simple symmetry are that the ternary cubic form $C(\xi, \eta, \zeta)$ should resolve itself into a rational linear factor and a rational quadratic factor, and that the ternary cubic form $H(x, y, z)$ should resolve itself into three linear factors. These conditions admit of being further developed (see Dr. Salmon's 'Higher Plane Curves,' pp. 190 and 202 *seqq.*); it is sufficient for our purpose to observe that the coefficients of the Jacobian $H(x, y, z)$, no less than those of $C(\xi, \eta, \zeta)$, depend solely on the coefficients of the forms ψ_1, ψ_2, ψ_3, *i.e.* on the integral numbers entering into the given linear relations.

The conditions for an ellipsoidal symmetry are that $C(\xi, \eta, \zeta)$ should resolve itself into three rational linear factors, and that $H(x, y, z)$ should resolve itself into three factors.

Two special cases of the general theory (which, however, are not cases of symmetry) deserve attention.

(1) There may exist in the parallelepipepal system a quadratic cone and a plane, such that every line of the system lying in the plane has a line of the system at right angles to it lying in the cone.

(2) Or, again, the parallelepipedal system may have an infinite number of pairs of perpendicular lines all lying in the same plane; and it may also have at the same time a second set of such pairs lying on the surface of a quadratic cone, the plane of each pair of this second set passing through the polar line of the first-named pair with regard to the cone.

11. *Case of four linear relations between the coefficients.*

Here every line, without exception, of the parallelepipedal system has a line at right angles to it; and this distribution of pairs of perpendicular lines may exist without the presence of any symmetry whatever. The symmetry (if any) may be simple, or ellipsoidal, or spheroidal, but cannot be spherical.

The characteristic form $f(x, y, z)$ may be expressed by an equation of the type

$$f = \omega_1 f_1 + \omega_2 f_2,$$

the ratio of ω_1 and ω_2 being irrational, but the coefficients of the quadratic forms f_1 and f_2 being integral numbers. There is a simple symmetry when the

discriminantal cubic of $f_1 + \theta f_2$ has one rational root, and an ellipsoidal symmetry when it has three rational and unequal roots, a spheroidal symmetry when it has two equal roots. (It cannot have its three roots equal, because the cone $f(x, y, z) = 0$ is imaginary.)

We suppress the further discussion of these conditions, only observing that they may be so expressed as to show that they depend only on the coefficients of the four given relations, and not on the six coefficients A, B, C, A', B', C' themselves.

12. *Case of five linear relations between the coefficients.*

In this case the ratios of the coefficients are themselves evidently rational, and the parallelepipedal system has a spherical symmetry. It is also true, conversely, that when there is a spherical symmetry the ratios of the coefficients are rational.

We may mention that the question of the rationality or irrationality of the ratios of the crystallographic coefficients had attracted the attention of Gauss, who, as appears from the memoir of his life ('Gauss. Zum Gedächtniss,' von W. Sartorius v. Waltershausen: Leipzig, 1856), had in 1831 devoted himself with great ardour to the study of crystallography.*

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* Some of the demonstrations, which have been omitted in the present note, will be found in a paper inserted in the 'Proceedings of the London Mathematical Society,' vol. vii. p. 83 [No. XXXII. vol. ii. p. 191.]

XXXIV.

SUR LES INTÉGRALES ELLIPTIQUES COMPLÈTES.

[Atti della R. Accademia dei Lincei. Transunti Ser. iii. vol. i. pp. 42–44. Read January 7, 1877.]

En suivant la notation usuelle, posons $\omega = x + iy$, la quantité y étant positive,

$$K = \tfrac{1}{2}\pi \left(\Sigma_{-\infty}^{+\infty} e^{m^2 i\pi\omega}\right)^2, \qquad K' = \frac{\omega}{i} K,$$

$$k^2 = 1 - k'^2 = \phi^8(\omega) = 16 e^{i\pi\omega} \Pi_1^{\infty} \left[\frac{1 + e^{2mi\pi\omega}}{1 + e^{(2m-1)i\pi\omega}}\right]^8;$$

désignons aussi par $\Phi(k^2)$ et $\Psi(k^2)$ les intégrales rectilignes

$$\int_0^1 \frac{dy}{\sqrt{(1-x^2)(1-k^2x^2)}} \text{ et } \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}},$$

la valeur initiale de chacun des deux radicaux étant $+1$. Lorsque k^2 est réel, positif et plus grand que l'unité, la définition de la première intégrale présente une ambiguité ; celle de la seconde se trouve pareillement en défaut, lorsque k^2 est réel et négatif. Nous conviendrons donc, dans ces deux cas, de prendre pour $\Phi(k^2)$ celle des deux valeurs admissibles dans laquelle le coefficient de i est positif ; et pour $\Psi(k^2)$ celle des deux valeurs de l'intégrale correspondante dans laquelle le coefficient de i est négatif.

On sait que, si la partie réelle de ω s'évanouit, c'est à dire, si k^2 et k'^2 sont réels, positifs et moindres que l'unité, on a les deux équations

$$K = \Phi(k^2), \qquad K' = \Psi(k^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Mais il a été démontré par M. Hermite, que ces équations ne peuvent pas avoir

lieu pour toutes les valeurs de ω*. Il est vrai qu'en doit avoir, dans tous les cas,

$$K=\int_0 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad K'=\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}}; \quad \ldots \quad (2)$$

mais ces intégrales ne sauraient être, en général, rectilignes ; et la détermination du chemin de l'intégration a paru offrir quelque difficulté. On y parvient de la manière suivante.

Soient α, β, γ, δ des nombres entiers, positifs ou négatifs, qui satisfont à l'équation $\alpha\delta-\beta\gamma=1$, et aux congruences $\alpha\equiv\delta\equiv1$, mod 4 ; $\beta\equiv\gamma\equiv0$, mod 2 ; soit aussi $\Omega=X+iY$, Y étant positif, et les quantités réelles X et Y étant assujetties à vérifier les inégalités

$$-1<X\leqq1, \quad -X<X^2+Y^2\geqq X. \quad \ldots \quad (3)$$

On conclut facilement de la théorie de la reduction les formes binaires à déterminant négatif, qu'on peut toujours satisfaire, et cela d'une manière unique, à l'équation

$$\frac{\gamma+\delta\Omega}{\alpha+\beta\Omega}=\omega.$$

Les nombres α, β, γ, δ, et la quantité complexe Ω étant ainsi determinés, la théorie des transformations linéaires donne aussitôt

$$K(\omega)=\alpha K(\Omega)+i\beta K'(\Omega),$$
$$iK'(\omega)=\gamma K(\Omega)+i\delta K'(\Omega);$$

de plus l'on vérifie sans peine que les équations (1) subsistent, tant que ω satisfait aux inégalités (3). En effet, tant qu'on a

$$-1<x<1, \quad -x<x^2+y^2<x,$$

les fonctions k^2 et k'^2 ne peuvent pas atteindre aucune des valeurs pour lesquelles les intégrales rectilignes cessent d'être complètement determinées. Et, quant aux cas limites qui se présentent lorsqu'on a

$$x=1, \text{ ou bien } x^2+y^2=x,$$

les conventions, que nous avons adoptées pour lever les ambiguités dans les définitions des intégrales Φ et Ψ, ont été choisies de manière à faire accorder dans

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* Voyez la Note *sur la théorie des fonctions elliptiques* ajoutée au *Calcul différentiel* de Lacroix, tom. ii. pp. 420-425, Paris, 1862.

ces mêmes cas limites, les valeurs des intégrales Φ et Ψ avec celles des fonctions K et K'. On a donc

$$K(\Omega) = \Phi(\phi^8(\Omega)), \qquad K'(\Omega) = \Psi(\phi^8(\Omega)),$$

ou bien, en observant que $\phi^8(\Omega) = \phi^8(\omega) = k^2$,

$$K(\Omega) = \Phi(k^2), \qquad K'(\Omega) = \Psi(k^2);$$

d'où l'on tire finalement

$$K(\omega) = \alpha\Phi(k^2) + i\beta\Psi(k^2),$$
$$iK'(\omega) = \gamma\Phi(k^2) + i\delta\Psi(k^2).$$

Ces équations remplacent les équations (1), et font connaître le chemin que l'intégration doit suivre dans les équations (2).

Nous ferons remarquer, en terminant, que la transformation de la quantité complexe ω, qui donne la solution du problème, est la même que Jacobi avait employée pour la réduction des fonctions Θ, sauf la différence qui provient des congruences auxquelles nous avons assujetti les nombres α, β, γ, δ.

XXXV.

MÉMOIRE SUR LES ÉQUATIONS MODULAIRES.

[Atti della R. Accademia dei Lincei. Memorie della classe di Scienze fisiche, matematiche e naturali. Ser. iii. vol. i. pp. 136–149. Read February 4, 1877.]

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1. On connait les beaux résultats auxquels sont parvenus MM. Kronecker et Hermite, en étudiant les rapports qui existent entre les équations modulaires et les formes quadratiques binaires à déterminant négatif. Mais les points de rapprochement, qu'on a trouvés jusqu'ici entre la théorie des équations modulaires et celle des formes quadratiques à déterminant positif, ont été peu nombreux ; et, à cet égard, nous ne saurions citer que le Mémoire si remarquable de M. Kronecker, 'sur la solution de l'équation de Pell par le moyen des fonctions elliptiques.' Cependant, nous avons été conduits à reconnaître qu'il existe entre ces deux théories des liens très intimes. C'est ce que nous nous proposons de faire voir dans ce Mémoire, en démontrant que, si l'on désigne par N un nombre entier quelconque, et par $\Phi(k^2, \lambda^2) = \Phi(\lambda^2, k^2) = 0$ une des équations symétriques, qui définissent les transformations modulaires du $N^{ième}$ ordre, la courbe représentée par l'équation cartésienne

$$\Phi\left(\tfrac{1}{2} + X + iY,\ \tfrac{1}{2} + X - iY\right) = 0$$

aura la propriété singulière de présenter une véritable image géométrique du système complet des formes quadratiques réduites appartenantes au déterminant positif N. C'est en suivant la route tracée par les illustres géomètres que nous venons de nommer, que nous avons été conduits à ce résultat, qui nous a paru offrir une intéressante application de l'arithmétique à la géométrie, aussi bien qu'à la théorie des fonctions elliptiques.

2. Soit $\omega = x + iy$ une quantité complexe, la valeur de y étant positive. Posons $\omega = \dfrac{\gamma + \delta\Omega}{\alpha + \beta\Omega}$; α, β, γ, δ étant des nombres entiers qui satisfont à l'équation

$$\alpha\delta - \beta\gamma = 1,$$

et aux congruences

$$\left|\begin{matrix}\alpha, \beta\\ \gamma, \delta\end{matrix}\right| \equiv \left|\begin{matrix}1, 0\\ 0, 1\end{matrix}\right|, \text{ mod } 2\,;\ \alpha \equiv \delta \equiv 1, \text{ mod } 4.$$

En représentant, comme on fait ordinairement, les quantités complexes par les points d'un espace de deux dimensions, nous exprimerons la relation qui subsiste entre ω et Ω, en disant que les deux points correspondants sont *équivalents*. Cette définition de l'équivalence est plus restreinte, et par conséquent, moins naturelle que celle qu'on emploie ordinairement en arithmétique ; et c'est uniquement pour abréger le discours que nous l'admettons ici. Dans le plan xy, dont toutefois nous ne considérons que la partie située au dessus de l'axe des abscisses, traçons les deux droites $P = x - 1 = 0$, $P^{-1} = x + 1 = 0$, et les deux demi-cercles

$$Q = x^2 + y^2 - x = 0, \quad Q^{-1} = x^2 + y^2 + x = 0.$$

Soit Σ l'espace compris entre les deux droites, mais extérieur aux deux cercles ; P et Q étant censés appartenir à cet espace, mais P^{-1} et Q^{-1} en étant exclus. Cela posé, on aura les propositions suivantes, qu'on déduit sans peine de la théorie de la réduction des formes quadratiques binaires à déterminant négatif.

'Étant donné un point quelconque ω, il existe toujours un point *réduit* (c'est à dire, un point appartenant à l'espace *réduit* Σ), qui est équivalent à ω ; et il n'en existe qu'un seul.' 'La substitution réduisante est aussi unique.'

Pour abréger, nous conviendrons de nommer *normales* les substitutions telles que $\left|\begin{matrix}\alpha, \beta\\ \gamma, \delta\end{matrix}\right|$.

3. Soit $N = b^2 - ac$; a, b, c étant des nombres entiers. L'équation

$$a + 2bx + c(x^2 + y^2) = 0$$

appartient à un cercle réel ; nous représenterons ce cercle par $[a, b, c]$, et la forme quadratique correspondante par (a, b, c). Nous conviendrons d'appeler *cercle rationnel* tout cercle tel que $[a, b, c]$; mais nous ne considérerons toujours que les demi-circonférences situées au-dessus de l'axe des x. Lorsque N est un carré, on peut avoir $c = 0$; dans ce cas le cercle rationnel devient une droite.

Soit (A, B, C) une forme équivalente à (a, b, c) par la substitution normale $\left|\begin{matrix}\alpha, \beta\\ \gamma, \delta\end{matrix}\right|$; les cercles correspondants $[A, B, C]$ et $[a, b, c]$ seront aussi équivalents par la même substitution. En effet, l'équation $\omega = \dfrac{\gamma + \delta\Omega}{\alpha + \beta\Omega}$ établit, entre les points ω et Ω des deux cercles $[a, b, c]$, $[A, B, C]$, cette espèce de correspondance géométrique qui a été appelée *affinité circulaire* par Moebius, et qui, à la vérité, ne diffère point essentiellement de la relation si connue de l'inversion. Il est bon

de remarquer que la transformation par affinité circulaire du cercle $[a, b, c]$ dans le cercle $[A, B, C]$, est en même temps une transformation homographique. Ainsi, lorsque N est non-carré, les substitutions automorphiques de la forme quadratique (a, b, c) sont représentées géométriquement par des transformations homographiques du cercle $[a, b, c]$ dans lui-même. Et de même que les substitutions automorphiques normales sont les puissances, positives ou négatives, d'une seule d'entr'elles ; de même les transformations homographiques, que nous avons à considérer par rapport au cercle $[a, b, c]$, proviennent de la répétition, dans les deux sens, d'une seule transformation fondamentale.

En supposant que $[a, b, c]$ soit un cercle primitif, et en désignant par t, u les moindres nombres positifs qui satisfont à l'équation $t^2 - Nu^2 = 1$, u étant pair, t impair, on trouve que l'ellipse, régulatrice de cette transformation fondamentale, est représentée par l'équation

$$a + 2bx + cx^2 + ct^2y^2 = 0.$$

L'excentricité de cette ellipse est donnée par l'équation $\sqrt{\frac{1+e}{1-e}} = t + u\sqrt{N}$; d'où l'on voit que les transformations homographiques, qui correspondent aux substitutions automorphiques normales, sont semblables pour tous les cercles primitifs du même déterminant.

4. Soit $\sigma = 2$ ou $= 1$, selon que le plus petit nombre u, qui satisfait à l'équation $t^2 - Nu^2 = 1$, est pair ou impair. Il est facile de vérifier que la restriction, que nous avons dû apporter à la définition de l'équivalence, entraîne la répartition des formes de chaque classe proprement primitive en 3σ classes subalternes, qui satisfont, en nombre égal, aux conditions exprimées par les congruences

$$(A) \quad a \equiv c \equiv 1, \text{ mod } 2,$$
$$(B) \quad a \equiv 0, c \equiv 1, \text{ mod } 2,$$
$$(C) \quad a \equiv 1, c \equiv 0, \text{ mod } 2.$$

Pareillement, lorsque $N \equiv 1$, mod 8, chaque classe de formes improprement primitives se partage en six classes subalternes, dont il y a toujours deux qui satisfont à chacun des systèmes

$$(A') \quad a \equiv c \equiv 0, \text{ mod } 4,$$
$$(B') \quad a \equiv 0, c \equiv 2, \text{ mod } 4,$$
$$(C') \quad a \equiv 2, c \equiv 0, \text{ mod } 4.$$

Enfin, lorsque $N \equiv 5$, mod 8, chaque classe improprement primitive contient $2\sigma'$ classes subalternes, σ' étant $= 1$, ou $= 3$, selon que l'équation $t^2 - Nu^2 = 4$ est résoluble ou irrésoluble en nombres impairs ; de plus, chacune des congruences

$$(A''), \quad b \equiv 1, \text{ mod } 4, \qquad (B''), \quad b \equiv -1, \text{ mod } 4,$$

est satisfaite par σ' de ces classes subalternes.

Soient h, h' les nombres des classes proprement et improprement primitives, qui appartiennent au déterminant N; soient H, H' les nombres correspondants des classes subalternes de cercles. En observant que les deux fermes (a, b, c), $(-a, -b, -c)$ ne correspondent qu'à un seul cercle $[a, b, c]$, et que ces deux formes appartiennent toujours à des classes subalternes différentes, bien que, dans certains cas, elles peuvent être comprises dans la même classe, on parvient à établir les deux équations

$$H = \tfrac{3}{2}\sigma h, \quad H' = \sigma' h'.$$

5. Maintenant, aux points du cercle primitif $[a, b, c]$ substituons les points réduits correspondants. Ce cercle se changera en un assemblage d'arcs circulaires réduits, dont la totalité formera une ligne L, qui en apparence sera brisée, mais dont on mettra en évidence la continuité, en repliant sur lui-même l'espace Σ, de manière à former une surface fermée tricuspide, les droites P, P^{-1} étant réunies ensemble, et aussi les cercles Q, Q^{-1}. Il importe surtout de savoir quels sont les cercles équivalents à $[a, b, c]$ qui traversent Σ; et quelle est la loi qui, dans la ligne composée L, gouverne la succession des arcs réduits. Voici la solution de ce problème pour le cas d'un déterminant non-carré.

Soit θ l'une ou l'autre des racines de l'équation $a + 2b\theta + c\theta^2 = 0$. Développons θ en fraction continue, en prenant toujours pour quotient intégral le nombre pair, positif ou négatif, qui est le plus rapproché du quotient complet correspondant. Soit

$$\theta_1 = 2\epsilon_1\mu_1 + \frac{1}{2\epsilon_2\mu_2} + \ldots + \frac{1}{2\epsilon_{2s}\mu_{2s}} + \frac{1}{\theta_1},$$

la période de la fraction continue; les nombres entiers $\mu_1, \mu_2, \ldots, \mu_{2s}$ étant positifs; $\epsilon_1, \epsilon_2, \ldots, \epsilon_{2s}$ désignant des unités positives ou négatives; et le quotient complet θ_1 étant de rang impair dans le développement de θ. Posons aussi

$$\theta_1 = 2\epsilon_1\mu_1 + \frac{1}{\theta_2}, \ \theta_2 = 2\epsilon_2\mu_2 + \frac{1}{\theta_3}, \ldots, \theta_{2s} = 2\epsilon_{2s}\mu_{2s} + \frac{1}{\theta_1};$$

et considérons les $\Sigma\mu$ quantités irrationnelles

$$\begin{array}{llll}
\theta_1, & \theta_1 - 2\epsilon_1, & \theta_1 - 4\epsilon_1, \ldots, & \theta_1 - 2(\mu_1 - 1)\epsilon_1, \\
\dfrac{1}{\theta_2}, & \dfrac{1}{\theta_2 - 2\epsilon_2}, & \dfrac{1}{\theta_2 - 4\epsilon_2}, \ldots, & \dfrac{1}{\theta_2 - 2(\mu_2 - 1)\epsilon_2}, \\
\theta_3, & \theta_3 - 2\epsilon_3, & \theta_3 - 4\epsilon_3, \ldots, & \theta_3 - 2(\mu_3 - 1)\epsilon_3, \\
\ldots & \ldots & \ldots & \ldots \\
\dfrac{1}{\theta_{2s}}, & \dfrac{1}{\theta_{2s} - 2\epsilon_{2s}}, & \dfrac{1}{\theta_{2s} - 4\epsilon_{2s}}, \ldots, & \dfrac{1}{\theta_{2s} - 2(\mu_{2s} - 1)\epsilon_{2s}}.
\end{array}$$

Chacune de ces $\mu_1+\mu_2+\ldots$ quantités est la racine d'une équation quadratique équivalente à $a+2b\theta+c\theta^2=0$; par conséquent, les cercles correspondants sont équivalents à $[a, b, c]$. Or, tous ces cercles traversent l'espace Σ, et la ligne L est composée des parties de leur circonférences qui se trouvent dans l'intérieur de cet espace, ces parties étant prises dans l'ordre indiqué par le développement.

6. Les formes quadratiques correspondantes aux cercles dont nous venons de parler, diffèrent des formes réduites de Gauss, (1°) parceque nous nous servons ici d'une fraction continue avec des quotients pairs ; (2°) parceque nous admettons parmi les formes réduites, non seulement les formes réduites *principales* qui correspondent aux quotients complets $\theta_1, \frac{1}{\theta_2}, \ldots$, mais aussi les formes *intermédiaires*, correspondantes aux racines

$$\theta_1-2\epsilon_1, \ldots, \frac{1}{\theta_2-2\epsilon_2}, \ldots.$$

(3°) parceque nous prenons pour les racines des équations qui correspondent aux quotients de rang pair, les quantités

$$\frac{1}{\theta_2}, \frac{1}{\theta_4}, \ldots, \frac{1}{\theta_{2s}},$$

au lieu des quantités $-\theta_2, -\theta_4, \ldots, -\theta_{2s}$. Il est presque inutile d'ajouter que, pour avoir la suite des arcs réduits, on pourrait se servir, au lieu du développement en fraction continue, de l'algorithme des formes contigües de Gauss, en y apportant une légère modification.

7. Lorsque le déterminant est carré, les arcs reduits, équivalents à un cercle donné, forment toujours une suite continue ; mais cette suite, au lieu d'être périodique, commence avec un arc passant par un des points singuliers de la surface tricuspide, et se termine de la même manière. Désignons les points $(0, \infty)$, $(0, 0)$ par p et q ; et l'un ou l'autre des deux points équivalents $(1, 0)$, $(-1, 0)$ par r. La suite des cercles réduits, équivalents à un cercle proprement primitif donné, aura pour ses points extrêmes rr, qq, pp, selon que le cercle donné satisfait aux congruences (A), (B), (C) de l'article 4. Pareillement, dans l'ordre improprement primitif, les points extrêmes de la suite des cercles réduits, équivalents à un cercle donné, seront pq, qr, rp, selon que l'équation de ce cercle satisfait aux congruences (A'), (B'), (C') de l'article cité. Pour déterminer complètement la ligne L qui correspond à un cercle donné $[a, b, c]$, dont le déterminant est un nombre carré, il suffira de connaître les équations des deux cercles extrêmes de L, et d'en déduire la substitution normale unique qui transforme l'un d'eux dans l'autre. Soit, en effet,

$$\omega=\frac{\gamma+\delta\Omega}{\alpha+\beta\Omega},$$

cette substitution ; on en déduira le développement fini

$$\omega = 2\epsilon_1\mu_1 + \frac{1}{2\epsilon_2\mu_2} + \ldots + \frac{1}{2\epsilon_{2s}\mu_{2s}} + \frac{1}{\Omega};$$

où il faut remarquer qu'on peut avoir $\mu_1 = 0$, $\mu_{2s} = 0$. Ce développement remplacera la fraction continue périodique de l'art. 6, et fera connaître tous les cercles de L dans leur ordre naturel de succession. Tout se réduit donc à trouver les équations des deux cercles extrêmes de L. Pour cela, soit

$$(a, b, c)\,(x, y)^2 = m\,(px + p'y)\,(qx + q'y),$$

m étant le plus grand diviseur commun de a, $2b$, c. En désignant toujours par $\left|\begin{matrix}\alpha, \beta\\ \gamma, \delta\end{matrix}\right|$ une substitution normale, et en regardant comme inconnus les nombres entiers α, β, γ, δ, η, η', λ, λ', l'équation indéterminée

$$\left|\begin{matrix}p, p'\\ q, q'\end{matrix}\right| \times \left|\begin{matrix}\alpha, \beta\\ \gamma, \delta\end{matrix}\right| = \left|\begin{matrix}\eta, \eta'\\ \lambda, \lambda'\end{matrix}\right|$$

admet une solution unique, dans laquelle les valeurs absolues de η, η' ne surpassent pas l'unité, et celles de λ, λ' ne surpassent pas $\pm\,(pq' - p'q) = \frac{2}{m}\sqrt{N}$. Le cercle $[m\eta\lambda, \frac{1}{2}m(\eta\lambda' + \eta'\lambda), m\eta'\lambda']$ est l'un des deux cercles cherchés ; l'autre peut s'obtenir en échangeant entr'eux dans la solution précédente, les deux facteurs de (a, b, c). Mais, un des deux cercles extrêmes étant trouvé, il vaut mieux partir de l'équation nouvelle

$$\left|\begin{matrix}\lambda, \lambda'\\ \eta, \eta'\end{matrix}\right| \times \left|\begin{matrix}\alpha, \beta\\ \gamma, \delta\end{matrix}\right| = \left|\begin{matrix}\eta_1, \eta'_1\\ \lambda_1, \lambda'_1\end{matrix}\right|,$$

puisque ainsi on est conduit immédiatement à la substitution $\left|\begin{matrix}\alpha, \beta\\ \gamma, \delta\end{matrix}\right|$ qui transforme le premier cercle dans le second. On tire aussi de cette équation la conclusion, très importante pour notre but actuel, que si l'on connaît la substitution $\left|\begin{matrix}\alpha, \beta\\ \gamma, \delta\end{matrix}\right|$, et les deux points extrêmes de L, on a tout ce qu'il faut pour pouvoir déterminer les équations des deux cercles extrêmes et, par conséquent, les équations de tous les cercles réduits, pris dans leur ordre naturel.

8. Le nombre N étant quelconque, les arcs réduits équivalents à un cercle donné sont de six espèces différentes, qu'on peut distinguer entr'elles par les symboles $(P\,P^{-1})$, $(Q\,Q^{-1})$, $(P\,Q)$, $(P\,Q^{-1})$, $(P^{-1}\,Q)$, $(P^{-1}\,Q^{-1})$, qui indiquent les différentes parties du contour de Σ, sur lesquelles se trouvent les points d'entrée et de sortie de l'arc que l'on considère. Lorsque N est carré, les cercles

extrêmes restent exclus de cette classification; on pourrait, au besoin, les exprimer par les symboles $(p\,Q)$, $(p\,Q^{-1})$, $(q\,P)$, $(q\,P^{-1})$, $(r\,P^{-1})$, $(r\,Q^{-1})$, $(r'P)$, $(r'Q)$. Il est aussi convenable de distinguer entre deux symboles tels que (P, Q) et (Q, P), pour pouvoir indiquer le sens dans lequel l'arc réduit est censé d'être parcouru. Cela posé, la table suivante fera connaître l'espèce de l'arc réduit qui, dans une fraction continue quelconque, correspond à un quotient donné.

r impair; $s = 1, 2, \ldots, \mu_r - 1$.

ϵ_{r-1}	ϵ_r	θ_r	$\theta_r - 2s\epsilon_r$
$+1$	$+1$	$(Q^{-1}\,P)$	$(P^{-1}\,P)$
$+1$	-1	$(Q^{-1}\,P^{-1})$	$(P^{-1}\,P)$
-1	$+1$	$(Q\,P)$	$(P^{-1}\,P)$
-1	-1	$(Q\,P^{-1})$	$(P\,P^{-1})$

r pair; $s = 1, 2, \ldots, \mu_r - 1$.

ϵ_{r-1}	ϵ_r	θ_r	$\theta_r - 2s\epsilon_r$
$+1$	$+1$	$(P^{-1}\,Q)$	$(Q^{-1}\,Q)$
$+1$	-1	$(P^{-1}\,Q^{-1})$	$(Q\,Q^{-1})$
-1	$+1$	$(P\,Q)$	$(Q^{-1}\,Q)$
-1	-1	$(P\,Q^{-1})$	$(Q\,Q^{-1})$

Réciproquement, étant donné le tracé des arcs réduits, équivalents à un cercle quelconque, la table servira pour retrouver la fraction continue, et, par conséquent, les équations des cercles réduits. Mais on peut obtenir le même résultat sans faire usage de la table. Posons

$$|\,P\,| = \begin{vmatrix} 1, & 0 \\ 2, & 1 \end{vmatrix}; \quad Q = \begin{vmatrix} 1, & 2 \\ 0, & 1 \end{vmatrix};$$

et observons que toute substitution normale $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$ peut se mettre, et cela d'une manière unique, sous la forme

$$|\,P\,|^{\epsilon_1\mu_1} \times |\,Q\,|^{\epsilon_2\mu_2} \ldots |\,Q\,|^{\epsilon_{2s}\mu_{2s}};$$

les exposants $\epsilon_1\mu_1, \epsilon_2\mu_2, \ldots, \epsilon_{2s}\mu_{2s}$, (dont le premier et le dernier peuvent s'évanouir) étant les mêmes nombres qui se présentent dans le développement

$$\omega = \frac{\gamma + \delta\,\Omega}{\alpha + \beta\,\Omega} = 2\epsilon_1\mu_1 + \frac{1}{2\epsilon_2\mu_2} + \ldots + \frac{1}{2\epsilon_{2s}\mu_{2s}} + \frac{1}{\Omega}.$$

D'un autre côté, un arc réduit qui se termine en P, P^{-1}, Q, Q^{-1} est toujours suivi par un arc réduit qui commence en P^{-1}, P, Q^{-1}, Q; et les substitutions qui correspondent à ces quatre cas sont respectivement

$$|P|,\ |P|^{-1},\ |Q|,\ |Q|^{-1}.$$

Il suit de là que, pour avoir les exposants $\epsilon_1\mu_1$, $\epsilon_2\mu_2$, ..., il suffira de compter les arcs réduits, en faisant attention à leurs points d'entrée et de sortie. Ainsi,

par exemple, dans le cas d'un déterminant non-carré, supposons qu'on commence la période avec un arc réduit (θ) qui prend son origine en Q ou Q^{-1} et qui se termine en P^{ϵ_1}; μ_1 sera le nombre des arcs réduits, y compris (θ) lui même, qu'on aura à parcourir avant de venir à un arc qui se termine en Q ou Q^{-1}; supposons encore que le premier arc réduit, qui n'aboutit pas en P^{ϵ_1}, se termine en Q^{ϵ_2}; μ_2 sera le nombre des arcs réduits qui se terminent en Q^{ϵ_2}, avant qu'on arrive à un arc qui aboutit en P ou P^{-1}; et ainsi de suite.

On remarquera que les arcs des deux premières espèces correspondent aux quotients intermédiaires

$$\theta_1 - 2\epsilon_1, \ldots, \frac{1}{\theta_2 - 2\epsilon_2}, \ldots;$$

tandis que ceux des quatre dernières correspondent aux quotientes complets θ_1, $\frac{1}{\theta_2}$.... On peut ajouter que tout cercle qui coupe le cercle $x^2+y^2=1$ est un cercle réduit d'un de ces quatre espèces, et réciproquement.

9. Pour faire l'application de ce qui précède aux fonctions modulaires, soit toujours $\omega = x+iy$, et posons, avec M. Hermite, $k^2 = \phi^8(\omega)$, $k'^2 = \psi^8(\omega)$; faisons aussi $\phi^8(\omega) = \frac{1}{2} + X + iY$, $\psi^8(\omega) = \frac{1}{2} - X - iY$, X et Y étant des quantités réelles. À chaque point ω du plan xy ou, si l'on veut, à chaque point seulement de l'espace réduit Σ, faisons correspondre le point $X+iY$ du plan illimité XY; c'est pour mettre en évidence la symétrie des figures, que nous avons choisi pour origine O des axes rectangulaires OX, OY le point $\phi^8(\omega) = \psi^8(\omega) = \frac{1}{2}$, qui a pour correspondant le point $\omega = i$. On sait que, dans une telle transformation, les parties infinitésimales correspondantes des deux figures sont, en général, semblables. Dans le cas actuel, il n'y a exception que pour les trois points singuliers p, q, r, auxquels correspondent respectivement les points $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$, et l'infini du plan XY. De plus, si l'on ne considère que l'espace réduit, la correspondance sera parfaitement déterminée; de sorte qu'à chaque point réel de Σ il correspondra toujours un seul point réel de XY, et réciproquement.

Maintenant il est facile de voir que, si l'on suppose réelles les quantités a, b, c, d, x, y, l'équation

$$-x+iy = \frac{c+d(x+iy)}{a+b(x+iy)}$$

entraîne les deux suivantes

$$a = d,$$

$$c + 2dx + b(x^2+y^2) = 0.$$

De cette seule observation on tire immédiatement le théorème que voici :

'Tous les cercles rationnels du plan xy sont représentés, dans le plan XY, par des courbes algébriques.'

'Les équations des ces courbes se déduisent des équations modulaires en posant $k^2 = \frac{1}{2} + X + iY$, $\lambda^2 = \frac{1}{2} + X - iY$.'

10. Considérons d'abord quelques cas particuliers, et désignons par A_1, A_2 les points $(\frac{1}{2}, 0)$, $(-\frac{1}{2}, 0)$. Le cercle $x^2+y^2=x$, et les droites $x=0$, $x=1$ sont représentés dans le plan XY par les trois parties de l'axe des X, depuis $+\infty$ jusqu'à A_1, depuis A_1 jusqu'à A_2, et depuis A_2 jusqu'à $-\infty$. L'axe des Y depuis $-\infty$ jusqu'à $+\infty$, correspond au cercle $x^2+y^2=1$, qui est le seul dont la demi-circonférence soit entièrement comprise dans l'espace Σ. Enfin, les deux cercles équivalents $x^2+y^2 = \pm 2x$, et les deux droites équivalentes $2x = \pm 1$, sont représentés respectivement par les cercles $(X+\frac{1}{2})^2 + Y^2 = 1$, $(X-\frac{1}{2})^2 + Y^2 = 1$.

On voit que les cercles rationnels de déterminant $+1$ sont représentés par des droites et des cercles dans le plan XY; ce sont les seuls cercles rationnels qui jouissent de cette propriété. Ces cercles divisent l'espace Σ en douze parties distinctes; la considération de cette division, et de la division correspondante du plan XY, est très importante pour la théorie des transformations du premier ordre. On peut aussi remarquer que deux points, symétriques par rapport à une droite de déterminant $+1$, ou inverses par rapport à un cercle de déterminant $+1$, sont remplacés dans le plan XY par deux points qui ont la même propriété par rapport à la ligne correspondante.

11. Ceci suffit pour les cercles de déterminant $+1$, tant proprement qu'improprement primitifs. La table suivante, dans laquelle nous avons posé

$$R^2 = X^2 + Y^2, \quad R_1^2 = (X-\tfrac{1}{2})^2 + Y^2, \quad R_2^2 = (X+\tfrac{1}{2})^2 + Y^2,$$

donne les résultats correspondants pour les déterminants 2, 3, 4, 5.

$n = 2$

Fraction continue	Cercles réduits	Courbe modulaire
$[2, 2]$	$(x-1)^2+y^2=2$ $(x+1)^2+y^2=2$	$R^4 = \frac{1}{2}(X^2-Y^2)$
$[4, -2]$	$(x-2)^2+y^2=2$ $x^2+y^2=2$ $(x+2)^2+y^2=2$	$R_1^4 = 16R_2^2$
$[-2, 4]$	$2(x+1)^2+2y^2=1$ $2(x-1)^2+2y^2=1$ $2x^2+2y^2=1$	$R_2^4 = 16R_1^2$

$n = 3$

$[4, -4]$	$(x-2)^2 + y^2 = 3$ $x^2 + y^2 = 3$ $(x+2)^2 + y^2 = 3$ $3x^2+3y^2 = 1$	$16\,(R^2-\frac{3}{4}) \times [(R^2+\frac{1}{4})^2-1]+X^4 = 0$
$[2, 2, -2, 2]$	$(x-1)^2+ y^2 = 3$ $(x+1)^2+ y^2 = 3$ $3\,(x+1)^2+3y^2 = 1$ $3\,(x-1)^2+3y^2 = 1$	$2^7 R^2 R_2^2 (2R_1^2-R_2^2-1)-(R_2^2-1)^4 = 0$
$[2, -2, 2, 2]$	$2\,(x^2+y^2)-2x-1 = 0$ $2\,(x^2+y^2)+6x+3 = 0$ $2\,(x^2+y^2)-6x+3 = 0$ $2\,(x^2+y^2)+2x-1 = 0$	$2^7 R_1^2 R_2^2 (2R_2^2-R_1^2-1)-(R_1^2-1)^4 = 0$

$n = 4$

$\omega = 2+\Omega$	$(x-1)^2+y^2 = 4$ $(x+1)^2+y^2 = 4$	$(1+2^7 R_1^2 R_2^2)^2-2^{14} R_1^2 R_2^2 R^4$ $= 2^{13}(Y^2-X^2) R^4+2^8.3.7.R^4$
$\omega = \dfrac{1}{2+\dfrac{1}{\Omega}}$	$3\,(x^2+y^2)+2x-1 = 0$ $3\,(x^2+y^2)-2x-1 = 0$	$-2^{14}X^2Y^2+2^7.3^2(Y^2-X^2)+3^4 = 0$
$\omega = 4+\Omega$	$(x-2)^2+y^2 = 4$ $x^2+y^2 = 4$ $(x+2)^2+y^2 = 4$	$(R_1^4+2^7 R_2^2)^2-2^{10} R_2^2 [(X+\frac{3}{2})^2+Y^2]^2 = 0$
$\omega = -2+\dfrac{1}{2+\dfrac{1}{-2+\Omega}}$	$3x^2+3y^2+4x = 0$ $3x^2+3y^2-8x+4 = 0$ $3x^2+3y^2+8x+4 = 0$ $3x^2+3y^2-4x = 0$	
$\omega = \dfrac{1}{-4+\dfrac{1}{\Omega}}$	$4x+1 = 0$ $4x^2+4y^2 = 1$ $4x-1 = 0$	$(R_2^4+2^7 R_1^2)^2-2^{10} R_1^2 [(X-\frac{3}{2})^2+Y^2] = 0$
$\omega = \dfrac{1}{2+\dfrac{1}{-2+\dfrac{1}{2+\dfrac{1}{\Omega}}}}$	$4x-3 = 0$ $4x^2+4y^2+8x+3 = 0$ $4x^2+4y^2-8x+3 = 0$ $4x+3 = 0$	

$n = 5$

$[4, 4]$	$(x-2)^2+ y^2 = 5$ $x^2+ y^2 = 5$ $(x+2)^2+ y^2 = 5$ $5x^2+5y^2 = 1$	$[(1+4R^2)^2-16X^2]$ $\times[4R^2(9+4R^2)^2-9X^2(4R^2+45)]$ $+X^6 = 0$
$[6, -2, 2, -2]$	$(x-3)^2+ y^2 = 5$ $(x-1)^2+ y^2 = 5$ $(x+1)^2+ y^2 = 5$ $(x+3)^2+ y^2 = 5$ $5(x-1)^2+5y^2 = 1$ $5(y+1)^2+5y^2 = 1$	$(R_2^2-1)^6+2^9 R_1^2 R_2^2$ $\times[(1+R_2^2)(2^5 R_2^2-7(1-R_2^2)^2+3.2^8 Y^2)$ $+(2X+1)(2^5 R_2^2+78(1-R_2^2)^2-2^9 Y^2)]$ $= 0$
$[-2, 2, -2, 6]$	$4(x^2+y^2)+ 6x+1=0$ $4(x^2+y^2)-10x+5=0$ $4(x^2+y^2)+10x+5=0$ $4(x^2+y^2)- 6x+1=0$ $4(x^2+y^2)+ 2x-1=0$ $4(x^2+y^2)- 2x-1=0$	$(R_1^2-1)^6+2^9 R_1^2 R_2^2$ $\times[(1+R_1^2)(2^5 R_1^2-7(1-R_1^2)^2+3.2^8 Y^2)$ $-(2X-1)(2^5 R_1^2+78(1-R_1^2)^2-2^9 Y^2)]$ $= 0$
$[2, -2, -2, 2]$	$2(x^2+y^2)-2x-2 = 0$ $2(x^2+y^2)+6x+2 = 0$ $2(x^2+y^2)+2x-2 = 0$ $2(x^2+y^2)-6x+2 = 0$	$[(1-4R^2)^3+16Y^3]$ $\times[9Y^2(45-4R^2)-4R^2(9-4R^2)^3]$ $-Y^6 = 0$

12. Maintenant, pour éclaircir le théorème général de l'article 9, il faut rappeler quelques résultats relatifs aux équations modulaires.

(i) Soit N un nombre impair ; désignons par $F(k^2, \lambda^2, N) = 0$ l'équation modulaire normale pour les transformations du $N^{\text{ième}}$ ordre, dans laquelle, lorsque N admet des diviseurs carrés, nous supposerons qu'on ait supprimé les facteurs correspondants aux transformations d'un ordre inférieur. En posant $k^2 = \Phi^8(\omega)$, on sait que les racines de cette équation sont comprises dans la formule $\lambda^2 = \Phi^8\left(\frac{\gamma\omega+2k}{\gamma'}\right)$, dans laquelle γ, γ' sont des diviseurs conjugués de N ; k désignant un terme quelconque d'un système de résidus pour le module γ', et les nombres γ, γ', k étant assujettis à ne pas avoir un diviseur commun. Cette équation est symétrique par rapport à k^2 et λ^2 ; en outre, elle ne change pas lorsqu'on substitue k^2 et λ^2 deux fonctions semblables, prises parmi les six fonctions anharmoniques que voici

$$k^2,\quad 1-k^2,\quad \frac{1}{k^2},\quad \frac{1}{1-k^2},\quad \frac{k^2}{k^2-1},\quad \frac{k^2-1}{k^2};$$

$$\lambda^2,\quad 1-\lambda^2,\quad \frac{1}{\lambda^2},\quad \frac{1}{1-\lambda^2},\quad \frac{\lambda^2}{\lambda^2-1},\quad \frac{\lambda^2-1}{\lambda^2}.$$

En substituant des fonctions dissemblables, on obtient le système complet des équations modulaires du $N^{\text{ième}}$ ordre. Nous écrirons ces équations, comme il suit :

(i) $F(k^2, 1-\lambda^2)=0$, (ii) $F\left(k^2, \frac{1}{\lambda^2}\right)=0$,

(iii) $F\left(k^2, \frac{\lambda^2}{\lambda^2-1}\right)=0$, (iv) $F(k^2, \lambda^2)=0$,

(v) $F\left(k^2, \frac{\lambda^2}{\lambda^2-1}\right)=0$; (vi.) $F\left(k^2, \frac{1}{1-\lambda^2}\right)=0$,

elles correspondent, comme on sait, aux six formes différentes

$$\left|\begin{matrix}0,1\\1,0\end{matrix}\right|,\ \left|\begin{matrix}1,1\\0,1\end{matrix}\right|,\ \left|\begin{matrix}1,0\\1,1\end{matrix}\right|,\ \left|\begin{matrix}1,0\\0,1\end{matrix}\right|,\ \left|\begin{matrix}1,1\\1,0\end{matrix}\right|,\ \left|\begin{matrix}0,1\\1,1\end{matrix}\right|,$$

que peut avoir une substitution de déterminant impair par rapport au module 2. Les équations (v) et (vi) s'échangent entr'elles lorsqu'on permute k^2 et λ^2; les quatre premières sont symétriques par rapport à k^2 et λ^2. Donc en écrivant $k^2=\frac{1}{2}+X+iY$, $\lambda^2=\frac{1}{2}+X-iY$, dans les équations (i), (ii), (iii), (iv) on aura les équations réelles de quatre courbes géométriques, que nous appellerons désormais la *première*, la *seconde*, la *troisième* et la *quatrième courbe modulaire*; nous observons toutefois que lorsque $N\equiv 3$, mod 4, la quatrième courbe se réduit aux deux points conjugués $(\pm\frac{1}{2}, 0)$. Des équations (v) et (vi) on déduit les équations de deux courbes imaginaires conjuguées, dont nous ne nous occuperons pas dans ce Mémoire.

(ii) Soit $N=2^\mu$. Dans ce cas, on a l'équation modulaire normale $f(k^2, \lambda^2, 2^\mu)=0$, dont les racines sont données par la formule $k^2=\phi^8(\omega)$, $\lambda^2=\phi^8\left(\frac{2h+\omega}{2^\mu}\right)$, en désignant par h un terme quelconque d'un système de résidus pour le module 2^μ. Cette équation n'est pas symétrique par rapport à k^2, λ^2; mais elle jouit des deux propriétés de ne pas se changer (1°) lorsqu'on écrit $\frac{1}{k^2}$ pour k^2, (2°) lorsqu'on écrit $\frac{\lambda^2}{\lambda^2-1}$ pour λ^2. Il suit de là que les trente-six substitutions anharmoniques ne donnent que neuf équations différentes :

(i) $f\left(\frac{k^2}{k^2-1}, \frac{1}{\lambda^2}\right)=0,$ (iii) $f(1-k^2, \lambda^2)=0,$

(ii) $f(k^2, 1-\lambda^2)=0,$

(iv) $f(k^2, \lambda^2)=0,$ (v) $f(1-k^2, 1-\lambda^2)=0,$

(vi) $f\left(1-k^2, \frac{1}{\lambda^2}\right)=0,$ (vii) $f\left(1-\lambda^2, \frac{1}{k^2}\right)=0,$

(viii) $f\left(k^2, \frac{1}{\lambda^2}\right)=0,$ (ix) $f\left(\lambda^2, \frac{1}{k^2}\right)=0,$

qui correspondent respectivement aux neuf formes différentes

$$\left|\begin{matrix}1,1\\1,1\end{matrix}\right|, \left|\begin{matrix}0,1\\0,0\end{matrix}\right|, \left|\begin{matrix}0,0\\1,0\end{matrix}\right|, \left|\begin{matrix}1,0\\0,0\end{matrix}\right|, \left|\begin{matrix}0,0\\0,1\end{matrix}\right|, \left|\begin{matrix}0,0\\1,1\end{matrix}\right|, \left|\begin{matrix}1,0\\1,0\end{matrix}\right|, \left|\begin{matrix}1,1\\0,0\end{matrix}\right|, \left|\begin{matrix}0,1\\0,1\end{matrix}\right|,$$

que peut avoir une substitution de déterminant pair (la forme $\left|\begin{matrix}0,0\\0,0\end{matrix}\right|$ étant exclue) par rapport au module 2. En effet, les racines de ces équations sont données par les formules :

(i) $\lambda^2=\phi^8\left(\frac{2h+1+\omega}{2^\mu+2h+1+\omega}\right),$ (iii) $\lambda^2=\phi^8\left(\frac{-1+2h\omega}{2^\mu\omega}\right),$

(ii) $\lambda^2=\phi^8\left(\frac{-2^\mu}{2h+\omega}\right),$

(iv) $\lambda^2=\phi^8\left(\frac{2h+\omega}{2^\mu}\right),$ (v) $\lambda^2=\phi^8\left(\frac{2^\mu\omega}{1+2h\omega}\right),$

(vi) $\lambda^2=\phi^8\left(\frac{-1+2h\omega}{-1+(2^\mu+2h)\omega}\right),$ (vii) $\lambda^2=\phi^8\left(\frac{-1-(2h+1)\omega}{2^\mu\omega}\right),$

(viii) $\lambda^2=\phi^8\left(\frac{2h+\omega}{2^\mu+2h+\omega}\right),$ (ix) $\lambda^2=\phi^8\left(\frac{2^\mu\omega}{1+(2h+1)\omega}\right).$

Les équations (i), (ii), (iii) sont symétriques par rapport à k^2 et λ^2 ; en y écrivant $k^2=\frac{1}{2}+X+iY$, $\lambda^2=\frac{1}{2}+X-iY$, comme ci-dessus, on obtient les équations de la première, de la seconde, et de la troisième courbe modulaire. Les équations (iv) et (v), (vi) et (vii), (viii) et (ix) s'échangent entr'elles, lorsqu'on échange k^2 et λ^2 ; par conséquent, elles ne fournissent que des courbes imaginaires conjuguées.

(iii) Soit enfin $N=2^\mu n$, n étant impair. Dans ce cas encore il y a neuf équations modulaires ; on les obtient successivement, en éliminant z de l'équation $F(z, \lambda^2, n)=0$, et de chacune des neuf équations modulaires de l'ordre 2^μ, dans lesquelles on remplace λ^2 par z. En effet, en désignant par N_1 et N_2 deux nombres premiers entr'eux, et par $f_1(k^2, \lambda^2, N_1)=0$, $f_2(k^2, \lambda^2, N_2)=0$ deux équations modulaires appartenantes aux ordres N_1 et N_2 respectivement, le résultat

de l'élimination de z des deux équations $f_1(k^2, z, N_1) = 0$, et $f_2(z, \lambda^2, N_2) = 0$, est toujours une des équations modulaires de l'ordre $N_1 \times N_2$. Si N_1, N_2 avaient un diviseur commun, cette proposition serait encore vraie ; mais le résultat serait encombré de facteurs étrangers, qu'on peut assigner *à priori*, et qui appartiennent à des transformations d'ordre inférieur. Dans le cas actuel, on vérifie facilement que les éliminations indiquées fournissent le système complet des équations modulaires pour les transformations de l'ordre $2^\mu \times n$. Les trois premières de ces équations sont les seules symétriques, les autres étant conjuguées par couples. On en déduit (comme dans le cas précédent) les équations de trois courbes modulaires réelles, et de trois couples de courbes imaginaires conjuguées.

Le théorème suivant résulte de cette discussion :

'En désignant par N un nombre quelconque positif, les cercles proprement primitifs de déterminant N, qui appartiennent aux classes subalternes (A), (B), (C), sont représentés respectivement dans le plan (XY) par la première, la seconde, et la troisième courbe modulaire.'

'Lorsque $N \equiv 1$, mod 4, les cercles improprement primitifs sont représentés par la quatrième courbe.'

13. Soit, en premier lieu, N un nombre non-carré. Les courbes modulaires ne peuvent rencontrer la ligne droit à l'infini que dans les deux points cycliques imaginaires ; de plus, aucune de ces courbes ne peut avoir une branche réelle passant par A_1 ou A_2 : toutefois le point A_1 peut appartenir, comme point conjugué, à la seconde courbe, le point A_2 à la troisième, et tous les deux à la quatrième. Chacune des trois premières courbes est composée de $\frac{1}{3}H = \frac{1}{2}\sigma\mathrm{h}$ parties fermées, entièrement distinctes entr'elles, et dont la forme générale est celle d'une spirale lemniscatique entrelacée, qui s'entortille alternativement autour des deux points A_1, A_2. Chaque spirale peut être considérée à volonté comme représentant, soit une classe subalterne de cercles proprement primitifs, soit un cercle unique choisi arbitrairement dans cette classe subalterne. De même, lorsque $N \equiv 1$, mod 4, la quatrième courbe modulaire est composée de $H' = \sigma'\,\mathrm{h}'$ spirales, qui représentent respectivement les H' classes subalternes des cercles improprement primitifs.

Soit, en second lieu, $N = n^2$ un nombre carré. Désignons par Δ le nombre des solutions de la congruence $a^2 + 1 \equiv 0$, mod $2n$, de sorte qu'on ait $\Delta = 0$, si n est pair, ou divisible par un nombre premier de la forme $4m + 3$. Comme dans le cas précédent, chaque courbe modulaire est composée d'un certain nombre de spirales distinctes ; mais ici chaque spirale doit passer soit par le point A_1, soit par le point A_2 ; ou bien elle doit avoir un point à l'infini ; de plus une spirale peut être simple, ou multiple ; c'est à dire qu'elle peut satisfaire à ces conditions

soit une fois, soit plusieurs fois. Dans chacune des trois premières courbes il y a Δ spirales simples, et $\frac{1}{6}(H-3\Delta)=\frac{1}{2}(h-\Delta)$ spirales doubles; chaque spirale ayant soit un point, soit deux points, à l'infini dans la première courbe, et passant soit une fois, soit deux fois, par le point A_1 dans la seconde courbe, et par le point A_2 dans la troisième courbe. Enfin, dans la quatrième courbe, qui n'existe que lorsque n est impair, il y a Δ spirales triples, $\frac{1}{6}(H'-3\Delta)=\frac{1}{2}(h'-\Delta)$ spirales sextuples; chaque spirale ayant, soit un point, soit deux points à l'infini, et passant, en outre, soit une fois, soit deux fois, par chacun des deux points A_1, A_2.

Une spirale simple représente une seule classe subalterne de cercles; les deux arcs réduits, qui appartiennent aux cercles extrêmes, étant représentés par les parties de la spirale les plus voisines, de part et d'autre, soit du point à l'infini, soit de celui des points A_1, A_2 qui appartient à la spirale. Une spirale double représente, au contraire, deux classes subalternes, comprises dans la même classe, mais non équivalentes entr'elles; les deux parties de la spirale, qui correspondent à ces deux classes subalternes, étant séparées l'un de l'autre par le point double A_1 ou A_2, ou bien par les deux points à l'infini. On remarquera que lorsqu'une spirale double a deux points à l'infini, ces deux points sont toujours distincts, ayant chacun une asymptote à distance finie; et de même, lorsqu'une spirale double passe deux fois par le même point A, les points extrêmes de chacune des deux parties dans lesquelles elle est divisée, bien que réunis au point A, appartiennent à deux branches qui s'y croisent à un angle fini. Les spirales triples et sextuples de la quatrième courbe modulaire donnent lieu à des observations tout-à-fait semblables, que nous pouvons passer sous silence.

Quelque soit le nombre N, toutes les courbes modulaires sont symétriques par rapport à l'axe des X; la première et la quatrième sont aussi symétriques par rapport à l'axe des Y; la seconde et la troisième sont symétriques entr'elles par rapport à ce même axe; de plus, ces deux courbes sont les inverses de la première par rapport aux cercles $R_1^2=1$, $R_2^2=1$; enfin, lorsque $N\equiv 1$, mod 4, la première courbe se change dans la quatrième par la substitution $X=iY$, $Y=-iX$. Ajoutons que dans chaque courbe modulaire, quelques unes des spirales sont elles-mêmes symétriques par rapport à l'axe des X; pareillement, dans la première et dans la quatrième courbe, il y a certaines spirales qui sont symétriques par rapport à l'axe des Y, ou bien qui ont le point 0 pour centre. Mais pour abréger, nous supprimons la discussion de ces particularités, qui dépendent de l'ambiguité des classes et de la résolubilité de l'équation $t^2-Nu^2=-1$ en nombres entiers.

14. Imaginons que dans le plan XY on ait opéré une coupure, suivant l'axe des X, depuis A_2 jusqu'à $-\infty$, et depuis A_1 jusqu'à $+\infty$, en y comprenant les

deux points A_1, A_2 eux-mêmes. Chaque spirale modulaire sera divisée dans un certaine nombre de parties, que nous nommerons *les spires de la spirale*, et que nous considérerons séparemment. Il est évident que, si l'on remplace le cercle rationnel, qui est représenté par la spirale, par l'assemblage des arcs réduits qui lui sont équivalents, chacun de ces arcs réduits aura pour image géométrique dans le plan XY une certaine spire de la spirale. Donc, en négligeant, pour abréger, les spires qui ont un point à l'infini, et celles qui passent par un des points A_1, A_2, on pourra en distinguer six espèces différentes, dont voici la description générale.

Une spire de première espèce $(P^{-1}P)$ prend son origine dans un point du segment $(-\infty A_2)$ et se dirige vers la partie inférieure du plan; ensuite elle traverse le segment A_2O pour passer dans la partie supérieure du plan, et aboutit à un point de $(-\infty A_2)$, différent en général du point d'origine. Les spires de seconde espèce $(Q^{-1}Q)$ sont symétriques, par rapport à l'axe des Y, aux spires de première espèce.

Une spire de troisième espèce (PQ) commence dans un point de $(-\infty A_2)$ et aboutit à un point de $A_1\infty$, en restant toujours au-dessus de l'axe des X. Une spire de quatrième espèce (PQ^{-1}) prend son origine dans un point de $(-\infty A_2)$ et se dirige vers la partie supérieure du plan, mais ensuite elle traverse A_2A_1, pour continuer son chemin au-dessous de l'axe des X, et se termine à un point de $(A_1\infty)$.

Enfin les spires de la cinquième et de la sixième espèce, $(P^{-1}Q)$ et $(P^{-1}Q^{-1})$, sont respectivement symétriques par rapport à l'axe des X, aux spires de la quatrième et de la troisième espèce.

On voit que les différentes espèces se distinguent entr'elles par des caractères qui se rapportent à la géométrie de situation. Et il résulte de ce qui a été dit dans les articles 5, 6, 7, que, si la courbe modulaire était une fois décrite, on n'aurait qu'à suivre des yeux le tracé d'une spirale quelconque pour avoir, en premier lieu, la fraction continue qui correspond à cette spirale, et pour retrouver ensuite le système complet des formes quadratiques réduites, représentées respectivement par les différentes spires dont la spirale se compose. Voici le principal résultat auquel nous voulions parvenir dans ce Mémoire. Nous nous proposons, dans une autre occasion, d'exposer quelques applications, que nous avons faites des principes précédents, et qui nous semblent propres à montrer le parti qu'on peut espérer d'en tirer en diverses questions d'arithmétique et de la théorie des fonctions elliptiques.

[The following abstract of the preceding paper was published in the Transunti, Ser. iii. vol. i. pp. 68–69.]

Il Socio L. Cremona presenta una Memoria: *Sur les équations modulaires* del signor Henry J. Stephen Smith professore all'Università di Oxford.

L'auteur se propose dans ce Mémoire d'étudier la liaison qui existe entre les équations modulaires, et les formes quadratiques binaires à déterminant positif. Cette liaison, qui peut-être n'avait pas été remarquée jusqu'ici, est à la verité purement analytique; mais on parvient à la mettre en évidence, en regardant d'une part les équations modulaires comme définissant des courbes géométriques, et en se servant, de l'autre part, d'une nouvelle représentation géométrique des formes quadratiques, dont voici le principe. À chaque forme (a, b, c) du déterminant positif $D = b^2 - ac$, on fait correspondre un cercle $a + 2bx + c\,(x^2 + y^2) = 0$, tracé dans un plan (xy) dont toutefois on ne considère que la partie située au dessus de l'axe des x. On appelle *espace réduit* la partie de ce plan comprise entre les deux droites $x = \pm 1$, mais extérieure aux cercles $x^2 + y^2 = \pm x$; et l'on regarde comme *arc réduit* tout arc de cercle qui se trouve en dedans de l'espace réduite. Cela posé, la période des formes réduites appartenantes à une classe donnée est représentée par les arcs réduits correspondants, dont l'assemblage forme une ligne en apparence brisée, mais qui peut être envisagée comme continue. En suivant la notation usuelle des fonctions elliptiques, qu'on pose $\omega = x + iy = \dfrac{iK'}{K}$, $k^2 = \phi^8(\omega) = \frac{1}{2} + X + iY$, et qu'on fasse correspondre aux points $x + iy$ de l'espace réduit les points $X + iY$ d'un nouveau plan illimité (XY), par le moyen de l'équation $\frac{1}{2} + X + iY = \phi^8(x + iy)$, dans laquelle on suppose réelles les quantités x, y, X, Y. Les lignes brisées correspondantes aux différentes classes de formes quadratiques de déterminant D se trouveront représentées dans le plan (XY) par autant de spirales formées distinctes, dont l'ensemble formera une courbe géométrique complète. L'équation cartésienne de cette courbe sera simplement $F(\frac{1}{2} + X + iY, \frac{1}{2} + X - iY) = 0$, en désignant par $F(k^2, \lambda^2) = 0$ une des équations modulaires pour les transformations elliptiques de l'ordre D. On tire de là ce résultat remarquable que si, sans penser aux formes quadratiques de déterminant D, on trace la courbe modulaire, on aura sous les yeux une image exacte du système complet de ces formes; de sorte que, par un simple dénombrement des spirales, et des différentes spires dont chaque spirale se compose, on pourra déter-

miner (1) le nombre des classes non équivalentes ; (2) le développement en fraction continue, propre à chaque classe, développement qui donne, comme on sait, le système complet de formes réduites, représentantes de la classe. Le cas d'un déterminant carré est compris (sauf quelques particularités) dans la théorie générale.

Il est bon de remarquer que les notions arithmétiques de *classe*, d'*équivalence*, et de *forme réduite*, doivent subir une légère modification pour les adapter à la théorie des fonctions modulaires. On donne dans le Mémoire toutes les explications nécessaires à cet égard.

Au lieu des équations modulaires entre k^2 et λ^2, on aurait pu introduire les equations plus simples de Jacobi entre $\sqrt[4]{k}$ et $\sqrt[4]{\lambda}$; mais la théorie arithmétique en deviendrait un peu plus compliquée.

Les mêmes principes peuvent s'appliquer avantageusement à d'autres questions de la théorie des fonctions elliptiques, parmi lesquelles on peut signaler le problème posé par Jacobi dans la note ajoutée à la page 75 des 'Fundamenta.' L'auteur espère de soumettre prochainement à l'Académie la solution qu'il pense avoir trouvé de ce problème difficile.

XXXVI.

ON THE SINGULARITIES OF THE MODULAR EQUATIONS AND CURVES.

[Proceedings of the London Mathematical Society, vol. ix. pp. 242–272. Read February 14 and April 11, 1878.]

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1. IT is proposed, in this paper, to examine the characteristic singularities of the modular equations and curves. The method employed is applicable to all the modular equations hitherto considered by geometers;* but, for brevity, the discussion is confined to the equations containing the squares of the modules, and to the case in which the order of the transformation is uneven.

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* The modular equations considered by Jacobi in the 'Fundamenta Nova' are (2) the equation between $u = \phi(\omega)$, and $v = \phi(\Omega)$, (see Art. 4 of this paper, equations (3) and (4),) and (5) the equation between u^8 and v^8, of which the characteristics are discussed here. M. Kronecker, in his researches on the modules which admit of complex multiplication, would seem to have also employed (3) the equation between u^2 and v^2, and (4) the equation between u^4 and v^4. (See the account of these researches in the Reports of the British Association for 1865, pp. 332 and 358 [vol. i. pp. 301 and 336]; see also Professor Cayley, 'Phil. Trans.,' vol. clxiv. p. 450.) M. Joubert ('Comptes Rendus,' vol. xlviii. pp. 290–294) was the first to consider (6) the equation between $u^8(1-u^8)$ and $v^8(1-v^8)$. Dr. Felix Müller, in his Inaugural Dissertation (Berlin, 1867), drew attention to the equation (7) between

$$T(\omega) = \frac{(1-u^8+u^{16})^3}{u^{16}(1-u^8)^2} \text{ and } T(\Omega) = \frac{(1-v^8+v^{16})^2}{v^{16}(1-v^8)^2};$$

and the discussion of this equation has recently been resumed by Professor Klein ('Mathematische Annalen,' vol. xiv. p. 112, May 1878). These geometers have expressed $T(\omega)$ and $T(\Omega)$ rationally in terms of a third indeterminate, in the cases $N = 2, 4; 3, 5, 7, 13$; the deficiency of the equation (7) being zero in these six cases; but neither of them has given any example of the equation in its explicit form. Writing x for $T(\omega)$ and y for $T(\Omega)$, I find, when $N = 3$,

$$x(x+2^7.3.5^3)^3 + y(y+2^7.3.5^3)^3 - 2^{16}x^3y^3 + 2^{11}.3^2.31x^2y^2(x+y) - 2^2.3^3.9907xy(x^2+y^2)$$
$$+ 2.3^4.13.193.6367x^2y^2 + 2^8.3^6.5^3.4471xy(x+y) - 2^{15}.5^6.22973xy = 0.$$

The equations (2)... (7) are symmetrical with regard to the two indeterminates, and, the number N

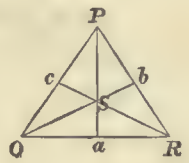
2. We represent by q, the square of the modulus of a given elliptic function; by p, the square of the transformed modulus, the transformation being primary,* and of the uneven order N; by

$$F(p, q, 1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the modular equation subsisting between p and q; in connection with this equation, we consider the modular curve C, of which the trilinear equation

$$F(\alpha, \beta, \gamma) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is obtained by writing $p = \frac{\alpha}{\gamma}$, $q = \frac{\beta}{\gamma}$. We denote by P, Q, R the vertices of the triangle $\alpha\beta\gamma$; and by S the point $\alpha = \beta = \gamma$, or $p = q = 1$; the three intersections (PS, QR), (QS, RP), (RS, PQ) we represent by a, b, c. We regard p and q as the parameters of two pencils of lines $\alpha - p\gamma$, $\beta - q\gamma$, of which the centres are Q and P, and between the rays of which a correspondence is established by the equation (1); we observe that, to the values 0, 1, ∞ of either parameter, there answer in the two pencils respectively the rays QR, QS, QP; PR, PS, PQ; and that, by a known property of the modular equation, the pairs of rays (QR, PR), (QS, PS), (QP, PQ) are corresponding rays in the two pencils, either ray of any of these



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being uneven, they are of the order $A + B$ (Art. 3) in the indeterminates separately, and of the order $2A$ in the indeterminates jointly. I have recently found that the Eulerian functions $\chi(\omega)$ and $\chi(\Omega)$, defined by the equation

$$\chi(\omega) = \sqrt[6]{2} \cdot e^{\frac{i\pi\omega}{24}} \times \Pi_1^{\infty}(1 - (-1)^s e^{s i\pi\omega}),$$

satisfy an equation (1) having the same properties; for the cube $\chi(\omega)$ this had already been shown by M. Kœnigsberger ('Borchardt's Journal,' vol. lxxii. p. 182 sqq). Writing $u_1 = \chi(\omega)$, $v_1 = \chi(\Omega)$, we have, in the cases $N = 5, 7, 11$,

$$u_1^6 + 2^{\frac{4}{3}} u_1^5 v_1^5 - 2^{\frac{2}{3}} u_1 v_1 + v^6 = 0,$$

$$u_1^{12} + 8 \cdot 2^{\frac{1}{3}} u_1^{11} v_1^{11} - 44 u_1^9 v_1^9 + 44 \cdot 2^{\frac{2}{3}} u_1^7 v_1^7 - 44 \cdot 2^{\frac{1}{3}} u_1^5 v_1^5 + 22 u_1^3 v_1^3 - 2 \cdot 2^{\frac{2}{3}} u_1 v_1 + v_1^{12} = 0,$$

$$u_1^8 - 16 u_1^7 v_1^7 + 7 u_1^4 v_1^4 - 2 u_1 v_1 + v_1^8 = 0.$$

The function $\chi(\omega)$ is a twenty-fourth root of $u^8(1 - u^8)$; the formulæ relating to its linear transformation have been given by M. Hermite ('Comptes Rendus,' 1858, vol. xlvi. p. 721). In respect of simplicity of form, the equations (1), (2)... (7) appear to arrange themselves in this numerical order; but, in respect of simplicity of algebraical theory, the order is reversed, as the deficiency decreases from (1) to (7).

* A transformation $\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right|$ of the uneven order N is primary when it satisfies the congruence $\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right| \equiv \left|\begin{matrix} 1, & 0 \\ 0, & 1 \end{matrix}\right|$, mod 2. For the theory of the elliptic multiplier it is convenient to fix the signs of a, b, c, d by the additional condition $a \equiv 1$, mod 4; but for our present purpose this restriction is unnecessary.

pairs being the only ray answering to the other ray of the same pair; the modular curve C is the locus of the intersections of corresponding rays in the two pencils.*

We denote by m and n the order and class of C; by H its deficiency; by K and I its cuspidal and inflexional indices; by D and T its discriminantal order and class; by $F(p)=E(q)$ the highest, by $E'(p)=E'(q)$ the lowest, exponents of p and q which present themselves in the equation (1).

Arts. 3–8. *N not divisible by a square.*

3. We confine ourselves, in the first instance, to the case in which N is not divisible by any square; and we represent, in this case, by A the sum of the divisors of N which surpass $\sqrt{N}$; by B the sum of the divisors of N which are less than $\sqrt{N}$; by ν the number of divisors of either sort, so that 2ν is the whole number of divisors of N. We then have the formulæ

$$\begin{aligned}
&\text{(i.)} && m=2A,\\
&\text{(ii.)} && n=3A-B,\\
&\text{(iii.)} && H=\tfrac{1}{2}(A+B)-3\nu+1,\\
&\text{(iv.)} && K=2A+2B-6\nu,\\
&\text{(v.)} && I-K=3(A-B),\\
&\text{(vi.)} && D=4A^2-5A+B,\\
&\text{(vii.)} && T-D=(A-B)(5A-B).
\end{aligned}$$

To these we may add the equations

$$\text{(viii.)} \quad \begin{cases} E(p)=E(q)=A+B, \\ E'(p)=E'(q)=2B. \end{cases}$$

Of these formulæ (i.) and (viii.) are well-known; of the remainder, it will suffice to attend to (iii.) and (iv), because, when the values of m, H, and K are given, the values of n, I, D, and T are known from the equations of Plücker.

4. *The Deficiency.*—The demonstration of the formulæ (iii.) and (iv.) depends on the simultaneous expression of the modular parameters p and q as

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* In a paper which I hope shortly to lay before the Society, I have discussed, with some fulness of detail, the relation of the algebraical singularities of the parametric equation $F(p, q, 1)=0$ to the characteristic singularities of the curves of which the equations are included in the formula $F\left(\frac{A}{B}, \frac{C}{D}, 1\right)=0$, A, B, C, D being the equations of straight lines. The discussion comprises an examination of the effect of any quadric transformation on the singularities of a curve.

transcendental functions of the quotient of the periods of the given elliptic function. As we have already developed these considerations elsewhere,* we

* In an Introduction (now in the press) to Mr. J. W. L. Glaisher's Tables of the Theta Functions [No. XLIII. of the present volume]. The method indicated in this article has been already employed by Professor Klein, in the paper to which reference has already been made ('Math. Ann.,' vol. xiv. p. 111; see especially §§ 6–8, 7–13), and by Professor Dedekind, in a letter addressed to M. Borchardt (June 1877, see 'Borchardt's Journal,' vol. lxxxiii. p. 265, especially §§ 1–4 and 7). In the year 1873, I submitted to the Academy of Sciences in Paris a Mémoire 'Sur les Équations Modulaires' ('Comptes Rendus,' August 1873, vol. lxxvii. p. 472). In this Mémoire (which was ultimately presented, without alteration, to the Accademia dei Lincei, and was printed in their 'Atti,' vol. i. series 3, p. 136, February 1877) [vol. ii. pp. 224, etc.], I had employed the same method (see Arts. ii, ix, and x of the Mémoire) to establish the relation which exists between the modular equations of order N and the binary quadratic forms of the positive determinant N. The Mémoire was devoted to that theory alone, as I attached more importance to it than to any other result relating to the modular equations at which I had then arrived. But I had already in the year 1873 obtained:—(i) a proof of the existence of the modular equations, simpler perhaps than that of M. Dedekind, and based solely on those elementary properties of the function $\phi(\omega)$, which were deduced from the theorem of Fourier by Cauchy and Poisson, without employing any elliptic formulæ; (ii) a determination in the simpler cases of the Plückerian characteristics of the modular curves; (iii) a solution of one part at least of the problem relating to complex modules, proposed by Jacobi in Art. 32 of the 'Fundamenta Nova.' I communicated to Professor Cayley, in 1873, the formulæ for the deficiency of the equations (2) ... (5) when N is an uneven prime (see his Memoir on the Transformation of Elliptic Functions, presented to the Royal Society in that year); the formulæ for the cuspidal index I obtained by transforming into normal developments the parametric developments which give the deficiency (see Art. 5 of this paper): thus, the order of the curves being known, all their Plückerian characteristics were determined. The case when N is a product of uneven primes presents no greater difficulty than the case when N is a prime; and I had (in fact) obtained the formula for this more general case as early as 1873. The case when N is divisible by a square, and still more the case when N is itself a square, appeared to involve some difficulty; and these I left untouched till the spring of the present year, when I found that the introduction of the arithmetical function f' (see Art. 9 of this paper) caused the supposed difficulty to disappear. To the more exact determination of the indices characteristic of each special singularity of the modular curves, I was guided by the methods employed in a former paper on the Higher Singularities of Plane Curves.

A complete system of formulæ, analogous to that given in the present paper for the modular equation (5), I have already obtained for the equations (2), (3), and (4); with the equation (7), and with the eight equations between corresponding powers of $\chi(\omega)$ and $\chi(\Omega)$, I have not advanced equally far, but I have not found that they offer any peculiar difficulty.

In the Mémoire 'Sur les Équations Modulaires,' I have confined myself (as in the present paper) to the equation (5) between the squares of the modules. At the time when the Mémoire was written, I was well acquainted with the characteristic property of the function $T(\omega)$; viz. that it is unchanged by any linear transformation of the elliptic functions; and I even thought of employing it in the Mémoire instead of the function $\phi^8(\omega)$. I had conjectured (erroneously however) that the modular curves T derived from the equation (7) would represent ordinary periodic continued fractions with positive integral quotients, in the same way in which the modular curves C derived from the equation (5) represent periodic continued fractions with even quotients. But I was deterred from employing

shall in this place assume the results of the discussion as known, and shall confine ourselves to their application to the formulæ (iii.) and (iv.).

Denoting by x and y real quantities, of which y is essentially positive, and by ω the complex variable $x+iy$, which is thus subject to the restriction that the coefficient of i in its imaginary part is positive, we define the function $\phi(\omega)$ by the equation

$$\phi(\omega) = \sqrt{2}\,.e^{\frac{i\pi\omega}{8}} \prod_{s=1}^{s=\infty} \frac{1+e^{2si\pi\omega}}{1+e^{(2s-1)i\pi\omega}}; \quad \ldots\ldots\ldots \quad (3)$$

and we consider the $A+B$ quantities Ω, of which the values are given by the equation

$$\Omega = \frac{g'\omega + 2k}{g}, \quad \ldots\ldots\ldots \quad (4)$$

g, g' being any two conjugate divisors of N, and k being any term of a complete system of residues for the modulus g. We then have the fundamental theorem,

'If $q=\phi^8(\omega)$, the $A+B$ corresponding values of p are included in the formula

$$p = \phi^8(\Omega);$$

or, which is the same thing,

$$F(p, q, 1) = \Pi[p - \phi^8(\Omega)],$$

the sign of multiplication extending to the $A+B$ values of Ω.'

It results from the discussion to which we have referred, that, if we regard q as an independent complex variable, and represent its values in the usual

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the equation (7) by the consideration that there was not a single calculated example of it; indeed, at that time I was not acquainted with the researches of Dr. Felix Müller, and did not know that the equation had attracted any attention. I have since found that the curves T do not precisely represent the reduced forms of Gauss, but instead a system of forms determined by a different regulative principle. I am disposed to think (notwithstanding the considerations mentioned in the note on Art. 1), that there is some advantage in continuing to regard the equation between the squares of the modules, as the principal modular equation, rather than either the equation (1) or (7). At least, as far as concerns the arithmetical theory to which the Mémoire relates, and which I have since extended to the equations (1), (2), and (7), (the theory of the equations (3), (4), and (6) hardly requiring a separate discussion), the modular curves (5) present phenomena in some respects simpler than those presented by the curves (7), and in all respects simpler than those presented by the curves (1) ... (4).

Both in the Mémoire and in this paper, I have given especial attention to the case in which N is a square, because the solution of the problem of Jacobi for the transformations of order N depends on a consideration of the spaces into which modular curves of order N^2 divide a plane. A note published in the 'Transunti' of the Accademia dei Lincei (vol. i. p. 42, 7 Jan. 1877) [vol. ii. pp. 221 etc.], contains what is in fact a solution of Jacobi's problem for the case $N=1$; to this particular case of the general problem, the attention of geometers had been called by M. Hermite in the note appended to the second volume of M. Serret's edition (1862) of the Differential Calculus of Lacroix, pp. 421–425.

manner by the points of a plane, p, considered as a function q, has no spiral points other than the three $q=0$, $q=1$, $q=\infty$ (it will be remembered that in the plane of double algebra the infinitely distant is a point). Thus, if we cause q to describe any closed contour, not including one of the three points $0, 1, \infty$, the values of p will undergo no interchange; but each root of the equation $F(p, q, 1)=0$ will return, when the contour is closed, to the same value which it had at the beginning; although the contour may include points (other than $0, 1, \infty$) at which two of the values of p become equal. But the case is different if we cause q to describe a closed contour round one of the three exceptional points. At each of these points all the values of p become equal to one another, and to the value of q indicated by the point. As the general result is the same for each of the three points, it will suffice to attend to one of them only; for example, to the point $q=0$. If, then, we cause q to describe a closed contour round 0, the g values of p, or of the expression $\phi^s\left(\frac{g'\omega+2k}{g}\right)$, which contain the divisor g in the denominator, change into one another cyclically, and thus the $\Sigma g = A+B$ roots of the equation arrange themselves in 2ν cycles, corresponding to the different divisors of N; or, which is the same thing, the developments obtained by expanding the different values of p in series proceeding by ascending powers of q are singular; being, in fact, of the type

$$p=\lambda q^{\frac{g'}{g}}+\ldots. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Similarly, at the points $+1$ and ∞, we have singular developments of the types

$$p-1=\lambda(q-1)^{\frac{g'}{g}}+\ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{1}{p}=\lambda\left(\frac{1}{q}\right)^{\frac{g'}{g}}+\ldots. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

As these are all the singular developments that can exist, we infer that, if $W(p)$ represent the number of the spiral points of p, each point being reckoned with its proper multiplicity,

$$W(p)=3\Sigma(g-1)=3(A+B)-6\nu.$$

Substituting, in the equation of Riemann,

$$2H=W(p)-2E(p)+2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

the value of $E(p)$ (equation viii., Art. 3), and the value just obtained for $W(p)$, we find

$$H=\tfrac{1}{2}(A+B)-3\nu+1,$$

which is the equation (iii.).

5. *The Cuspidal Index.*—To determine the cuspidal index of C, we first consider the developments (5) which appertain to the point R. Since N is not a square, we cannot have $g=g'$; if $g'>g$, we have the normal developmen t*

$$\frac{\alpha}{\gamma}=\lambda\left(\frac{\beta}{\gamma}\right)^{\frac{g'}{g}}+\ldots; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

if $g>g'$, we have the normal development

$$\frac{\beta}{\gamma}=\mu\left(\frac{\alpha}{\gamma}\right)^{\frac{g}{g'}}+\ldots. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The branches corresponding to the developments (9) and (10) have, for their cuspidal indices, $g-1$ and $g'-1$ respectively. Hence each of the lines QR, PR is touched at the point R by a set of branches of which the aggregate cuspidal index is $B-\nu$.

In the same manner, it will be found that at the point S each of the lines QS, PS is touched by a set of branches having the same cuspidal indices as the branches which touch QR or PR at R.

Lastly, from the developments (7), we deduce normal developments of the types

$$\frac{\gamma}{\alpha}=\mu\left(\frac{\beta}{\alpha}\right)^{\frac{g'}{g'-g}}+\ldots, \quad g'>g; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{\gamma}{\beta}=\mu\left(\frac{\alpha}{\beta}\right)^{\frac{g'}{g-g'}}+\ldots, \quad g>g'; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and from these we infer that the line PQ is touched at each of the points P and Q by a set of branches of which the aggregate cuspidal index is $A-B-\nu$.

Since C can have no other cuspidal branches, we find

$$\begin{aligned} K&=2(B-\nu)+2(B-\nu)+2(A-B-\nu)\\ &=2(A+B)-6\nu, \end{aligned}$$

which is the formula (iv.).

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* If A, B, C represent straight lines forming a triangle, a development of the type

$$\frac{A}{C}=\lambda_1\left(\frac{B}{C}\right)^{\alpha_1}+\lambda_2\left(\frac{B}{C}\right)^{\alpha_2}+\ldots,$$

in which $\alpha_1, \alpha_2, \ldots$ are positive and increasing, and α_1 is greater than unity, is termed a normal development; A is, of course, the tangent to the branch, B is any line passing through the point to which the development refers.

6. *The discriminant of* $F(p, q, 1)$.—The values of m, $E(p)$, $E(q)$, $E'(p)$, $E'(q)$ are inferred from the equation (1), by a method, due (as it would seem) to M. Kronecker, of which examples are given in the Report on the Theory of Numbers ('Reports of the British Association' for 1865, p. 349 sqq.).* This method is also applicable to the discriminant of the equation (1);† *i.e.* to the expression

$$\nabla(q) = \Pi[\phi^8(\Omega_1) - \phi^8(\Omega_2)]^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where the sign of multiplication extends to every pair of values of Ω. Let

$$(A+B)^2 = A_2 + B_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where A_2 comprises all the terms in $(A+B)^2$ which are greater than N, and B_2 all the terms which are less than N; as for the terms which are equal to N (of which the sum is evidently $2\nu N$), we divide them equally between A_2 and B_2; thus, if $g_1, g_2, \ldots g_t$ are all the divisors of N (unity and N included), we have

$$\left.\begin{aligned} A_2 &= \nu N + \Sigma_{r=1}^{r=t} \Sigma_{s=1}^{s=t} g_r g_s, \quad g_r g_s > N; \\ B_2 &= \nu N + \Sigma_{r=1}^{r=t} \Sigma_{s=1}^{s=t} g_r g_s, \quad g_r g_s < N; \\ A_2 + B_2 &= \Sigma_{r=1}^{r=t} \Sigma_{s=1}^{s=t} g_r g_s = (A+B)^2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Applying the method of M. Kronecker, we find that the highest power of q in $\nabla(q)$ is $2A_2 - A - B$, and that the lowest power of q is $2B_2 - A - B$. By a known property of the modular equation,

$$F(1-p, 1-q, 1) = F(p, q, 1);$$

hence $\nabla(q)$ must be divisible by $1-q$ as often as by q; we have, therefore,

$$\nabla(q) = (q-q^2)^{2B_2 - A - B} \times \chi(q); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where $\chi(q)$ is a rational and integral function of q, not divisible by q or $1-q$, and of the order $2A_2 - 4B_2 + (A+B)$.

By another property of the modular equation we have the identity

$$(pq)^{A+B} \times F\left(\frac{1}{p}, \frac{1}{q}, 1\right) = F(p, q, 1).$$

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* [Vol. i. p. 325.]

† It has been applied by M. Koenigsberger ('Vorlesungen über die Theorie der Elliptischen Funktionen,' vol. ii. p. 154) to the discriminant of the modular equations between u and v, in the case in which N is not divisible by any square; the result had already been given by M. Hermite in his Mémoire sur la Théorie des Équations Modulaires ('Comptes Rendus,' vol. xlviii. p. 1079). The discriminant of the modular equation between $\chi^3(\omega)$ and $\chi^3(\Omega)$ has been similarly treated by M. Krause ('Math. Annalen,' vol. xii. pp. 1–3).

If therefore we write $p \div p'$ for p, and $q \div q'$ for q, the dialytic discriminant of the bipartite binary quantic

$$(p'q')^{A+B} \times F\left(\frac{p}{p'}, \frac{q}{q'}, 1\right)$$

is symmetric with regard to q and q'; *i.e.* it is of the form

$$[qq'(q'-q)]^{2B_2-A-B} \times \chi(q, q'), \quad \ldots\ldots\ldots \quad (17)$$

where $\chi(q, q')$ is symmetric and of the order $2A_2 - 4B_2 + A + B$.

It will be observed that the order of the discriminant (17) is

$$\begin{aligned} &3(2B_2 - A - B) + 2A_2 - 4B_2 + A + B \\ &= 2(A_2 + B_2) - 2(A + B) \\ &= 2(A + B)(A + B - 1), \end{aligned}$$

as it ought to be; and that the equation $\nabla(q) = 0$ is to be regarded as having $2B_2 - A - B$ infinite roots, and as having lost the same number of dimensions.

We may add that $\chi(q)$ is a perfect square. For, if $q - q_0$, be any factor of $\chi(q)$, all the developments of p, proceeding by powers of $q - q_0$, have integral exponents; the exponent of $q - q_0$ in $\chi(q)$ is the sum of the discriminantal indices of these developments taken in pairs; and this sum is always even.

7. *The discriminantal indices of P, Q, R, S.*—We next examine the discriminantal indices in the curve C of the points P, Q, R, S. Representing these indices by $D(P)$, $D(Q)$, $D(R)$, $D(S)$, we have

$$\left.\begin{aligned} D(P) &= 2A^2 - A_2 - A = D(Q), \\ D(R) &= 2B_2 - 2A \quad\;\; = D(S). \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (18)$$

To establish these formulæ, it will suffice to consider the points P and R. At P we have ν superlinear branches of the aggregate order $A - B$ touching PQ; we may symbolize the branch of which (11) is the normal development by (g', g), where $gg' = N$, and $g' > \sqrt{N} > g$. The discriminantal index of the branch (g', g) taken by itself is $g'(g' - g - 1)$; the joint discriminantal index of the two branches (g', g) and (g_1', g_1) is $2g'(g_1' - g_1)$, if $g' > g_1'$, and consequently $g < g_1$. Hence we have

$$D(P) = \Sigma g'(g' - g - 1) + 2\Sigma\Sigma g'(g_1' - g_1), \quad \ldots\ldots \quad (19)$$

the summations extending to all values of g' and g_1' which satisfy the inequalities

$$g' > \sqrt{N}, \quad g_1' < g', \quad g_1' > \sqrt{N}. \quad \ldots\ldots\ldots \quad (20)$$

Attending to these inequalities, we find

$$\begin{aligned} \Sigma g'^2 + 2\Sigma\Sigma g' g_1' &= A^2, \\ \Sigma g' &= A, \\ \Sigma g' g + 2\Sigma\Sigma g' g_1 &= A_2 - A^2; \end{aligned}$$

and substituting these values in (19), we obtain the value of $D(P)$, given by the formula (18).

Again, at the point R, we first consider the branches touching PR, the normal developments of which are of the type (10). For the discriminantal index of these branches, taken singly and in pairs, we have the expression

$$\Sigma g(g'-1)+2\Sigma\Sigma g_1 g',$$

the summations extending to all divisors g and g_1 which satisfy the inequalities

$$g>\sqrt{N},\quad g_1<g,\quad g_1>\sqrt{N}. \qquad \ldots\ldots\ldots \quad (21)$$

The discriminantal index of the branches touching QR is evidently the same as that of the branches touching PR; and as the aggregate order of each of the two sets of branches is B, they intersect one another in B^2 points, and the part of $D(R)$ which arises from their crossing one another at R is $2B^2$. We have, therefore,

$$D(R)=2\Sigma g(g'-1)+4\Sigma\Sigma g_1 g'+2B^2;$$

but, attending to the inequalities (21), we also have

$$\Sigma g=A,\quad \Sigma g' g+2\Sigma g_1 g'=B_2-B^2,$$

whence, in accordance with (18),

$$D(R)=2B_2-2A.$$

8. *The intersections with C of the polar curves of P and Q.*—Since the branches which touch PR at R are of the aggregate class $A-B$, the line PR, considered as a tangent drawn from P, counts $A-B$ times as a tangent at R. Similarly PS counts $A-B$ times as a tangent at S. Again, since the branches which touch PQ at P and at Q are of the aggregate order $A-B$, and of the aggregate class B, PQ, considered as a tangent drawn from P, counts $A-B+B=A$ times as a tangent at P, and B times as a tangent at Q. Thus the three lines PR, PS, PQ count as

$$(A-B)+(A-B)+(A+B)=3A-B$$

tangents from P; *i.e.* no other tangents can be drawn to C from P. Again, the polar curve of P intersects C at P, Q, R, S, in

$$\begin{aligned} D(P)+A &= 2A^2-A_2,\\ D(Q)+B &= 2A^2-A_2+B-A,\\ D(R)+A-B &= 2B_2-B-A,\\ D(S)+A-B &= 2B_2-B-A, \end{aligned}$$

points respectively; or, in all, in $4A^2+4B_2-2A_2-3A-B$ points. The whole number of intersections of C by any one of its first polars is $2A(2A-1)$; hence

the polar of P intersects C in

$$2A(2A-1)-(4A^2+4B_2-2A_2-3A-B)=2A_2-4B_2+A+B$$

points, other than P, Q, R, S. These $2A_2-4B_2+A+B$ intersections correspond in the discriminant of $F(p, q, 1)$ to the factors of $\chi(q)$, of which the aggregate order is the same.

As the intersections other than those at P, Q, R, S correspond to the factor $\chi(q)$, so also the intersections at R correspond to the factor q^{2B_2-A-B}, and the intersections at S to the factor $(1-q)^{2B_2-A-B}$. But it is proper to observe that the remaining intersections at P and Q (of which the aggregate number is $4A^2-2A_2+B-A$) surpass the order of the remaining factor q'^{2B_2-A-B} of the complete dialytic discriminant; the difference

$$4A^2-2A_2+B-A-(2B_2-A-B)=2A(2A-1)-2(A+B)(A+B-1)$$

being, as it ought to be, equal to the difference between the number of intersections of C by any one of its polars, and the order of the dialytic discriminant. We reserve, for a future communication to the Society, a complete discussion of the relations which subsist between the exponents of the factors of the dialytic discriminant of any parametric equation, and the corresponding intersections of the locus curve by the polars of the centres of the generating pencils.

The points, other than P, Q, R, S, in which C is intersected by the polar of P, are all ordinary double (or it may be multiple) points, free from superlinearity, and having tangents which do not pass through P. For C has no superlinear branches beside those at P, Q, R, S, and the only tangents which pass through P are PQ, PR, PS. The same thing is also evident from what has been said in Art. 6 of the exponents of the factors of $\chi(q)$.

Arts. 9–11. *N divisible by a square.*

9. *Definition of certain Arithmetical Functions.*—We now pass to the general case in which N is any uneven number whatever. Let

$$N=a_1^{\alpha_1}a_2^{\alpha_2}a_3^{\alpha_3}\ldots,$$

a_1, a_2, ... being different uneven primes; let g, g', as before, be two conjugate divisors of N, and let η be the greatest common divisor of g and g'. We resolve g into the product of two factors γ_1 and γ_2, of which γ_1 contains only those prime divisors of g which do not occur in η and g'; and γ_2 contains only those prime divisors of g which do occur in η and g'. Representing by $f(z)$ the number of numbers prime to any given number z and not surpassing it, we write

$$f'(g)=\gamma_1 f(\gamma_2),$$

and we observe that we have the equations

$$\frac{f(\eta)}{\eta} = \frac{f'(g)}{g} = \frac{f'(g')}{g'};$$

each of these quotients being equal to

$$\Pi\left(1-\frac{1}{\epsilon}\right),$$

if ϵ denote any prime divisor of η. We still retain the symbols ν, A, B, A_2, B_2; but with extended significations, which we proceed to explain.

We define ν by the equation

$$2\nu = \Sigma f(\eta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

the sign of summation extending to every divisor g of N. We observe that, in general, each term $f(\eta)$ occurs twice in $\Sigma f(\eta)$, because η is the same for each of any two conjugate divisors; but that, if $N=\theta^2$ is a perfect square, the term $f(\theta)=f'(\theta)$ occurs only once in $\Sigma f(\eta)$.

Again, we define A and B by the equations

$$\left.\begin{aligned} A &= \Sigma f'(g), \quad g \geqq \sqrt{N}, \\ B &= \Sigma f'(g), \quad g \leqq \sqrt{N}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

the summations extending to all divisors g of N which satisfy the inequalities (23) respectively; when $N=\theta^2$ is a perfect square, we divide the term $f'(\theta)=f(\theta)$ equally between A and B. We thus have in every case

$$A+B=\Sigma f'(g),$$

the summation extending to every divisor g of N.

Lastly, we define A_2 and B_2 by the equations

$$\left.\begin{aligned} A_2 &= \Sigma\Sigma f'(g_1)f'(g_2), \quad g_1 g_2 \geqq N, \\ B_2 &= \Sigma\Sigma f'(g_1)f'(g_2), \quad g_1 g_2 \leqq N; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

in which g_1 and g_2 are any two divisors of N (the same or different) which satisfy the inequalities specified; so that, if g_1 and g_2 are different, the term $f'(g_1)f'(g_2)$ occurs twice in A_2, or in B_2, as the case may be. If $g_1 g_2 = N$, we divide the double term $2f'(g_1)f'(g_2)$, corresponding to these two divisors, equally between A_2 and B_2; if, in particular, $N=\theta^2$ is a perfect square, the single term $[f(\theta)]^2$ is to be divided equally between A_2 and B_2. It is evident that we have, in every case,

$$A_2+B_2=(A+B)^2.$$

The sums $2\nu=\Sigma f(\eta)$, and $A+B=\Sigma f'(g)$, may be conveniently expressed in terms of the prime divisors of N. Observing (1) that the terms of the product

$$\Pi[1+\alpha+\alpha^2+\ldots\alpha^a]$$

represent, after development, all the divisors of N, and (2) that

$$f'(g) = f'(h_1) \times f'(h_2),$$

if g be a product of two relatively prime factors h_1 and h_2, we find

$$\begin{aligned}\Sigma f'(g) &= \Pi\left[f'(1) + f'(a) + \ldots f'(a^a)\right] \\ &= \Pi\left[1 + (a-1) + a(a-1) + \ldots + a^{a-2}(a-1) + a^a\right] \\ &= \Pi\left[a^{a-1} + a^a\right];\end{aligned}$$

whence

$$A + B = N \times \Pi\left(1 + \frac{1}{a}\right). \quad \ldots\ldots\ldots \quad (25)$$

Again, if we write $f''(g)$ for $f(\eta)$, and give to h_1, h_2 the same signification as before, $f''(g)$ satisfies the equation

$$f''(g) = f''(h_1) \times f''(h_2);$$

whence we infer that

$$\Sigma f(\eta) = \Pi\left[f''(1) + f''(a) + \ldots + f''(a^a)\right].$$

First, let $a = 2\mu + 1$; then

$$f''(1) + f''(a) + \ldots f''(a^{2\mu+1}) = 2\left[1 + (a-1) + a(a-1) + \ldots + a^{\mu-1}(a-1)\right] = 2a^\mu.$$

Secondly, let $a = 2\mu$; then

$$\begin{aligned}f''(1) + f''(a) + \ldots + f''(a^{2\mu}) &= 2\left[1 + (a-1) + a(a-1) + \ldots + a^{\mu-2}(a-1)\right] + a^{\mu-1}(a-1) \\ &= a^\mu + a^{\mu-1}.\end{aligned}$$

If therefore $N = \Pi b^{2\beta+1} \times \Pi c^{2\gamma}$, where $b, \ldots, c, \ldots$, are different prime numbers, we have

$$\Sigma f(\eta) = 2\nu = \Pi 2b^\beta \times \Pi c^\gamma\left(1 + \frac{1}{c}\right). \quad \ldots\ldots\ldots \quad (26)$$

It will be observed that the definitions which we have now given of the symbols ν, A, B, A_2, B_2, coincide, in the case in which N is not divisible by any square, with the definitions of Arts. 3 and 6.

10. *Case when N is not a square.*—Excluding, for the present, the case in which N is a perfect square, we have to show that, in all other cases, the formulæ of Arts. 3–8 hold without further modification. For brevity, we shall establish only a few of the assertions included in this general statement, as the method to be pursued with regard to all of them is the same.

(i.) If $\Phi(N)$ is the sum of the divisors of N, and e_1, e_2, ... are the primes of which the squares divide N, the order of the irreducible modular equation of order N is

$$\Phi(N) - \Sigma\Phi\left(\frac{N}{e^2}\right) + \Sigma\Phi\left(\frac{N}{e_1^2 e_2^2}\right) - \ldots,$$

(See M. Joubert, 'Comptes Rendus,' vol. i. p. 1041; Report on the Theory of Numbers, *loc. cit.*, p. 332*). But this expression has for its value $N\Pi\left(1+\frac{1}{a}\right)$; *i.e.* the order of the modular equation in p or in q is $A+B$ (equation 25).

(ii.) If we write $q=\phi^8(\omega)$, the $A+B$ corresponding values of p are given by the equation

$$p=\phi^8\left(\frac{g'\omega+2k}{g}\right),$$

in which k is any one of the $\frac{f(\eta)}{\eta}g=f'(g)$ residues of g which are prime to η, the greatest common divisor of g and g'. If, as in Art. 4, we cause q to describe a closed contour round 0, $f'(g)$ values of p, which answer to any given divisor g, arrange themselves in $f(\eta)$ cycles each containing $\frac{g}{\eta}$ roots; and thus the developments (5), which appertain to the simultaneous values $p=0$, $q=0$, assume, in the general case which we are now considering, the form

$$p=\lambda(q)^{\frac{g'}{\eta}+\frac{g}{\eta}}+\ldots; \quad \ldots\ldots\ldots \quad (27)$$

the least common denominator of the exponents being $\frac{g}{\eta}$. It will be observed that there are $f(\eta)$ developments, in which g and g' have the same values; the coefficients λ having different values in these $f(\eta)$ developments. Similarly, there are $\Sigma f(\eta)$ developments of each of the types

$$(p-1)=\lambda(q-1)^{\frac{g'}{\eta}+\frac{g}{\eta}}+\ldots, \quad \ldots\ldots\ldots \quad (28)$$

$$\frac{1}{p}=\lambda\left(\frac{1}{q}\right)^{\frac{g'}{\eta}+\frac{g}{\eta}}+\ldots. \quad \ldots\ldots\ldots \quad (29)$$

Hence
$$W(p)=3\Sigma f(\eta)\left[\frac{g}{\eta}-1\right]$$
$$=3\Sigma f'(g)-3\Sigma f(\eta)=3(A+B)-6\nu;$$

and, consequently, as in Art. 4,

$$H=\tfrac{1}{2}(A+B)-3\nu+1.$$

(iii.) From the developments (27), (28), (29), we can deduce the normal developments of the six sets of branches which touch PR and QR at R, PS and QS at S, PQ at P and Q. Each set comprises ν branches; if h^2 is the greatest square dividing N, h of these are linear in each of the first four sets; all

* [Vol. i. p. 302.]

the rest are superlinear. It will suffice here to determine the cuspidal and discriminantal indices of the branches touching PQ at P. The normal developments of these branches are of the type

$$\left.\frac{\gamma}{\alpha}=\mu\left(\frac{\beta}{\alpha}\right)^{\frac{g'}{\eta}+\frac{g'-g}{\eta}}+\ldots\right\}\quad\ldots\ldots\ldots\quad(30)$$
$$g'>g.$$

Hence their aggregate cuspidal index is

$$\Sigma f(\eta)\left(\frac{g'}{\eta}-\frac{g}{\eta}-1\right),\quad g<\sqrt{N};$$

or, which is the same thing,

$$\overset{g>\sqrt{N},}{\Sigma f'(g)}-\overset{g<\sqrt{N},}{\Sigma f'(g)}-\overset{g<\sqrt{N},}{\Sigma f(\eta)}=A-B-\nu.$$

To obtain the discriminantal index, we first consider a single group of $f(\eta)$ branches corresponding to given values of g, g'. The discriminantal index of one of these branches, taken by itself, is

$$\frac{g'}{\eta}\left(\frac{g'}{\eta}-\frac{g}{\eta}-1\right);$$

the joint discriminantal index of two different branches of the group is

$$2\frac{g'}{\eta}\left(\frac{g'}{\eta}-\frac{g}{\eta}\right);$$

so that the aggregate discriminantal index of the group (g', g) is

$$f(\eta)\times\frac{g'}{\eta}\left(\frac{g'}{\eta}-\frac{g}{\eta}-1\right)+f(\eta)[f(\eta)-1]\times\frac{g'}{\eta}\left(\frac{g'}{\eta}-\frac{g}{\eta}\right)$$
$$=f'(g')[f'(g')-f'(g)-1].$$

We next consider the two groups (g', g) and (g_1', g_1) consisting respectively of $f(\eta)$ and $f(\eta_1)$ branches. If $g'>g_1'$, or, which is the same thing, if $g'g_1>N$, the joint discriminantal index of the two groups is

$$2f(\eta)f(\eta_1)\times\frac{g'}{\eta}\left(\frac{g_1'}{\eta_1}-\frac{g_1}{\eta_1}\right)=2f'(g')[f'(g_1')-f'(g_1)].$$

Thus the aggregate discriminantal index of the branches touching PQ at P is given by the equation

$$D(P)=\Sigma f'(g')[f'(g_2')-f'(g)-1]+2\Sigma\Sigma f'(g')[f'(g_1')-f'(g_1),$$

the summations extending to all values of g' and g_1' which satisfy the inequalities

$$g'>\sqrt{N},\quad g_1'<g',\quad g_1'>\sqrt{N}.$$

But, as in Art. 7,

$$\Sigma[f'(g')]^2+2\Sigma\Sigma f'(g')f'(g_1')=A^2,\quad \Sigma f'(g')=A,$$
$$\Sigma f'(g')f'(g)+2\Sigma\Sigma f'(g')f'(g_1)=A_2-A^2;$$

whence, as before, $$D(P)=2A^2-A_2-A.$$

11. *Case when N is a square.*—The case in which $N=\theta^2$ is a perfect square requires separate consideration, because the modular curve of order θ^2 meets the line PQ in $f(\theta)$ points distinct from one another and from P and Q; and again, at each of the points R and S, it has $f(\theta)$ linear branches, of which the tangents are different from one another, and from the lines RP, RQ, SP, SQ. Thus some of the characteristics of the singularities at $PQRS$ are changed; and with them some of the characteristic indices of the curve.

We write θ' for $f(\theta)=f'(\theta)$. It will be found, on referring to Art. 10, (i.) and (ii.), that

$$E(p)=E(q)=A+B,\quad m=2A,$$
$$H=\tfrac{1}{2}(A+B)-3\nu+1,$$

as in the case when N is not a square. Again, the cuspidal index of each of the four sets of branches which touch PR and QR at R, PS and QS at S, is, as before, $B-\nu$; but the cuspidal index of the branches at P and Q is $A-B-\nu+\frac{1}{2}\theta'$ instead of $A-B-\nu$. For this index is

$$\overset{g>\sqrt{N}}{\Sigma}f'(g')-\overset{g<\sqrt{N}}{\Sigma}f'(g)-\overset{g<\sqrt{N}}{\Sigma}f(\eta),$$

(see Art. 10, iii.); and

$$\overset{g>\sqrt{N}}{\Sigma}f'(g)=A-\tfrac{1}{2}\theta',$$
$$\overset{g<\sqrt{N}}{\Sigma}f'(g)=B-\tfrac{1}{2}\theta',$$
$$\overset{g<\sqrt{N}}{\Sigma}f(\eta)=\nu-\tfrac{1}{2}\theta'.$$

To find the discriminantal indices of $PQRS$, we denote by $\overline{A}$, $\overline{B}$, $\overline{A}_2$, $\overline{B}_2$ the numbers obtained by omitting in ABA_2B_2 the terms depending on θ; we thus have

$$\overline{A}=A-\tfrac{1}{2}\theta',\quad \overline{B}=B-\tfrac{1}{2}\theta',$$
$$\overline{A}_2=A_2-2\theta'A+\tfrac{1}{2}\theta'^2,$$
$$\overline{B}_2=B_2-2\theta'B+\tfrac{1}{2}\theta'^2.$$

Using these expressions, we find, as in Art. 10, (iii.),

$$D(P)=D(Q)=2\overline{A}^2-\overline{A}_2-\overline{A};$$

or, substituting for $\overline{A}$ and $\overline{A}_2$ their values,

$$D(P) = D(Q) = 2A^2 - A_2 - A + \tfrac{1}{2}\theta'.$$

To determine $D(R)$, we have:—(i.) For the discriminantal index of the set of branches touching either PR or QR, $\overline{B}_2 - \overline{A} - \overline{B}^2$; (ii.) for the joint discriminantal index of these two sets of branches, $2\overline{B}^2$; (iii.) for the joint discriminantal index of the θ' linear branches, $\theta'(\theta' - 1)$; (iv.) for the joint discriminantal index of the linear branches taken with the branches touching either PR or QR, $2\theta'\overline{B}$.

Hence $$D(R) = D(S) = 2[\overline{B}_2 - \overline{A} - \overline{B}^2] + 2\overline{B}^2 + \theta'(\theta' - 1) + 4\theta'\overline{B}$$
$$= 2(B_2 - A),$$

as in the case when N is not a square.

There is no change in the expressions for the order of $\nabla(q)$, and for the exponents of the factors q and $1 - q$ in $\nabla(q)$ (see Art. 6); and these expressions agree with the values which we have obtained for $D(R) = D(S)$, and for $D(P) = D(Q)$. For PR, touching the curve at R, counts as $A - B$ tangents drawn from P; and hence the order of q in the discriminant ought to be, what in fact it is, $2(B_2 - A) + (A - B) = 2B_2 - A - B$.

And again, PQ, considered as drawn from P, counts as $A - \frac{1}{2}\theta'$ tangents at P, and as $B - \frac{1}{2}\theta'$ tangents at Q. Thus, the number of intersections of C by the polar of P, which lie on the line PQ, is

$$2(2A^2 - A_2 - A\tfrac{1}{2} + \theta') + (A - \tfrac{1}{2}\theta') + (B - \tfrac{1}{2}\theta') = 4A^2 - 2A_2 + B - A;$$

and this number is, as it ought to be, the excess of $2A(2A - 1)$ above the order of $\nabla(q)$; *i.e.*, the excess of the whole number of intersections above the intersections lying on PQ.

Arts. 12–14. *Formulæ applicable to all values of N.*

12. If, in the formulæ relating to the case when N is a square, we omit the terms containing the symbol θ' defined by the equation

$$\theta' = f(\theta) = f'(\theta') = f(\surd N),$$

we obtain the corresponding formulæ for the case when N is not a square. We shall henceforward denote by θ' a number which is equal to zero when N is not a square, and which is equal to $f(\surd N)$ when N is a square; and we shall treat the two cases simultaneously, except when it is necessary to call attention to the difference between them.

13. *The discriminantal class of the superlinear branches.*—In the paper on the Higher Singularities of Plane Curves* (Arts. 12 and 13), it has been shown

* Proceedings of the Society, vol. vi. p. 153 [vol. ii. p. 112].

that, if d and t are the order and class of a superlinear branch, D and T its discriminantal order and class, we have the equation

$$T - D = t^2 - d^2.$$

And again, that if there be a second superlinear branch of the order d' and class t' touching the first, and if we represent by $\overline{T}$ and $\overline{D}$ the joint discriminantal indices (of order and class) appertaining to the two branches, we have the equation

$$\overline{T} - \overline{D} = 2(tt' - dd').$$

Combining these two results, we obtain the theorem—

'If any number of branches touch one another at the same point, the difference between the discriminantal order and class of the singularity is equal to the difference between the squares of its order and of its class.'

Employing a notation explained in Art. 14, we apply this theorem to determine the discriminantal class of the branches (PPQ), (PQQ), (PRR), (QRR), (PSS), (QSS). We thus find

$$\begin{aligned} T(PPQ) - D(PPQ) &= T(PQQ) - D(PQQ) \\ &= (B - \tfrac{1}{2}\theta')^2 - (A - B)^2. \quad \ldots\ldots \quad (31) \end{aligned}$$

$$\begin{aligned} T(PRR) - D(PRR) &= T(QRR) - D(QRR) \\ &= T(PSS) - D(PSS) = T(QSS) - D(QSS) \\ &= (A - B)^2 - (B - \tfrac{1}{2}\theta')^2; \quad \ldots\ldots\ldots \quad (32) \end{aligned}$$

so that

$$\begin{aligned} T(PPQ) = T(PQQ) &= B_2 - B^2 - A - \theta' B + \tfrac{1}{2}\theta' + \tfrac{1}{4}\theta'^2 \\ &= \overline{B}_2 - \overline{B}^2 - \overline{A} \\ &= D(PRR) = D(QRR) = D(PSS) = D(QSS), \end{aligned}$$

and

$$\begin{aligned} T(PRR) = T(QRR) &= T(PSS) = T(QSS) \\ &= 2A^2 - A_2 - A + \tfrac{1}{2}\theta \\ &= 2\overline{A}^2 - \overline{A}_2 - \overline{A} = D(PQQ) = D(PPQ). \end{aligned}$$

14. *Summary of the results.*—For convenience of reference, we exhibit the preceding results in a tabular form.

Characteristics and Singularities of the Modular Curve C.

I. *Explanation of the symbols*:

(1) The order of the transformation is the uneven number N.

(2) g and g' are conjugate divisors of N; h^2 is the greatest square dividing N.

(3) η is the greatest common divisor of g and g'.

(4) $f(\eta)$ is the number of numbers not surpassing η and prime to it.

(5) $f'(g)$ and $f'(g')$ are defined by the equation

$$\frac{f'(g)}{g} = \frac{f(\eta)}{\eta} = \frac{f'(g')}{g'}.$$

(6) $$2\nu = \Sigma f(\eta), \quad A+B = \Sigma f'(g), \quad A_2+B_2 = (A+B)^2.$$

In these equations the summations Σ extend to all divisors g of N; A comprehends all the terms $f'(g)$ in which $g > \sqrt{N}$, and also, if $N=\theta^2$, the term $\frac{1}{2}\theta' = \frac{1}{2}f(\theta)$; A_2 comprehends all the terms of $\Sigma f'(g_1) \times \Sigma f'(g_2)$, in which $g_1 g_2 > N$, and one-half of every term in which $g_1 g_2 = N$, g_1 and g_2 denoting any two divisors of N, the same or different. The definitions of B and B_2 follow from those of A and A_2.

(7) m, n, K, I, D, T, H denote respectively the order, the class, the cuspidal index, the inflexional index, the discriminantal order, the discriminantal class, and the deficiency of the curve.

(8) The symbol (XXY) or (YXX) denotes a branch, or an aggregate of branches, touching the line XY at the point X.

(9) The symbols $O(XXY)$, $C(XXY)$, $K(XXY)$, $I(XXY)$, $D(XXY)$, $T(XXY)$ denote the order, class, cuspidal index, inflexional index, discriminantal order, discriminantal class of the branches (XXY). The symbols $O(X)$, $K(X)$, $D(X)$, $C(XY)$, $I(XY)$, $T(XY)$ are to be similarly interpreted with regard to the branches which pass through a given point X or touch a given line XY. Lastly, the symbols $D(XXY, XXZ)$ and $T(XXY, XYY)$ denote respectively twice the number of points common to the branches (XXY), (XXZ), and twice the number of tangents common to the branches (XXY), (XYY).

II. *Characteristics of the Curve.**

$$\begin{aligned}
m &= 2A, \quad n = 3A - B - \theta', \\
H &= \tfrac{1}{2}(A+B) - 3\nu + 1, \\
K &= 2(A+B) - 6\nu + \theta', \\
I &= 5A - B - 6\nu - 2\theta', \\
I - K &= 3A - 3B - 3\theta', \\
D &= 4A^2 - 5A + B + \theta', \\
T &= (3A - B - \theta')^2 - 5A + B + \theta', \\
T - D &= (3A - B - \theta')^2 - 4A^2.
\end{aligned}$$

* Several of the formulæ which follow may be more simply expressed by using the symbols $\overline{A}, \overline{B}, \overline{A}_2, \overline{B}_2$ of Art. 11, and by writing $\overline{\nu} = \nu - \frac{1}{2}\theta'$.

III. *Characteristics of the Special Singularities.*

(i.) *Characteristics of* (PPQ) *and* (PQQ).

$$O(PPQ) = A - B; \quad C(PPQ) = B - \tfrac{1}{2}\theta',$$
$$K(PPQ) = A - B - \nu + \tfrac{1}{2}\theta', \quad I(PPQ) = B - \nu,$$
$$D(PPQ) = 2A^2 - A_2 - A + \tfrac{1}{2}\theta',$$
$$T(PPQ) = B_2 - B^2 - A - \theta' B + \tfrac{1}{2}\theta' + \tfrac{1}{4}\theta'^2.$$

The number of distinct branches is $\nu - \frac{1}{2}\theta'$. They are all superlinear; viz. corresponding to every divisor g of N, which is less than $\sqrt{N}$, there are in (PPQ) $f(\eta)$ superlinear branches, each of the order $\frac{g' - g}{\eta}$, and of the class $\frac{g}{\eta}$; (PQQ) is of the same type as (PPQ).

(ii.) *Characteristics of* (PRR), (QRR), (PSS), (QSS).

All these singularities are of the same type.

$$O(PRR) = B - \tfrac{1}{2}\theta'; \quad C(PRR) = A - B,$$
$$K(PRR) = B - \nu,$$
$$I(PRR) = A - B - \nu + \tfrac{1}{2}\theta',$$
$$D(PRR) = B_2 - B^2 - A - \theta' B + \tfrac{1}{2}\theta' + \tfrac{1}{4}\theta'^2,$$
$$T(PRR) = 2A^2 - A_2 - A + \tfrac{1}{2}\theta'.$$

The number of distinct branches in (PRR) is $\nu - \frac{1}{2}\theta'$; of these, $h - \theta'$ are linear (Art. 10, iii.); the characteristics of (PRR) and (PPQ) are reciprocal; viz. corresponding to any divisor g of N, which is less than $\sqrt{N}$, there are in (PRR) $f(\eta)$ branches, each of the order $\frac{g}{\eta}$ and of the class $\frac{g' - g}{\eta}$.

(iii.) *Characteristics of* (PQ).

$$C(PQ) = 2C(PPQ) = 2B - \theta';$$
$$I(PQ) = 2I(PPQ) = 2B - 2\nu,$$
$$T(PQ) = T(PPQ) + T(PQQ) + T(PPQ, PQQ)$$
$$= 2T(PPQ) + 2(B - \tfrac{1}{2}\theta')^2$$
$$= 2B_2 - 2A - 4\theta' B + \theta'(\theta' + 1).$$

(iv.) *Characteristics of* (R) *and* (S).

These are the same for the two points.

(1) $$O(R) = O(PRR) + O(QRR) + O(\theta)$$
$$= 2B.$$

(2) $$K(R) = K(PRR) + K(QRR)$$
$$= 2B - 2\nu.$$

$$(3)\quad D(R) = D(PRR) + D(QRR) + D(PRR, QRR) + D(\theta) + D(\theta, PRR) + D(\theta, QRR)$$
$$= 2[B_2 - B^2 - A - \theta' B + \tfrac{1}{2}\theta' + \tfrac{1}{4}\theta'^2] + 2(B - \tfrac{1}{2}\theta')^2 + \theta'(\theta' - 1) + 4\theta'(B - \tfrac{1}{2}\theta')$$
$$= 2(B_2 - A).$$

The symbol (θ) is used to represent the θ' linear branches which pass through R, having tangents distinct from one another and from PR, QR.

(v.) *Tangents to the Curve from PQRS.*

(1) PQ, considered as drawn from P, counts as $A - \frac{1}{2}\theta'$ tangents at P, and as $B - \frac{1}{2}\theta'$ tangents at Q; PR counts as $A - B$ tangents at R; thus PQ, PR, PS count as $3A - B - \theta' = n$ tangents drawn from P.

(2) The tangents to the branches (PRR), (QRR), and (θ) count as $2(A - \frac{1}{2}\theta') + 2\theta' = 2A + \theta'$ tangents drawn from R. Thus, there are $A - B - 2\theta'$ other tangents which can be drawn to the curve from R.*

(vi.) *Intersections with the Curve of the sides of the quadrangle PQRS.*

(1) PQ meets the curve in $A - \frac{1}{2}\theta'$ points at P, and in as many at Q; and in θ' non-singular points distinct from either P or Q.

(2) PR meets the curve in $A - B$ points at P; at R it meets the branches (PRR) in $A - \frac{1}{2}\theta'$ points; the branches (QRR) in $B - \frac{1}{2}\theta'$ points; the branches (θ) in θ' points; in all in $2A$ points. The same statements hold, *mutatis mutandis*, for the lines QR, PS, QS.

(3) RS meets the curve $2(B - \frac{1}{2}\theta') + \theta' = 2B$ points at R, and in as many at S; and also in $2(A - 2B)$ other points.†

IV. *Residual singularities of the Curve.*

Designating by K_1, I_1, D_1, T_1 the parts of the indices K, I, D, T which arise from the singularities connected with the points and lines of the quadrangle

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* If $\chi\left(\frac{1}{M}, q\right) = 0$ represent the equation of the multiplier, which is of the order $A + B$ in $\frac{1}{M}$, and of the order $\frac{1}{2}(A - B)$ in q, the values of q appertaining to the points of contact of these tangents are determined by the equations $\chi\left(\sqrt{n}, \frac{q}{q-1}\right) = 0$, $\chi\left(-\sqrt{n}, \frac{q}{q-1}\right) = 0$. When $N = \theta'^2$, the first of these equations has θ' roots equal to zero, and θ' infinite roots; both these sets of roots are to be rejected.

† At each of these points we have $p = q$. The equation $F(p, p, 1) = 0$ is divisible by $[p(p-1)]^{2B}$; the remaining roots, which are $2A - 4B$ in number, give the intersections of the curve by RS at points other than R and S. These roots may be determined by the method (due to M. Kronecker) described in the Report on the Theory of Numbers, Arts. 131–133 [vol. i. p. 325].

PQRS, we find, from the preceding formulæ,

$$\begin{cases} K_1 = 2(A+B) - 6\nu + \theta', \\ I_1 = 4A - 2B - 6\nu + 2\theta', \\ D_1 = 4A^2 + 4B_2 - 2A_2 - 6A + \theta', \\ T_1 = 8A^2 + 2B_2 - 4A_2 - 6A - 4\theta' B + 3\theta' + \theta'^2; \end{cases}$$

and for the residual singularities we have

$$\begin{cases} K_2 = 0, \\ I_2 = A + B - 4\theta', \\ D_2 = 2A_2 - 4B_2 + A + B, \\ T_2 = 4A^2 + 4B^2 + A_2 - 5B_2 + 6\theta'(B-A) + A + B - 2\theta'. \end{cases}$$

ARTS. 15, 16.—*Case when N is a square. The Linear Branches* (θ).

15. The developments appertaining to the θ' linear branches U, which intersect PQ at points other than P and Q, are

$$-\frac{1}{p} = \frac{e^{iu}}{q} + \frac{e^{iu}(1+e^{iu})}{2q^2} + \frac{e^{iu}(1+e^{iu})(21+11e^{iu})}{64q^2} + \ldots, \quad . \quad . \quad . \quad (33)$$

where $u = \dfrac{2l+1}{\theta}\pi$, $2l+1$ being any term of a system of residues prime to 2θ.

We hence obtain the normal developments

$$\frac{\beta - \frac{1}{2}\gamma + e^{iu}(\alpha - \frac{1}{2}\gamma)}{\beta} = 5(1-e^{2iu})\frac{\gamma^2}{\beta^2} + \ldots, \quad . \quad . \quad . \quad . \quad . \quad (34)$$

so that the tangents of the θ' branches are the lines

$$\beta - \tfrac{1}{2}\gamma + e^{iu}(\alpha - \tfrac{1}{2}\gamma) = 0,$$

which meet one another in the point $\alpha = \beta = \frac{1}{2}\gamma$; *i.e.* in the point O in which RS intersects ab.

The developments appertaining to the θ' linear branches at R are

$$p = e^{iv}q + \tfrac{1}{2}e^{iv}(1-e^{iv})q^2 + \ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where $v = \dfrac{2h}{\theta}\pi$, h being any term of a system of residues prime to θ; so that the tangents are $\alpha - e^v\beta = 0$,

none of the branches being inflected at R.

Similarly the developments appertaining to the θ' linear branches at S are

$$p - 1 = e^{iv}(q-1) + \tfrac{1}{2}e^{iv}(1-e^{iv})(q-1)^2 + \ldots, \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and the tangents are $\alpha - \gamma = e^{iv}(\beta - \gamma)$,

there being no inflexion.

The two sets of tangents at R and S meet PQ in the same points in which it is intersected by the linear branches U; for, if

$$2l+1+2h=(2k+1)\,\theta,$$

we have $u+v=(2k+1)\,\pi$, whence $e^{iv}=-e^{-iu}$.

16. The developments (33–36) may be obtained as follows, with the help of formulæ established in the Report on the Theory of Numbers already cited.

If $\omega=1+\frac{i}{\sigma}$, where σ is positive and increases without limit,

$$q=\phi^8(\omega)=1-\phi^{-8}(i\sigma)$$

increases without limit; and the limit of $q\div 16e^{\pi\sigma}$ is unity. The corresponding values of p are comprised in the formula

$$\phi^8\left(\frac{g'\left(1+\frac{i}{\sigma}\right)+2k}{g}\right),$$

where g and g' are conjugate divisors of N, and g, g', k have no common divisor. But this expression may be exhibited in a form from which the dimensions of $\phi(\Omega)$, as compared with $\phi(\omega)$, may be inferred; viz., we have (Report, *loc. cit.*, p. 350)

$$\phi^8\left(\frac{g'\left(1+\frac{i}{\sigma}\right)+2k}{g}\right)=\phi^{-8}\left(\frac{d'i\sigma+2l+1}{d}\right),$$

where d' is the greatest common divisor of $g'+2k$ and g, $d=\frac{N}{d'}$, and $2l+1$ is determined by a certain congruence for the modulus d. In order that the development of $\frac{1}{p}$, in a series proceeding by powers of $\frac{1}{q}$, should correspond to a branch intersecting PQ elsewhere than at P or Q, $\frac{1}{p}$ and $\frac{1}{q}$ must be of the same dimensions. But

$$\text{Lim. } p\div e^{\frac{d'}{d}\pi\sigma}=1, \quad \text{Lim. } q\div e^{\pi\sigma}=1;$$

hence $d'=d$, or N is necessarily a square, and $d=d'=\theta$. Since $\theta=d'$ is the greatest common divisor of $g'+2k$ and g, let $g=\lambda\theta$, $g'=\frac{\theta}{\lambda}$; then θ divides $\frac{\theta}{\lambda}+2k$; *i.e.*, $\frac{\theta}{\lambda}$ divides g, g', and $2k$, which are relatively prime. Hence $\lambda=\theta$, $g=\theta^2=N$, $g'=1$. Now there are just θ' values of $2k$ for which θ is the greatest common divisor of θ^2 and $1+2k$; viz., if $2\mu+1$ be any number less than 2θ and prime to θ, the θ' values of $2k$ are included in the formula $2k=(2\mu+1)\,\theta-1$;

and it will be found that the congruence determining $2l+1$ is

$$(2\mu+1)(2l+1) \equiv -1, \text{ mod } \theta.$$

Hence we have, for the θ' values of $2k$ which we are considering,

$$\phi^8\left(\frac{\frac{i}{\sigma}+1+2k}{N}\right) = \phi^{-8}\left(i\sigma+\frac{2l+1}{\theta}\right) = \phi^{-8}\left(i\sigma+\frac{u}{\pi}\right).$$

Expanding the values of

$$\frac{1}{q} = -\frac{\phi^8(i\sigma)}{1-\phi^8(i\sigma)} \quad \text{and of} \quad \frac{1}{p} = \phi^8\left(i\sigma+\frac{u}{\pi}\right)$$

by means of the formula

$$\phi^8(\omega) = 16e^{i\pi\omega}(1-8e^{i\pi\omega}+44e^{2i\pi\omega}-\ldots), \quad \ldots\ldots \quad (37)$$

which arises from the expansion of (3); and equating the coefficients of like powers of $e^{-\pi\sigma}$ in the series

$$\frac{1}{p} = \frac{A}{q}+\frac{B}{q^2}+\frac{C}{q^3}+\ldots$$

we obtain the developments (33).

Similarly the developments (35), which appertain to the linear branches at R, may be obtained by substituting the expansions of $p = \phi^8\left(i\sigma+\frac{v}{\pi}\right)$ and $q = \phi^8(i\sigma)$ in the assumed series

$$p = Aq+Bq^2+\ldots.$$

Arts. 17–19. *The Six Modular Curves.*

17. If we represent by $\epsilon(x)$ any one of the six anharmonic functions

$$x, \quad 1-x, \quad \frac{1}{x}, \quad \frac{1}{1-x}, \quad \frac{x}{x-1}, \quad \frac{x-1}{x}, \quad \ldots\ldots \quad (38)$$

the modular equation (1) is unchanged by the simultaneous substitution of $\epsilon(p)$ for p and $\epsilon(q)$ for q. Hence, if $\epsilon_1(x)$, $\epsilon_2(x)$ denote any two, the same or different, of the functions (38), the thirty-six substitutions

$$F[\epsilon_1(p), \quad \epsilon_2(q), \quad 1] \quad \ldots\ldots\ldots \quad (39)$$

give only six different equations. As representatives of these, we take the following:

(i.) $F(1-p, q, 1) = 0,$

(ii.) $F\left(p, \frac{1}{q}, 1\right) = 0,$

$$\text{(iii.)} \quad F\left(1-p, \frac{1}{1-q}, 1\right) = 0,$$

$$\text{(iv.)} \quad F(p, q, 1) = 0,$$

$$\text{(v.)} \quad F\left(1-p, \frac{1}{q}, 1\right) = 0,$$

$$\text{(vi.)} \quad F\left(p, \frac{1}{1-q}, 1\right) = 0.$$

The equation $F(p, q, 1) = 0$ is symmetric with regard to p and q; and it will be found that the equations (i.), (ii.), (iii.) possess the same property; thus, for example, the equations

$$F\left(p, \frac{1}{q}, 1\right) = 0, \quad \text{and} \quad F\left(q, \frac{1}{p}, 1\right) = 0,$$

are the same, because

$$F(x, y, 1) = F(y, x, 1) = (xy)^{A+B} F\left(\frac{1}{x}, \frac{1}{y}, 1\right).$$

The fifth and sixth equations, on the other hand, are changed, each into the other, by the interchange of p and q.

18. Denoting by X and Y rectangular Cartesian coordinates, and writing in the equations (i), ..., (vi.),

$$\left.\begin{aligned} p &= \tfrac{1}{2} + X - iY \\ q &= \tfrac{1}{2} + X + iY, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (40)$$

we obtain the equations of six curves, which, in the Mémoire 'Sur les Équations Modulaires,'* we have called the first, second, third, fourth, fifth, and sixth modular curves. The equations of the first four of these curves are real, as appears from the symmetry of the equations (i.)—(iv.) with regard to p and q; the equations of the fifth and sixth curves are imaginary and conjugate to one another.

The first and fourth curves are each of them symmetric with regard to both axes; the fourth curve is its own inverse (anallagmatic) with regard to each of the two real circles

$$(X \pm \tfrac{1}{2})^2 + Y^2 = 1;$$

and the first curve with regard to each of the two imaginary circles

$$X^2 + (Y \pm \tfrac{1}{2} i)^2 = -1.$$

The second and third curves are symmetric with regard to the axis of X, and symmetric to one another with regard to the axis of Y; the fifth and sixth (imaginary) curves are symmetric with regard to the axis of Y, and symmetric

* [Vol. ii. p. 224.]

to one another with regard to the axis of X. The second and third curves are the inverses of the first, with regard to the circles

$$(X-\tfrac{1}{2})^2+Y^2=1, \quad (X+\tfrac{1}{2})^2+Y^2=1,$$

respectively; similarly the two imaginary curves are the inverses of the fourth curve with regard to the two imaginary circles

$$X^2+(Y\pm\tfrac{1}{2}i)^2=-1.$$

Lastly, the substitution $X=iY', \quad Y=-iX'$

changes the first curve into the fourth, the second into the fifth, the third into the sixth, and *vice versâ*.

These assertions are the geometrical equivalents of the properties of the modular equation stated in Art. 17; it will suffice to verify one of them. The equation of the first modular curve is

$$F(\tfrac{1}{2}-X+iY,\ \tfrac{1}{2}+X+iY,\ 1)=0;$$

its inverse with regard to the circle $(X+\frac{1}{2})^2+Y^2=1$ is obtained by writing

$$\tfrac{1}{2}-X+iY=\frac{1}{\frac{1}{2}-X'-iY'},$$

$$\tfrac{1}{2}-X-iY=\frac{1}{\frac{1}{2}-X'+iY'},$$

so that

$$\tfrac{1}{2}+X+iY=1-\frac{1}{\frac{1}{2}-X'+iY'}.$$

The equation of the inverse curve is therefore

$$F\left(\frac{1}{\frac{1}{2}-X'-iY'},\ 1-\frac{1}{\frac{1}{2}-X'+iY'},\ 1\right)=0;$$

and this is identical with the equation

$$F\left(\tfrac{1}{2}+X'-iY',\ \frac{1}{\frac{1}{2}+X'+iY'},\ 1\right)=0;$$

i.e. with the equation of the second modular curve, because

$$F\left(\frac{1}{1-y},\ 1-\frac{1}{1-x},\ 1\right)=0$$

is identical with

$$F\left(x, \frac{1}{y}, 1\right)=0.$$

19. The equations of the first and fourth modular curves are included in the general equation

$$F(\alpha, \beta, \gamma)=0;$$

viz. to obtain the first curve, we write

$$\alpha = \tfrac{1}{2} - X + iY, \quad \beta = \tfrac{1}{2} + X + iY, \quad \gamma = 1;$$

and, to obtain the fourth curve, we write

$$\alpha = \tfrac{1}{2} + X - iY, \quad \beta = \tfrac{1}{2} + X + iY, \quad \gamma = 1.$$

Thus the theory of the singularities of these two curves is implicitly contained in the preceding discussion of the singularities of C.

In both curves the points P, Q are the cyclic points, and (ab, RS) or O is the origin: in the first curve ab and RS are the axes of X and Y; a, b being the points $(\pm\frac{1}{2}, 0)$, and R, S the points $(0, \pm\frac{1}{2}i)$: c is the point at an infinite distance on the axis of Y; in the fourth curve R, S are the points $(\pm\frac{1}{2}, 0)$, and a, b the points $(0, \pm\frac{1}{2}i)$, c being the point at an infinite distance on the axis of X. Both equations (as has been already said) are real; and it follows, from the theory explained in the Mémoire cited, that both of them represent real curves, except when $N \equiv 3$, mod 4; in which case the fourth curve reduces itself to the pair of conjugate points $(\pm\frac{1}{2}, 0)$.

When N is not a square, both curves are completely and parabolically cyclic, having at each cyclic point ν branches, of the aggregate order $A - B$ and class B, touching the line at an infinite distance.

When N is a square, each of the two curves has θ' real infinite branches. The fourth curve has also θ' real branches passing through each of the points $(\pm\frac{1}{2}, 0)$; (these two points always belong to the curve, though, when N is not a square, only as isolated points:) the tangents to the θ' branches are parallel to the asymptotes of the curve. Similarly the first modular curve acquires θ' linear, but imaginary, branches at each of the points $(0, \pm\frac{1}{2}i)$; the tangents to these branches being imaginary lines parallel to the real asymptotes.

The equations of the asymptotes of the first and fourth curves are respectively

$$\left.\begin{aligned} Y \cos \tfrac{1}{2}u - X \sin \tfrac{1}{2}u &= 0, \\ Y \sin \tfrac{1}{2}u + X \cos \tfrac{1}{2}u &= 0; \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (41)$$

u denoting $\dfrac{2l+1}{\theta}\pi$, as in Art. 15. And it may be inferred from the developments (33) and (33), given in that Article, that the rectangular hyperbola

$$(Y \cos \tfrac{1}{2}u - X \sin \tfrac{1}{2}u)\ (Y \sin \tfrac{1}{2}u + X \cos \tfrac{1}{2}u) = 5 \sin u$$

osculates at an infinite distance the branches asymptotic to the two lines (41).

Lastly, if $v=\frac{2h}{\theta}\pi$, as in Art. 15, the tangents of the fourth curve at the points $(\pm\frac{1}{2}, 0)$ are

$$Y\cos\tfrac{1}{2}v+(X\pm\tfrac{1}{2})\sin\tfrac{1}{2}v=0;$$

the tangents of the first curve at the imaginary points $(0, \pm\frac{1}{2}i)$ are

$$(Y\pm\tfrac{1}{2}i)\sin\tfrac{1}{2}v-X\cos\tfrac{1}{2}v=0:$$

and these tangents are parallel to the asymptotes of the curves to which they respectively appertain; because, if

$$2l+1+2h=(2k+1)\theta,$$
$$\tan\tfrac{1}{2}u=\cot\tfrac{1}{2}v.$$

The points $(\pm\frac{1}{2}, 0)$ and $(0, \pm\frac{1}{2}i)$ are foci, and indeed the only foci, of both curves: of these, the points $(0, \pm\frac{1}{2}i)$ lie on the first curve, and the two real points $(\pm\frac{1}{2}, 0)$ are its two foci (properly so called); the axis of Y being the only corresponding cyclic axis, or directrix. The points $(\pm\frac{1}{2}, 0)$ belong to the fourth curve (only as isolated points, when N is not a square), and this curve has, properly speaking, only the pair of imaginary foci $(0, \pm\frac{1}{2}i)$.

19. The second and third modular curves may be regarded as derived from the equation (1), by the substitution

$$\left.\begin{aligned} p&=\frac{\alpha}{\gamma}, \quad q=\frac{\gamma}{\beta},\\ \alpha&=\tfrac{1}{2}\pm(X-iY),\\ \beta&=\tfrac{1}{2}\pm(X+iY),\\ \gamma&=1. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (42)$$

X and Y being rectangular Cartesian coordinates, and the upper signs relating to the second curve, the lower to the third.

Thus the theory of each of these curves is comprehended in that of the curve C', of which the trilinear equation is

$$\left.\begin{aligned} &(\beta\gamma)^{B-A}\times F(\alpha\beta, \gamma^2, \beta\gamma)=0,\\ \text{or}\quad &(\alpha\gamma)^{B-A}\times F(\alpha\beta, \gamma^2, \alpha\gamma)=0. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (43)$$

The singularities of C' may be examined by the method already employed in the case of C. Attending, for brevity, only to the case in which N is not divisible by any square, we write $p=\frac{\alpha}{\gamma}$, $q=\frac{\gamma}{\beta}$, in the parametric developments of Art. 4, and we deduce, as follows, the normal developments of the singular branches of C'.

(i.) From (6) we obtain

$$\frac{\alpha-\gamma}{\gamma}=\lambda\left(\frac{\gamma-\beta}{\beta}\right)^{\frac{g'}{g}}+\ldots,$$

or, multiplying by $\frac{\gamma}{\beta}=1+\frac{\gamma-\beta}{\beta}$,

$$\frac{\alpha-\gamma}{\beta}=\lambda\left(\frac{\gamma-\beta}{\beta}\right)^{\frac{g'}{g}}+\ldots, \quad \ldots\ldots\ldots \quad (44)$$

which is itself a normal development, if $g'>g$, and gives rise, by reversion, to such a development, if $g'<g$. Hence C' has a singularity at S, having the same characteristics as the corresponding singularity of C.

(ii.) From (5) we infer

$$\frac{\alpha}{\gamma}=\lambda\left(\frac{\gamma}{\beta}\right)^{\frac{g'}{g}}+\ldots.$$

Here, when p and q are small, α must be small compared with γ, and γ compared with β; *i.e.* the coordinates of the point $(p=0,\ q=0)$ are $\alpha=0,\ \gamma=0$; and the normal development is

$$\frac{\alpha}{\beta}=\lambda\left(\frac{\gamma}{\beta}\right)^{\frac{g+g'}{g}}+\ldots. \quad \ldots\ldots\ldots \quad (45)$$

(iii.) Similarly from the development (7) we deduce

$$\frac{\gamma}{\alpha}=\lambda\left(\frac{\beta}{\gamma}\right)^{\frac{g'}{g}}+\ldots.$$

Thus the coordinates of the point $(p=\infty,\ q=\infty)$ are $\beta=0,\ \gamma=0$; and we find, after reversion and multiplication by $\frac{\gamma}{\alpha}$, the normal development

$$\frac{\beta}{\alpha}=\mu\left(\frac{\gamma}{\alpha}\right)^{\frac{g'+g}{g'}}+\ldots, \quad \ldots\ldots\ldots \quad (46)$$

which is of the same type as (45).

Thus the curve has at P and Q singularities of one and the same self-reciprocal type, not resembling the singularities which C has at the same points. The point R does not lie on C'.

ART. 20. *Characteristics and Singularities of the Modular Curve C'.*

I. *Characteristics of the Curve.*

$$\begin{aligned} m &= 2A + 2B, \quad n = 3A + B, \\ H &= \tfrac{1}{2}(A+B) - 3\nu + 1, \\ K &= 2A + 4B - 6\nu, \\ I &= 5A + B - 6\nu, \\ I - K &= 3(A - B), \\ D &= 4(A+B)^2 - 5A - 3B, \\ T &= (3A+B)^2 - 5A - 3B, \\ T - D &= (3A+B)^2 - 4(A+B)^2 \\ &= (A-B)(5A+3B). \end{aligned}$$

It will be noticed (1) that these formulæ do not contain θ', although the case when N is a square is included in them; (2) that, when N is not a square, $I-K$ has the same value for C' as for C.

II. *Characteristics of the Special Singularities.*

(i.) *Characteristics of* (PPR) *and* (QQR).

$$\begin{aligned} O(PPR) &= A + B = C(PPR), \\ K(PPR) &= A + B - 2\nu = I(PPR), \\ D(PPR) &= A_2 + 3B_2 - 2A - 2B = T(PPR). \end{aligned}$$

The number of distinct branches at each of the points P and Q is 2ν; viz., corresponding to every divisor g of N, there are in (PPR), $f(\eta)$ branches of the order $\frac{g}{\eta}$ and of the class $\frac{g'}{\eta}$; of the 2ν branches, h are linear, and, in particular, when N is a square, θ' of these are also non-inflexional.

(ii.) *Characteristics of* (PSS), (QSS), *and* (S).

These are the same for C' as for C (see Art. 15, III., (ii.) and (iv.)). When N is a square the equations of the tangents to the linear branches are

$$\alpha - \gamma = e^{i\omega}(\gamma - \beta)$$

(see Art. 15, equation (36)).

(iii.) *Tangents to the Curve from* $PQRS$.

(1) PR, considered as a tangent drawn from P, counts as $2A + 2B$ tangents at P; and PS counts as $A - B$ tangents; hence PR and PS are the only tangents from P.

(2) RP, considered as a tangent drawn from R, counts as $A+B$ tangents at P; and so does RQ at Q; thus there are $A-B$ other tangents which can be drawn to C' from R.

(3) Besides the tangents at S, there are $A-B-2\theta'$ other tangents which can be drawn to the curve from S (see Art. 14, III., (v.) (2)).

(iv.) *Intersections with the Curve of the sides of the quadrangle PQRS.*

(1) PQ meets the curve in $A+B$ points at P, and in $A+B$ points at Q. Thus it never meets the curve again, and touches it nowhere.

(2) PR and QR each meet the curve in $2A+2B$ points, touching it at P and Q respectively, and meeting it nowhere else.

(3) PS and QS meet the curve in $A+B$ points at S, and in $A+B$ points at P and Q respectively; thus they never meet the curve again.

(4) RS meets the curve in $2B$ points at S, and in $2A$ other points.*

III. *Residual Singularities of the Curve.*

Employing the notation of Art. 14, IV., we have

$$\begin{cases} K_1 = 2A + 4B - 6\nu, \\ I_1 = 4A - 6\nu + \theta', \\ D_1 = 2A_2 + 8B_2 - 6A - 4B, \\ T_1 = 4A^2 + 6B_2 - 6A - 4B + \theta'. \end{cases}$$

$$\begin{cases} K_2 = 0, \\ I_2 = A + B - \theta', \\ D_2 = 2A_2 - 4B_2 + A + B, \\ T_2 = 2A^2 - 2B^2 + 3A_2 - 3B_2 + A + B - \theta'. \end{cases}$$

The indices K_2, D_2 have the same values for C and C', because these indices refer to singularities which do not lie, in either figure, upon the fundamental triangle of the quadric transformation by which the curves are changed into one another. The equality of the indices I_2, when N is not a square, implies the theorem:

'Each of the first three modular curves has as many non-singular inflexional tangents as it has osculating circles, which pass through the point $(\frac{1}{2}, 0)$; or again, through the point $(-\frac{1}{2}, 0)$.'

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* The points of contact of the $A-B$ tangents, (iii.) (2), and of the $A-B-2\theta'$ tangents, (iii.) (3), and the $2A$ points of intersection, (iv.) (4), can be determined by methods similar to those indicated in the Notes on Art. 14.

Of the formulæ contained in the preceding enumeration, we shall demonstrate only one; viz., the expression for $D(PPR)$ or $D(P)$.

We have, as in Art. 10, (iii.),

$$D(P)=\Sigma f(\eta)\frac{g+g'}{\eta}\left(\frac{g}{\eta}-1\right)+\Sigma f(\eta)\,[f(\eta)-1]\frac{g(g+g')}{\eta^2}$$
$$+2\Sigma\Sigma_1 f(\eta)\,f(\eta_1)\frac{g_1(g+g')}{\eta\eta_1};$$

where g is any divisor of N, and g' is the conjugate divisor of g; g_1 is any divisor less than g, so that $g_1'>g'$; the summations Σ_1 and Σ extend to every value of g_1 and g respectively. Hence we find

$$D(P)=\Sigma f'(g)\,[f'(g)+f'(g')]-\Sigma[f'(g)+f'(g')]$$
$$+2\Sigma\Sigma_1 f'(g_1)\,[f'(g)+f'(g')]. \quad . \quad . \quad (47)$$

But we have, evidently,

$$\Sigma f'(g)\,f'(g)+2\Sigma\Sigma_1 f'(g_1)\,f'(g)=(A+B)^2=A_2+B_2;$$
$$\Sigma[f'(g)+f'(g')]=2A+2B;$$

and, observing that $g'g_1<N$, we also find

$$\tfrac{1}{2}\Sigma f'(g)\,f'(g')+\Sigma\Sigma_1 f'(g_1)\,f'(g')=B_2.$$

Introducing these values into the equation (47), we obtain

$$D(P)=A_2+3B_2-2A-2B,$$

in accordance with the formula II. (i.) *supra*.

XXXVII.

NOTE ON A MODULAR EQUATION

OF THE

TRANSFORMATION OF THE THIRD ORDER.

[Proceedings of the London Mathematical Society, vol. x. pp. 87–91. Read February 13, 1879.]

I HAVE given elsewhere* the modular equation for the transformation of the third order between

$$x = f(k^2) = \frac{(1-k^2+k^4)^3}{k^4(1-k^2)^2} \quad \text{and} \quad y = f(\lambda^2) = \frac{(1-\lambda^2+\lambda^4)^3}{\lambda^4(1-\lambda^2)^2};$$

viz.,

$$\begin{aligned} F(x, y) = {} & x(x+2^7.3.5^3)^3 + y(y+2^7.3.5^3)^3 \\ & - 2^{16}x^3y^3 + 2^{11}.3^2.31\,x^2y^2(x+y) \\ & - 2^2.3^3.9907\,xy(x^2+y^2) + 2.3^4.13.193.6367\,x^2y^2 \\ & + 2^8.3^5.5^3.4471\,xy(x+y) - 2^{15}.5^6.22973\,xy = 0. \end{aligned}$$

The following is the process by which the coefficients were determined:

It follows from the general theory of modular equations that $F(x, y)$ is symmetrical with respect to x and y; and that it is of the order 4 in x and y separately, and of the order 6 in x and y jointly. Hence $F(x, y)$ is of the form

$$\begin{aligned} & A_{3,3}x^3y^3 + A_{4,2}x^2y^2(x^2+y^2) + A_{4,1}xy(x^3+y^3) \\ & \quad + (x^4+y^4) + A_{3,2}x^2y^2(x+y) + A_{3,1}xy(x^2+y^2) \\ & \quad + A_{3,0}(x^3+y^3) + A_{2,2}x^2y^2 + A_{2,1}xy(x+y) \\ & \quad + A_{2,0}(x^2+y^2) + A_{1,1}xy + A_{1,0}(x+y) + A_{0,0}. \end{aligned}$$

Several of the coefficients may be conveniently found by employing the method of Sohncke; *i.e.* by substituting for x and y their expressions as series

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* 'Proceedings of the London Mathematical Society,' vol. ix. p. 243, note [vol. ii. p. 242].

proceeding by powers of q, and equating the coefficients of the powers of q to zero. We have, by a known formula,

$$k^2(1-k^2) = 2^4 q \prod_{m=1}^{m=\infty} (1+q^{2m-1})^{-24};$$

calling this quantity z, we have $x = \dfrac{(1-z)^3}{z^2}$;

we also write $$Z = 2^4 q^3 \prod_{m=1}^{m=\infty} (1+q^{6m-3})^{-24},$$

so that $$y = \frac{(1-Z)^3}{Z^2}.$$

Substituting these values, and equating to zero the coefficients of q^{-28}, q^{-26}, and q^{-24}, we find successively

$$A_{4,2} = 0, \quad A_{4,1} = 0, \quad A_{3,3} = -2^{16}.$$

To determine $A_{3,2}$, we observe that q^{-22} presents itself only in the terms

$$-2^{16} x^3 y^3 + A_{3,2} x^2 y^3.$$

Its coefficient in x^2y^3 is 2^{-40}; its coefficient in x^3y^3 is $3^2 . 31 \times 2^{-45}$; viz. this is the coefficient of q^2 in

$$\frac{1}{2^{24}}\left[\frac{1}{2^{24}}(1+q)^{144} - \frac{9}{2^{20}} q\,(1+q)^{120} + \frac{9}{2^{14}} q^2\right].$$

Hence $$A_{3,2} \times 2^{-40} + 2^{-45} . 3^2 . 31 . A_{3,3} = 0,$$

or $$A_{3,2} = 2^{11} . 3^2 . 31.$$

Again, to determine $A_{3,1}$ we consider the coefficient of q^{-20}; this power of q presents itself only in the terms

$$y^3 [A_{3,3} x^3 + A_{3,2} x^2 + A_{3,1} x].$$

It will be found that in the development of y^3 there is no power of q intermediate between q^{-18} and q^{-12}; hence, the coefficient of q^{-2} in

$$A_{3,3} x^3 + A_{3,2} x^2 + A_{3,1} x,$$

or the coefficient of q^4 in

$$A_{3,3}\left[\frac{1}{2^4}(1+q)^{144}(1+q^3)^{144} - \frac{3^2}{2^{20}} q\,(1+q)^{120}(1+q^3)^{120}\right.$$
$$\left. + \frac{3^2}{2^{14}} q^2 (1+q)^{196} - \frac{3 . 7}{2^{10}} q^3 (1+q)^{72} - \frac{3^2 . 7}{2^7} q^4\right]$$
$$+ A_{3,2}\left[\frac{1}{2^{16}} q^2 (1+q)^{96} - \frac{3}{2^{11}} q^3 (1+q)^{72} + \frac{3 . 5}{2^8} q^4\right] + \frac{A_{3,1}}{2^8} q^4,$$

is equal to zero. Substituting the values of $A_{3,3}$, $A_{3,2}$, and reducing, we find

$$A_{3,1} = -2^2 . 3^3 . 9907.$$

All the coefficients may successively be determined by this method, but the work becomes very laborious for the later coefficients. They may be more easily obtained by the consideration that, if ξ, η are any two corresponding values of k^2 and λ^2, $f(\xi)$ and $f(\eta)$ are corresponding values of x and y. Availing ourselves of this principle, we find

$$\text{(i.)} \quad F(0, y) = y(y + 2^7 . 3 . 5^3)^3 ;$$

$$\text{(ii.)} \quad F\left(\tfrac{27}{4}, y\right) = \left(y^2 - \frac{3^2}{2} \times 133283 y - \frac{3^3 . 11^3 . 23^3}{2^4}\right)^2 ;$$

$$\text{(iii.)} \quad F(y, y) = -2^{16} y \left(y - \frac{3^3 . 5^3}{16}\right)\left(y - \frac{5^3}{4}\right)^2 (y + 2^7)^2 .$$

From (1) we infer that $F(x, y)$ is of the form

$$\left.\begin{aligned} &y(y + 2^7 . 3 . 5^3)^3 + x(x + 2^7 . 3 . 5^3)^3 - 2^{16} x^3 y^3 \\ &+ 2^{11} . 3^2 . 31 x^2 y^2 (x + y) - 2^2 . 3^3 . 9907 xy (x^2 + y^2) \\ &+ A_{2,2} x^2 y^2 + A_{2,1} xy (x + y) + A_{1,1} xy = 0. \end{aligned}\right\} \quad \ldots\ldots \quad \text{(A)}$$

From (iii.) we find

$$\begin{aligned} A_{1,1} &= -2^{15} . 5^6 . 22973, \\ A_{2,1} &= 2^8 . 3^5 . 5^3 . 4471, \\ A_{2,2} &= 2^{11} . 1262587 + 2^8 . 3^3 . 9907 - 2 \\ &= 2 . 3^4 . 15974803 \\ &= 2 . 3^4 . 13 . 193 . 6367. \end{aligned}$$

This completes the determination of the coefficients; a verification is supplied by the equation (ii.); and it only remains to show how the equations (i.), (ii.), (iii.) are inferred from the modular equation between $\xi = k^2$ and $\eta = \lambda^2$; viz.:

$$\Phi(\xi, \eta) = (\xi^2 + 6\xi\eta + \eta^2)^2 - 16\xi\eta(4\xi\eta - 3\xi - 3\eta + 4)^2 = 0.$$

(i.) Let ρ denote either root of the equation $\rho^2 - \rho + 1 = 0$; if $k^2 = \rho$, we have $x = f(\rho) = 0$; we also find

$$\Phi(\rho, \eta) = (\eta - \rho)(\eta^3 - [128 - 253\rho]\eta^2 - [128 + 253\rho^2]\eta + 1).$$

If ω_1, ω_2, ω_3 are the roots of the cubic factor, the four roots of $F(0, y)$ are

$$f(\rho) = 0, \quad f(\omega_1), \quad f(\omega_2), \quad f(\omega_3).$$

But the cubic factor is one of the two conjugate factors of the expression

$$(\eta^2 - \eta + 1)^3 + 2^7 . 3 . 5^3 . \eta^2 (\eta - 1)^2.$$

Hence $$f(\omega_1) = f(\omega_2) = f(\omega_3) = -2^7 . 3 . 5^3,$$

and $$F(0, y) = y(y + 2^7 . 3 . 5^3)^3.$$

(ii.) Let $k^2=\frac{1}{2}$, so that $f(k^2)=\frac{27}{4}$; we find

$$\Phi\left(\tfrac{1}{2}, y\right)=(\eta-\eta^2)^2-48\tfrac{1}{2}\,(\eta-\eta^2)+\tfrac{1}{16}.$$

If $\eta_1, \eta_2, \eta_3, \eta_4$ are the roots of $\Phi\left(\frac{1}{2}, \eta\right)$, the four roots of $F\left(\frac{27}{4}, y\right)$ are $f(\eta_1)$, $f(\eta_2)$, $f(\eta_3)$, $f(\eta_4)$; *i.e.* if z_1, z_2 are the roots of $z^2-48\frac{1}{2}\,z+\frac{1}{16}$, these four roots are $z_1^{-2}(1-z_1)^3$, $z_2^{-2}(1-z_2)^3$, each taken twice. Hence

$$F\left(\tfrac{27}{4}, y\right)=[y-z_1^{-2}(1-z_1)^3]^2\,[y-z_2^{-2}(1-z_2)^3]^2;$$

and by the ordinary methods we find

$$[y-z_1^{-2}(1-z_1)^3]\,[y-z_2^{-2}(1-z_2)^3]^2=y^2-\frac{3^2}{2}\times 133283\,y-\frac{3^3\,.\,11^3\,.\,23^3}{2^4}.$$

(iii.) Since the only roots of the equation

$$f(\eta)=f(\xi)$$

are

$$\xi,\ 1-\xi,\ \frac{1}{\xi},\ \frac{1}{1-\xi},\ \frac{\xi-1}{\xi},\ \frac{\xi}{\xi-1},$$

the roots of the equation $F(y, y)=0$ are all of the form $f(\theta)$, where θ is a root of one of the six equations:

$$\Phi(\theta, \theta)=\theta^2(\theta-1)^2(\theta^2-\theta+1)=0, \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\Phi(\theta, 1-\theta)=(16\theta^2-16\theta+1)(4\theta^2-4\theta-1)^2=0, \quad \ldots\ldots\ldots \quad (2)$$

$$\theta^4\Phi\left(\theta, \frac{1}{\theta}\right)=(\theta-1)^2(\theta^2-6\theta+1)^2(\theta^2+14\theta+1)=0, \quad \ldots\ldots\ldots \quad (3)$$

$$(1-\theta)^4\Phi\left(\theta, \frac{1}{1-\theta}\right)=(\theta^2-\theta+1)\times[(1-\theta+\theta^2)^3+128\theta^2(1-\theta)^2]=0; \quad \ldots \quad (4)$$

$$\theta^4\Phi\left(\theta, \frac{\theta-1}{\theta}\right)=(\theta^2-\theta+1)\times[(\theta^2-\theta+1)^3+128\theta^2(\theta-1)^2]=0, \quad \ldots\ldots \quad (5)$$

$$(\theta-1)^4\Phi\left(\theta, \frac{\theta}{\theta-1}\right)=\theta^2(\theta^2-16\theta+16)(\theta^2+4\theta-4)^2=0. \quad \ldots\ldots \quad (6)$$

These equations are of order 8, the first and second having each two infinite roots. The 48 roots give, in all, five distinct values for $f(\theta)$ according to the following scheme:

A. $f(\theta)=\infty$; 12 roots, viz.:

$\theta=\infty$, 2 roots in (1) and in (2);
$\theta=0$, 2 roots in (1) and in (6);
$\theta=1$, 2 roots in (1) and in (3).

B. $f(\theta)=0$; 6 roots, viz.:

$$\theta^2-\theta+1=0, \text{ in (1), (4), and (5).}$$

C. $f(\theta)=\frac{3^3.5^3}{2^4}$; 6 roots, viz.:

$$16\theta^2-16\theta+1=0, \text{ in (2);}$$
$$\theta^2+14\theta+1=0, \text{ in (3);}$$
$$\theta^2-16\theta+16=0, \text{ in (6).}$$

D. $f(\theta)=-128$; 12 roots, viz.:

$$(\theta^2-\theta+1)^3+128\theta^2(\theta-1)^2=0, \text{ in (4) and (5).}$$

E. $f(\theta)=\frac{5^3}{2^2}$; 12 roots, viz.:

$$(4\theta^2-4\theta-1)^2=0, \text{ in (2);}$$
$$(\theta^2-6\theta+1)^2=0, \text{ in (3);}$$
$$(\theta^2+4\theta-4)^2=0, \text{ in (6).}$$

Hence the roots of $F(y,y)=0$ are $0, \frac{3^3.5^3}{2^4}$, each once; $\infty, -2^7, \frac{5^3}{2^2}$, each twice; *i.e.*:

$$F(y,y)=-2^{16}y\left(y-\frac{3^3.5^3}{2^4}\right)\left(y-\frac{5^3}{2^2}\right)^2(y+2^7)^2.$$

The multiplicity of the roots of $F(y,y)=0$ may be otherwise, and more simply, determined as follows. The form of $F(x,y)$ (see the equation (A), *supra*) shows that $F(y,y)$ has one root equal to zero and two roots equal to infinity); the multiplicities of the finite roots are determined by the equation

$$2^{16}\left[\frac{3^3.5^3}{2^4}+2\times\frac{5^3}{2^2}-2\times 2^7\right]=2A_{3,2}=2^{12}.3^2.31.$$

XXXVIII.

NOTE ON THE FORMULA FOR THE MULTIPLICATION OF FOUR THETA FUNCTIONS.

[Proceedings of the London Mathematical Society, vol. x. pp. 91–100.* Read February 13, 1879.]

THE normal formula for the multiplication of four Theta Functions is ('Proceedings of the London Mathematical Society,' vol. I., part viii., p. 4 †)

$$\left.\begin{aligned} &2\,\theta_{\mu_1,\,\mu_1'}(x_1)\,\theta_{\mu_2,\,\mu_2'}(x_2)\,\theta_{\mu_3,\,\mu_3'}(x_3)\,\theta_{\mu_4,\,\mu_4'}(x_4)\\ &\quad=\Pi_{j=1}^{j=4}\theta_{\sigma-\mu_j,\,\sigma'-\mu_j'}(s-x_j)+\Pi_{j=1}^{j=4}\theta_{\sigma-\mu_j,\,\sigma'-\mu_j'+1}(s-x_j)\\ &\quad+(-1)^{\sigma'}\Pi_{j=1}^{j=4}\theta_{\sigma-\mu_j+1,\,\sigma'-\mu_j'}(s-x_j)+(-1)^{\sigma'+1}\Pi_{j=1}^{j=4}\theta_{\sigma-\mu_j+1,\,\sigma'-\mu_j'+1}(s-x_j). \end{aligned}\right\} \quad (1)$$

In this formula we have

$$\theta_{\mu,\mu'}(x)=\Sigma_{m=-\infty}^{m=+\infty}(-1)^{m\mu'}q^{\frac{1}{4}(2m+\mu)^2}e^{i(2m+\mu)x}=\Sigma_{m=-\infty}^{m=+\infty}(-1)^{m\mu'}e^{\frac{1}{4}i\pi\omega(2m+\mu)^2}e^{i(2m+\mu)x}; \quad (2)$$

$$\left.\begin{aligned} 2s&=x_1+x_2+x_3+x_4,\\ 2\sigma&=\mu_1+\mu_2+\mu_3+\mu_4,\\ 2\sigma'&=\mu_1'+\mu_2'+\mu_3'+\mu_4'; \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3)$$

$q=e^{i\pi\omega}$ being a constant of which the analytical modulus is inferior to unity, and the indices μ, μ' being integral numbers, which render the sums 2σ, $2\sigma'$ even.

I. *The Eleven Cases of the Formula.*

Since

$$\left.\begin{aligned} \theta_{\mu+2,\,\mu'}(x)&=(-1)^{\mu'}\theta_{\mu,\,\mu'}(x),\\ \theta_{\mu,\,\mu'+2}(x)&=\theta_{\mu,\,\mu'}(x), \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4)$$

we need only attribute the values 1, 0 to the indices μ, μ'. We may also

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* [This paper is referred to on p. 75 of vol. x., and appears in the Contents of the volume under the title, 'On the formula for four Abelian functions answering to the formula for four Theta functions.']

† [Vol. i. p. 443.]

permute the four arguments x_1, x_2, x_3, x_4 in any way we please; thus the matrix $\left|\begin{matrix}\mu_1, \mu_2, \mu_3, \mu_4 \\ \mu_1', \mu_2', \mu_3', \mu_4'\end{matrix}\right|$ may have eleven different values, which are enumerated in the following table:

TABLE I.

A. $\sigma \equiv 0$, $\sigma' \equiv 0$, mod 2: Cases i.—iv.

$$\left|\begin{matrix}1&1&1&1\\1&1&1&1\end{matrix}\right|, \quad \left|\begin{matrix}1&1&1&1\\0&0&0&0\end{matrix}\right|, \quad \left|\begin{matrix}0&0&0&0\\1&1&1&1\end{matrix}\right|, \quad \left|\begin{matrix}0&0&0&0\\0&0&0&0\end{matrix}\right|.$$

B. $\sigma \equiv 0$, $\sigma' \equiv 1$, mod 2: Cases v., vi.

$$\left|\begin{matrix}1&1&1&1\\1&1&0&0\end{matrix}\right|, \quad \left|\begin{matrix}0&0&0&0\\1&1&0&0\end{matrix}\right|.$$

C. $\sigma \equiv 1$, $\sigma' \equiv 0$, mod 2: Cases vii., viii.

$$\left|\begin{matrix}1&1&0&0\\1&1&1&1\end{matrix}\right|, \quad \left|\begin{matrix}1&1&0&0\\0&0&0&0\end{matrix}\right|.$$

D. $\sigma \equiv 1$, $\sigma' \equiv 1$, mod 2: Cases ix.—xi.

$$\left|\begin{matrix}1&1&0&0\\1&1&0&0\end{matrix}\right|, \quad \left|\begin{matrix}1&1&0&0\\1&0&1&0\end{matrix}\right|, \quad \left|\begin{matrix}1&1&0&0\\0&0&1&1\end{matrix}\right|.$$

The formulæ appertaining to these eleven cases are all different from one another; *i.e.*, none of them can be derived from any other by a permutation of the four arguments x; we can, however, pass from any one of them to any other by means of the formula which connects any two different Theta functions; viz.,

$$\theta_{\mu+\nu, \mu'+\nu'}(x) = e^{i\pi\left[\nu\frac{x}{\pi}+\nu^2\frac{\omega}{4}-\frac{\mu\nu}{2}\right]} \times \theta_{\mu, \mu'}\left[x+\frac{\pi}{2}(\nu\omega+\nu')\right].$$

Using a notation with a single suffix, and writing

$$\theta_{0,1}(x) = \vartheta_0(x), \quad \theta_{1,1}(x) = \vartheta_1(x),$$
$$\theta_{1,0}(x) = \vartheta_2(x), \quad \theta_{0,0}(x) = \vartheta_3(x),$$

we may exhibit the eleven formulæ as follows.

TABLE II. *Formulæ for the Multiplication of Four Theta Functions.*

A.

(i.) $$2\vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1 = \vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1+\vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2 + \vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0-\vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3.$$

(ii.) $$2\vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2 = \vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2+\vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1 + \vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3-\vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0.$$

$$\text{(iii.)}\quad 2\vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0 = \vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0+\vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3 +\vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1-\vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2.$$

$$\text{(iv.)}\quad 2\vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3 = \vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3+\vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0 +\vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2-\vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1.$$

B.

$$\text{(v.)}\quad 2\vartheta_1\times\vartheta_1\times\vartheta_2\times\vartheta_2 = \vartheta_2\times\vartheta_2\times\vartheta_1\times\vartheta_1+\vartheta_1\times\vartheta_1\times\vartheta_2\times\vartheta_2 -\vartheta_3\times\vartheta_3\times\vartheta_0\times\vartheta_0+\vartheta_0\times\vartheta_0\times\vartheta_3\times\vartheta_3.$$

$$\text{(vi.)}\quad 2\vartheta_0\times\vartheta_0\times\vartheta_3\times\vartheta_3 = \vartheta_3\times\vartheta_3\times\vartheta_0\times\vartheta_0+\vartheta_0\times\vartheta_0\times\vartheta_3\times\vartheta_3 -\vartheta_2\times\vartheta_2\times\vartheta_1\times\vartheta_1+\vartheta_1\times\vartheta_1\times\vartheta_2\times\vartheta_2.$$

C.

$$\text{(vii.)}\quad 2\vartheta_1\times\vartheta_1\times\vartheta_0\times\vartheta_0 = \vartheta_0\times\vartheta_0\times\vartheta_1\times\vartheta_1+\vartheta_3\times\vartheta_3\times\vartheta_2\times\vartheta_2 +\vartheta_1\times\vartheta_1\times\vartheta_0\times\vartheta_0-\vartheta_2\times\vartheta_2\times\vartheta_3\times\vartheta_3.$$

$$\text{(viii.)}\quad 2\vartheta_2\times\vartheta_2\times\vartheta_3\times\vartheta_3 = \vartheta_3\times\vartheta_3\times\vartheta_2\times\vartheta_2+\vartheta_0\times\vartheta_0\times\vartheta_1\times\vartheta_1 +\vartheta_2\times\vartheta_2\times\vartheta_3\times\vartheta_3-\vartheta_1\times\vartheta_1\times\vartheta_0\times\vartheta_0.$$

D.

$$\text{(ix.)}\quad 2\vartheta_1\times\vartheta_1\times\vartheta_3\times\vartheta_3 = \vartheta_3\times\vartheta_3\times\vartheta_1\times\vartheta_1+\vartheta_0\times\vartheta_0\times\vartheta_2\times\vartheta_2 -\vartheta_2\times\vartheta_2\times\vartheta_0\times\vartheta_0+\vartheta_1\times\vartheta_1\times\vartheta_3\times\vartheta_3.$$

$$\text{(x.)}\quad 2\vartheta_1\times\vartheta_2\times\vartheta_0\times\vartheta_3 = \vartheta_3\times\vartheta_0\times\vartheta_2\times\vartheta_1+\vartheta_0\times\vartheta_3\times\vartheta_1\times\vartheta_2 +\vartheta_2\times\vartheta_1\times\vartheta_3\times\vartheta_0-\vartheta_1\times\vartheta_2\times\vartheta_0\times\vartheta_3.$$

$$\text{(xi.)}\quad 2\vartheta_2\times\vartheta_2\times\vartheta_0\times\vartheta_0 = \vartheta_0\times\vartheta_0\times\vartheta_2\times\vartheta_2+\vartheta_3\times\vartheta_3\times\vartheta_1\times\vartheta_1 -\vartheta_1\times\vartheta_1\times\vartheta_3\times\vartheta_3+\vartheta_2\times\vartheta_2\times\vartheta_0\times\vartheta_0.$$

For brevity, the arguments x_1, x_2, x_3, x_4, and the arguments $s-x_1, s-x_2, s-x_3, s-x_4$ are omitted in the left and right-hand members respectively.

It will be observed that of the 256 sets of values which may be attributed to the constituents of the matrix

$$\begin{vmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & \nu_2 & \nu_4 \end{vmatrix}$$

192 are excluded by the condition that 2σ and $2\sigma'$ are even; of the remaining 64, 4 are represented by the formulæ (i.)—(iv.), which are symmetrical with respect to all the arguments; 24 by the formula (x.), which is entirely unsymmetrical; and 6 by each of the formulæ (v.)—(ix.) and (xi.), which are symmetrical with respect to the arguments taken in pairs.

II. *Application of the Formula to the Abelian Functions.*

The Abelian functions are defined by the equations

$$\left.\begin{aligned} \mathrm{Al}_1(x) &= e^{-\zeta\left(\frac{\pi x}{2K}\right)^2} \times \frac{\vartheta_1\left(\frac{\pi x}{2K}\right)}{\frac{\pi}{2K}\vartheta_1'(0)}, \\ \mathrm{Al}_2(x) &= e^{-\zeta\left(\frac{\pi x}{2K}\right)^2} \times \frac{\vartheta_2\left(\frac{\pi x}{2K}\right)}{\vartheta_2(0)}, \\ \mathrm{Al}_0(x) &= e^{-\zeta\left(\frac{\pi x}{2K}\right)^2} \times \frac{\vartheta_0\left(\frac{\pi x}{2K}\right)}{\vartheta_0(0)}, \\ \mathrm{Al}_3(x) &= e^{-\zeta\left(\frac{\pi x}{2K}\right)^2} \times \frac{\vartheta_3\left(\frac{\pi x}{2K}\right)}{\vartheta_3(0)}; \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6)$$

the constants K and ζ being determined by the equations

$$\sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \ldots, \quad \ldots\ldots\ldots \quad (7)$$

$$\tfrac{1}{4}\zeta = \frac{q - 4q^4 + 9q^9 - 16q^{16} + \ldots}{1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \ldots} \quad \ldots\ldots\ldots \quad (8)$$

The formulæ of Table II. give rise to a corresponding system of formulæ for the multiplication of four Abelian functions. To obtain this second system, we have only to express the Theta functions as Abelian functions, and to attend to the equations

$$k^2 = \frac{\vartheta_2^4(0)}{\vartheta_3^4(0)}, \qquad k'^2 = \frac{\vartheta_0^4(0)}{\vartheta_3^4(0)}, \qquad \frac{\pi}{2K}\vartheta_{1,1}'(0) = i\frac{\vartheta_0(0)\,\vartheta_2(0)}{\vartheta_3(0)} \quad \ldots\ldots \quad (9)$$

TABLE III. *Formulæ for the Multiplication of four Abelian Functions.*

A.

$$\begin{aligned} (\mathrm{i.}) \quad 2\mathrm{Al}_1 \times \mathrm{Al}_1 \times \mathrm{Al}_1 \times \mathrm{Al}_1 = {} & \mathrm{Al}_1 \times \mathrm{Al}_1 \times \mathrm{Al}_1 \times \mathrm{Al}_1 + \frac{1}{k'^2}\,\mathrm{Al}_2 \times \mathrm{Al}_2 \times \mathrm{Al}_2 \times \mathrm{Al}_2 \\ & + \frac{1}{k^2}\,\mathrm{Al}_0 \times \mathrm{Al}_0 \times \mathrm{Al}_0 \times \mathrm{Al}_0 - \frac{1}{k^2 k'^2}\mathrm{Al}_3 \times \mathrm{Al}_3 \times \mathrm{Al}_3 \times \mathrm{Al}_3. \end{aligned}$$

$$\begin{aligned} (\mathrm{ii.}) \quad 2\mathrm{Al}_2 \times \mathrm{Al}_2 \times \mathrm{Al}_2 \times \mathrm{Al}_2 = {} & \mathrm{Al}_2 \times \mathrm{Al}_2 \times \mathrm{Al}_2 \times \mathrm{Al}_2 + k'^2\,\mathrm{Al}_1 \times \mathrm{Al}_1 \times \mathrm{Al}_1 \times \mathrm{Al}_1 \\ & + \frac{1}{k^2}\,\mathrm{Al}_3 \times \mathrm{Al}_3 \times \mathrm{Al}_3 \times \mathrm{A}_3 - \frac{k'^2}{k^2}\,\mathrm{Al}_0 \times \mathrm{Al}_0 \times \mathrm{Al}_0 \times \mathrm{Al}_0. \end{aligned}$$

$$\text{(iii.)}\quad 2\mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_1 = \mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_0 + \frac{1}{k'^2}\,\mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_3$$
$$+ k^2\mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_1 - \frac{k^2}{k'^2}\,\mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_2.$$

$$\text{(iv.)}\quad 2\mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_3 = \mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_3 + k'^2\,\mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_0$$
$$+ k^2\mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_2 - k^2k'^2\,\mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_1.$$

B.

$$\text{(v.)}\quad 2\mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_2\times\mathrm{Al}_2 = \mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_1\times\mathrm{Al}_1 + \mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_2\times\mathrm{Al}_2$$
$$+ \frac{1}{k^2}\,\mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_0\times\mathrm{Al}_0 - \frac{1}{k^2}\,\mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_3\times\mathrm{Al}_3.$$

$$\text{(vi.)}\quad 2\mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_3\times\mathrm{Al}_3 = \mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_0\times\mathrm{Al}_0 + \mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_3\times\mathrm{Al}_3$$
$$+ k^2\,\mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_1\times\mathrm{Al}_1 - k^2\,\mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_2\times\mathrm{Al}_2.$$

C.

$$\text{(vii.)}\quad 2\mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_0\times\mathrm{Al}_0 = \mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_1\times\mathrm{Al}_1 - \frac{1}{k'^2}\,\mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_2\times\mathrm{Al}_2$$
$$+ \mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_0\times\mathrm{Al}_0 + \frac{1}{k'^2}\,\mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_3\times\mathrm{Al}_3.$$

$$\text{(viii.)}\quad 2\mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_3\times\mathrm{Al}_3 = \mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_2\times\mathrm{Al}_2 - k'^2\,\mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_1\times\mathrm{Al}_1$$
$$+ \mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_3\times\mathrm{Al}_3 + k'^2\,\mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_0\times\mathrm{Al}_0.$$

D.

$$\text{(ix.)}\quad 2\mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_3\times\mathrm{Al}_3 = \mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_1\times\mathrm{Al}_1 - \mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_2\times\mathrm{Al}_2$$
$$+ \mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_0\times\mathrm{Al}_0 + \mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_3\times\mathrm{Al}_3.$$

$$\text{(x.)}\quad 2\mathrm{Al}_1\times\mathrm{Al}_2\times\mathrm{Al}_0\times\mathrm{Al}_3 = \mathrm{Al}_3\times\mathrm{Al}_0\times\mathrm{Al}_2\times\mathrm{Al}_1 + \mathrm{Al}_0\times\mathrm{Al}_3\times\mathrm{Al}_1\times\mathrm{Al}_2$$
$$+ \mathrm{Al}_2\times\mathrm{Al}_1\times\mathrm{Al}_3\times\mathrm{Al}_0 - \mathrm{Al}_1\times\mathrm{Al}_2\times\mathrm{Al}_0\times\mathrm{Al}_3.$$

$$\text{(xi.)}\quad 2\mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_0\times\mathrm{Al}_0 = \mathrm{Al}_0\times\mathrm{Al}_0\times\mathrm{Al}_2\times\mathrm{Al}_2 - \mathrm{Al}_3\times\mathrm{Al}_3\times\mathrm{Al}_1\times\mathrm{Al}_1$$
$$+ \mathrm{Al}_1\times\mathrm{Al}_1\times\mathrm{Al}_3\times\mathrm{Al}_3 + \mathrm{Al}_2\times\mathrm{Al}_2\times\mathrm{Al}_0\times\mathrm{Al}_0.$$

III. *Case when the Sum of the Four Arguments is zero.*

Putting $$s = x_1 + x_2 + x_3 + x_4 = 0,$$
and attending to the equation
$$\theta_{\mu,\mu'}(-x) = (-1)^{\mu\mu'}\theta_{\mu,\mu'}(x), \quad \ldots\ldots\ldots \quad (10$$
we find that in each of the formulæ (i.)—(xi.), one of the terms on the right-hand side cancels one of the two equal terms on the left, and that the four formulæ A., and the formulæ of the three pairs B., C., D., (ix.) and (xi.), become respectively coincident; the formula D. (x.) remains *sui generis*. Introducing the elliptic functions
$$\mathrm{sn}\,x = \frac{\mathrm{Al}_1(x)}{\mathrm{Al}(x)}, \quad \mathrm{cn}\,x = \frac{\mathrm{Al}_2(x)}{\mathrm{Al}(x)}, \quad \mathrm{dn}\,x = \frac{\mathrm{Al}_3(x)}{\mathrm{Al}(x)}$$

into the five formulæ thus obtained, we arrive at the following system :

$$\text{(i.)}\quad k^2k'^2\,\Pi\,.\,\mathrm{sn}\,x - k^2\,\Pi\,.\,\mathrm{cn}\,x + \Pi\,.\,\mathrm{dn}\,x - k'^2 = 0,$$

$$\text{(ii.)}\quad k^2\,\mathrm{sn}\,x_1\,\mathrm{sn}\,x_2\,\mathrm{cn}\,x_3\,\mathrm{cn}\,x_4 - k^2\,\mathrm{cn}\,x_1\,\mathrm{cn}\,x_2\,\mathrm{sn}\,x_3\,\mathrm{sn}\,x_4 - \mathrm{dn}\,x_1\,\mathrm{dn}\,x_2 + \mathrm{dn}\,x_3\,\mathrm{dn}\,x_4 = 0,$$

$$\text{(iii.)}\quad k'^2\,\mathrm{sn}\,x_1\,\mathrm{sn}\,x_2 - k'^2\,\mathrm{sn}\,x_3\,\mathrm{sn}\,x_4 + \mathrm{dn}\,x_1\,\mathrm{dn}\,x_2\,\mathrm{cn}\,x_3\,\mathrm{cn}\,x_4 - \mathrm{cn}\,x_1\,\mathrm{cn}\,x_2\,\mathrm{dn}\,x_3\,\mathrm{dn}\,x_4 = 0,$$

$$\text{(iv.)}\quad \mathrm{sn}\,x_1\,\mathrm{sn}\,x_2\,\mathrm{dn}\,x_3\,\mathrm{dn}\,x_4 - \mathrm{dn}\,x_1\,\mathrm{dn}\,x_2\,\mathrm{sn}\,x_3\,\mathrm{sn}\,x_4 + \mathrm{cn}\,x_3\,\mathrm{cn}\,x_4 - \mathrm{cn}\,x_1\,\mathrm{cn}\,x_2 = 0,$$

$$\text{(v.)}\quad \mathrm{sn}\,x_1\,\mathrm{cn}\,x_2\,\mathrm{dn}\,x_4 + \mathrm{dn}\,x_1\,\mathrm{cn}\,x_3\,\mathrm{sn}\,x_4 + \mathrm{dn}\,x_2\,\mathrm{sn}\,x_3\,\mathrm{cn}\,x_4 + \mathrm{cn}\,x_1\,\mathrm{sn}\,x_2\,\mathrm{dn}\,x_3 = 0.$$

The first of these, which alone is symmetrical with respect to the four arguments, is the formula given by Professor Cayley ('Proceedings,' p. 43). The formulæ (ii.), (iii.), (iv.) are symmetrical with respect to the two pairs x_1x_2, x_3x_4, and with respect to the two arguments of each pair ; thus each of these formulæ represents a set of three. Lastly, the formula (v.) remains unchanged when any two of the arguments are interchanged, provided that the other two are interchanged at the same time ; *i.e.* there are six formulæ of this type.

As a verification of these formulæ in a particular case, let $x_4 = 0$; we find

$$\left.\begin{aligned} &Dd_1d_2 - k^2Cc_1c_2 = k'^2,\\ &D + k^2Cs_1s_2 = d_1d_2.\\ &Dc_1c_2 - Cd_1d_2 = k'^2s_1s_2,\\ &C + Ds_1s_2 = c_1c_2,\\ &Dc_1s_2 - Sd_2 = -s_1c_2, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (11)$$

(where we have written, for brevity, $s_1 = \mathrm{sn}\,x_1$, $c_1 = \mathrm{cn}\,x_1$, $d_1 = \mathrm{dn}\,x_1$; $s_2c_2d_2$ and SCD having similar meanings with respect to the arguments x_2 and $x_1 + x_2 = -x_3$) ; and these equations are easily shown to be true by means of the formulæ for the addition of elliptic functions.

IV. *Formula for the Multiplication of Four Multiple Theta Functions.*

We define a double Theta function by the equation

$$\left.\begin{aligned} \theta(\mu, \mu' ; \nu, \nu' ; x, y) &= \Sigma_{m=-\infty}^{m=+\infty}\,\Sigma_{n=-\infty}^{n=+\infty}(-1)^{m\mu' + n\nu}\\ &\quad \times A^{\frac{1}{4}(2m+\mu)^2} \times B^{\frac{1}{2}(2m+\mu)(2n+\nu)} \times C^{\frac{1}{4}(2n+\nu)^2} \times e^{(2m+\mu)ix + (2n+\nu)iy}\\ &= \Sigma_{m=-\infty}^{m=+\infty}\,\Sigma_{n=-\infty}^{n=+\infty}(-1)^{m\mu' + n\nu'} e^{\frac{i\pi}{4}\phi(2m+\mu,\, 2n+\nu)} \times e^{i(2m+\mu)x + i(2n+\nu)y} \end{aligned}\right\}, \quad (12)$$

where ϕ is the quadratic form (a, b, c) ; $i\pi a$, $i\pi b$, $i\pi c$ being the hyperbolic

logarithms of A, B, C; and where the condition of convergence is that the real part of $i\phi$ must be a negative form of negative determinant.

Considering four different Theta functions, and writing

$$\left.\begin{aligned} 2s &= x_1+x_2+x_3+x_4, \quad 2t = y_1+y_2+y_3+y_4, \\ 2\sigma &= \mu_1+\mu_2+\mu_3+\mu_4, \quad 2\tau = \nu_1+\nu_2+\nu_3+\nu_4, \\ 2\sigma' &= \mu_1'+\mu_2'+\mu_3'+\mu_4', \quad 2\tau' = \nu_1'+\nu_2'+\nu_3'+\nu_4', \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

we have the formula

$$\left.\begin{aligned} 4\Pi\,.\,\theta(\mu_j, \mu_j'; \nu_j, \nu_j'; x_j, y_j) &= \Sigma(-1)^{\alpha\sigma'+\alpha\alpha'+\beta\tau'+\beta\beta'} \\ &\times \Pi\,.\,\theta(\sigma-\mu_j+\alpha, \sigma'-\mu_j'+\alpha'; \tau-\nu_j+\beta, \tau'-\nu_j'+\beta'; s-x_j, t-y_j); \end{aligned}\right\} \quad (14)$$

the signs of multiplication Π referring to the four values 1, 2, 3, 4 of the index j, and the sign of summation Σ in the right-hand member referring to the symbols α, α', β, β', each of which is to have the values 0, 1; so that the right-hand member contains sixteen terms. The formula may be demonstrated in the same manner as the corresponding formula for the single Theta functions (see the note already cited, 'Proceedings,' vol. I., part viii., Art. 2).*

If, instead of a double Theta function, we consider a multiple Theta function containing λ arguments $x, y, \ldots$, and depending on λ pairs of indices $\mu\mu', \nu\nu', \rho\rho', \ldots$, we have the equation of definition

$$\left.\begin{aligned} &\theta(|\mu, \mu'|; |x|) = \Sigma_{m=-\infty}^{m=+\infty} \Sigma_{n=-\infty}^{n=+\infty} \Sigma_{r=-\infty}^{r=+\infty} \cdots \\ &(-1)^{m\mu'+n\nu'+r\rho'+\cdots} \times e^{\frac{i\pi}{4}\phi(2m+\mu,\, 2n+\nu,\, 2r+\rho, \ldots)} \times \epsilon^{i(2m+\mu)x+i(2n+\nu)y+i(2r+\rho)z+\cdots} \end{aligned}\right\}, \quad (15)$$

where ϕ is a quadratic form, containing λ indeterminates, and such that the real part of $i\phi$ is definite and negative; and we obtain a formula similar to (14); viz. taking four Theta functions, such that the sums of the homologous indices are all even, we have

$$2^\lambda\,\Pi\,.\,\theta(|\mu_j, \mu_j'|; |x_j|) = \Sigma(-1)^{|\alpha\sigma'+\alpha\alpha'|} \times \Pi\,.\,\theta(|\sigma-\mu_j+\alpha, \sigma'-\mu_j'+\alpha'|; |s-x_j|, \quad (16)$$

where the symbols $\quad |\mu_j, \mu_j'|, \quad |\sigma-\mu_j+\alpha, \sigma'-\mu_j'+\alpha'|$

are placed by abbreviation for the λ pairs

$$\mu_j, \mu_j'; \quad \nu_j, \nu_j'; \quad \ldots$$

and $\qquad \sigma-\mu_j+\alpha, \sigma'-\mu_j'+\alpha'; \quad \tau-\nu_j+\beta, \tau'-\nu_j'+\beta'; \ldots,$

respectively; and the symbol $|\alpha\sigma'+\alpha\alpha'|$ represents the sum of the λ terms

$$(\alpha\sigma'+\alpha\alpha')+(\beta\tau'+\beta\beta')+\ldots.$$

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* [Vol. i. p. 444.]

Each of the 2λ indices $\alpha\alpha'$, $\beta\beta'$,... is to receive successively the values 0, 1; and the sign of summation Σ extends to every combination of these values, so that the right-hand member consists of $2^{2\lambda}$ products, each affected with its proper sign, of four Theta functions.

The multiple Theta function (15) satisfies the equations

$$\left.\begin{aligned}\theta(\mu+2, \mu'; \ldots; x\ldots) &= (-1)^{\mu'}\theta(\mu, \mu'; \ldots; x\ldots),\\ \theta(\mu, \mu'+2; \ldots; x\ldots) &= \theta(\mu, \mu'; \ldots; x\ldots);\end{aligned}\right\} \quad \ldots\ldots \quad (17)$$

$$\theta(\mu, \mu'; \ldots; -x\ldots) = (-1)^{\mu\mu'}\theta(\mu, \mu'; \ldots; x\ldots) \quad ; \quad \ldots\ldots \quad (18)$$

$$\left.\begin{aligned}&\theta(\mu, \mu'; \nu, \nu'; \rho, \rho'; \ldots x, y, z, \ldots)\\ &\quad = e^{\frac{1}{4}i\pi\phi(\mu,\nu,\rho,\ldots)} \times e^{i(\mu x+\nu y+\rho z+\ldots)} \times \theta(0,0; 0,0; 0,0; \ldots x_1, y_1, z_1, \ldots),\\ &\qquad x_1 = x + \tfrac{1}{2}\pi\mu' + \tfrac{1}{4}\pi\frac{d\phi}{d\mu}, \quad y_1 = y + \tfrac{1}{2}\pi\nu' + \tfrac{1}{4}\pi\frac{d\phi}{d\nu},\\ &\qquad z_1 = z + \tfrac{1}{2}\pi\rho' + \tfrac{1}{4}\pi\frac{d\phi}{d\rho}, \ldots\ldots,\end{aligned}\right\} \quad \ldots \quad (19)$$

which correspond to the equations (4), (10), (5), relating to a Theta function of one argument.

The equations (17) show that we need only attribute to the indices the values 0, 1; thus, the formula (16) may be regarded as comprehending 11^{λ} different equations, if we regard two equations as identical which may be deduced from one another by permutation of the four indices j; or, as comprehending $\dfrac{11.12.13\ldots10+\lambda}{1.2.3\ldots\lambda}$ different equations, if we also regard as identical two equations which may be deduced from one another by a simultaneous permutation of the λ arguments, and corresponding pairs of indices, in each of the Theta functions; viz. counting the different equations on this principle, we have as many of them as there are λ-combinations, with repetition, of the eleven matrices of Table I. But we can always pass from any one of the equations (16) to any other by means of the formula (19), which may be employed to express any one of the 4^{λ} Theta functions as a product of any other by an exponential factor; although (for brevity) we have supposed that the indices of one of the Theta functions compared in that formula are all equal to zero.

XXXIX.

DE FRACTIONIBUS QUIBUSDAM CONTINUIS.

[Collectanea Mathematica (in memoriam Dominici Chelini), Milan 1881, pp. 117–143.
The paper is dated 1879.]

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1. PROPOSITA æquatione

$$P_1P_2 - R^2 = 1$$

olim demonstravimus fore*

$$\begin{aligned} P_1 &= (q_1, q_2, \cdots, q_i, q_i, \cdots, q_2, q_1), \\ P_2 &= (q_2, q_3, \cdots, q_i, q_i, \cdots, q_3, q_2), \\ R &= (q_1, q_2, \cdots, q_i, q_i, \cdots, q_3, q_2), \\ &= (q_2, q_3, \cdots, q_i, q_i, \cdots, q_2, q_1), \end{aligned}$$

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* 'Journal für die reine und angewandte Mathematik,' vol. l. p. 91 [No. III. vol. i. p. 33].
Verum hoc theorema ante nos invenerat vir clarissimus Serret.

Numeratorem fractionis continuæ

$$q_1 + \frac{1}{q_2} + \frac{1}{q_3} + \cdots \frac{1}{q_n}$$

per formulam $(q_1, q_2, q_3, \ldots, q_n)$ denotamus; cujus proprietates a forma determinantali

$$\left|\begin{array}{l} q_1, -1, 0, 0, \ldots\ldots\ldots\ldots 0 \\ 1, q_2, -1, 0, \ldots\ldots\ldots\ldots 0 \\ 0, 1, q_3, -1, \ldots\ldots\ldots\ldots 0 \\ 0, 0, 1, q_4, \ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0, 0, 0, 0, \ldots, 1, q_{n-1}, -1 \\ 0, 0, 0, 0, \ldots, 0, 1, q_n \end{array}\right|$$

pendere in commentatiuncula supra memorata observavimus. Ipsam autem fractionem continuam ita significamus ut quotientium series uncinis quadratis includatur. Itaque habetur æquatio

$$[q_1, q_2, q_3, \cdots, q_n] = \frac{(q_1, q_2, \cdots, q_n)}{(q_2, \cdots, q_n)}.$$

Cœterum in fractione periodica utimur punctis superpositis ad distinguendos primos atque ultimos periodorum quotientes. Sic exempli gratia erit

$$[a, b, c, \ldots, \dot{q}_1, q_2, \cdots, \dot{q}_n]$$

fractio continua constans e quotientibus *heteroclitis* $a, b, c, \ldots$, et periodo $q_1, q_2, \ldots, q_n$ infinities repetita.

atque hinc haberi partitiones numerorum P_1 et P_2 in summas duorum quadratorum. Eadem fere methodo æquationes

$$P_1P_2 - 2R^2 = \pm 1, \quad P_1P_2 - 3R^2 = \pm 1,$$

tractari possunt ope lemmatis quod viro clarissimo J. J. Sylvester acceptum referimus; 'Sit $q, q_2, \ldots, q_i, q_i, \ldots, q_2, q_1$

quotientium series ordine symmetrico disposita, e quibus alterni per numerum μ multiplicentur; determinans

$$K = (q_1, \mu q_2, q_3, \ldots, \mu q_3, q_2, \mu q_1)$$

erit formæ $A^2 + \mu B^2$, designantibus A et B numeros inter se primos.'

Fit enim, si numerus i par est,

$$(q_1, \mu q_2, q_3, \ldots, \mu q_i) = (\mu q_1, q_2, \mu q_3, \ldots, q_i);$$

si vero idem numerus impar est,

$$(\mu q_1, q_2, \mu q_3, \ldots, \mu q_i) = \mu x (q_1, \mu q_2, \ldots, q_i).$$

Itaque erit aut

$$K = (q_1, \mu q_2, q_3, \ldots, \mu q_i)^2 + \mu + (q_1, \mu q_2, \ldots, q_{i-1})^2,$$

aut

$$K = (q_1, \mu q_2, q_3, \ldots, \mu q_{i-1})^2 + \mu (q_1, \mu q_2, \ldots, \mu q_{i-1}, q_i)^2.$$

Aperte autem confitemur totam hanc disquisitionem angustis finibus contineri; quam, annis ab hinc amplius viginti a nobis inchoatam, nunc demum longo post intervallo retractatam absolutamque, ideo arithmeticorum hominum judicio committere ausi sumus, quod elementis arithmeticæ incrementa etiam tenuissima afferre operæ pretium putamus. Quæstiones autem in eodem genere latius patentes et nos olim attigimus, et fusius tractavit vir clarissimus M.A. Stern in egregia commentatione.*

2. Initium operis facimus ab æquatione

$$P_1P_2 = 3R_1^2 + 1;$$

in qua fiat

$$R_2 = R_1 - q_1P_2,$$

$$P_3 = \frac{3R_2^2 + 1}{P_2} = P_1 - 6q_1R_1 + 3q_1^2P_2,$$

'Report on the Theory of Numbers,' Art. 123 (Report of the British Association for the Advancement of Science for 1863) [vol. i. p. 283].

'Journal für die reine und angewandte Mathematik,' vol. liii. pp. 1–102.

Præter autem Goepelii dissertationem (ibid. vol. xlv. pp. 1–13) maxime memorabilis est commentatio clarissimi viri C. Hermite (ibid. p. 191); quæ tamen a theoria continuarum fractionum paullo longius recedit.

et generaliter

$$R_n = R_{n-1} - q_{n-1} P_n,$$
$$P_{n+1} = \frac{3R_n^2 + 1}{P_n};$$

ubi quotientes integros q ita determinari volumus, ut fiant numeri residui R quam minimi. Ponimus autem in æquatione data numeros P_1, P_2 (atque adeo omnes numeros P) esse positivos, et non impariter pares; R_1 autem numerum esse sive positivum sive negativum, quem positive sumptum per $[R_1]$ denotabimus. Præterea statuimus, quod licet, $[R_1]$ esse minorem quam $\frac{1}{2}P_1$. Erit ergo $P_1 > P_2$, $P_2 < \frac{3}{2}[R_1]$; unde patet quotientem q_1 evanescere non posse, et fore $[R_2] < \frac{1}{2}P_2$, $P_3 < \frac{3}{2}[R_2]$; ideoque $[R_1] > [R_2]$, $P_2 > P_3$; sive generaliter

$$[R_s] > \tfrac{2}{3}P_{s+1}, \quad [R_s] < \tfrac{1}{2}P_s,$$
$$[R_1] > [R_2] < [R_3] \ldots$$
$$P_1 > P_2 > P_3, \ldots.$$

Itaque veniemus aliquando ad æquationem

$$P_i P_{i+1} - 3R_i^2 = 1$$

in qua $P_{i+1} = 1$, $q_i = R_i$, quæque adeo omnium postrema erit. Tum vero habebuntur æquationes

$$R_i = (q_i), \quad P_i = (q_i,\ 3q_i),$$
$$R_{i-1} = (q_{i-1},\ 3q_i,\ q_i),$$
$$P_{i-1} = (q_{i-1},\ 3q_i,\ q_i,\ 3q_{i-1}),$$
$$R_{i-2} = (q_{i-2},\ 3q_{i-2},\ q_i,\ 3q_i,\ q_{i-1}),$$
$$P_{i-2} = (q_{i-2},\ 3q_{i-1},\ q_i,\ 3q_i,\ q_{i-1},\ 3q_{i-2}),$$
. . . . . .
. . . . . .
$$R_1 = (q_1,\ 3q_2,\ q_3,\ \ldots,\ 3q_3,\ q_2),$$
$$P_1 = (q_1,\ 3q_2,\ q_3,\ \ldots,\ 3q_3,\ q_2,\ 3q_1);$$

quæ facile aliæ ex aliis probantur, ope formularum

$$R_{s-1} = q_{s-1} P_s + R_s,$$
$$P_{s-1} = \frac{3R_{s-1}^2 + 1}{P_s}.$$

Itaque ex data æquatione $P_1 P_2 = 3R_1^2 + 1$ elicimus repræsentationes numerorum P_1 et P_2 per quadratum et triplicem quadrati.

Exemplum. Sit æquatio data, $1999 \times 436 = 3 \times 529^2 + 1$: erit

$$\begin{aligned} 1999 &= (1, 3, 1, -6, 1, 3, -2, 3, 1, 3) \\ &= (1, 3, 1, -6)^2 + 3(1, 5, 1, -6, 1)^2 \\ &= 26^2 + 3 \times 21^2. \end{aligned}$$

$$\begin{aligned} 436 &= (3, 1, -6, 1, 3, -2, 3, 1) \\ &= (3, 1, -6, 1)^2 + 3(1, 3, -2)^2 \\ &= 17^2 + 3 \times 7^2. \end{aligned}$$

3. Statuamus in fractione continua

$$\frac{P_1}{3R_1} = [q_1, 3q_2, q_3, \ldots, 3q_3, q_2, 3q_1]$$

hunc haberi quotientium completorum ordinem

$$\theta_1, 3\theta_2, \theta_3, \ldots, \theta_i \text{ sive } 3\theta_i, 3\phi_i \text{ sive } \phi_i, \ldots, \phi_2, 3\phi_1,$$

ita ut fiat

$$\frac{P_1}{3R_1} = \theta_1 = q_1 + \frac{1}{3\theta_2}, \quad \theta_2 = q_2 + \frac{1}{3\theta_3}, \ldots$$

$$\theta_i = q_i + \frac{1}{3\phi_i}, \quad \phi_i = q_i + \frac{1}{3\phi_{i-1}}, \ldots$$

$$\phi_2 = q_2 + \frac{1}{3\phi_1}, \quad \phi_1 = q_1.$$

Quantitatum θ et ϕ ea est conditio ut semper evadat $[\theta] > \frac{2}{3}$, $[\phi] > \frac{1}{\sqrt{3}}$.

Quod ut certa demonstratione confirmetur, adnotamus primum, si inter quotientes q duæ signis diversis unitates juxta positæ inveniantur, continuari signum secundæ unitatis in proximo quotiente. Sit enim, exempli causa, $q_1 = 1$, $q_2 = -1$; numeri R_1 et $-R_2$ erunt positivi, quia signa numerorum q_s, R_s semper inter se conveniunt. Quum autem sit $P_2 < \frac{3}{2} R_1$, sequitur fore

$$-R_2 > \tfrac{1}{3}P_2, \quad P_3 = \frac{3R_2^2 + 1}{P_2} < -R_2, \quad R_3 = R_2 + P_3 < 0\,;$$

ideoque etiam $q_3 < 0$. Monemus tamen duos ultimos quotientes q_{i-1} et q_i unitates signis diversis esse posse; neque ulla lege adstringi signum quotientis duas unitates proxime antecedentis. His positis, apparet quantitatem ϕ_{s+1} satisfacere conditioni supra memoratæ, si eidem ab ipsis ϕ_s, $\phi_{s-1}, \ldots$ satisfiat. Quum enim sit

$$\phi_{s+1} = q_{s+1} + \frac{1}{3\phi_s}, \quad \phi_s = q_s + \frac{1}{3\phi_{s-1}}, \ldots$$

manifestum est signa ipsorum ϕ_{s+1}, ϕ_s,... congruere cum signis quotientium q_{s+1}, q_s,... Sit igitur, brevitatis causa, q_{s+1} numerus positivus; erit certe

$$\phi_{s+1} \geqq 1 + \frac{1}{3\phi_s} \geqq 1 + \frac{1}{-3} + \frac{1}{\phi_{s-1}} \geqq 1 + \frac{1}{-3} + \frac{1}{2}\,\frac{1}{3\phi_{s-2}};$$

quia quotiens tertius q_{s-1} unitas positiva esse nequit. Itaque ad extremum habebitur

$$\phi_{s+1} \geqq [1, -\dot{3}, \dot{2}], \quad \text{sive} \quad \phi_{s-1} \geqq \frac{1}{\sqrt{3}}.$$

Et similiter si est $[\theta_{s+1}] > \frac{2}{3}$, $[\theta_s] > \frac{2}{3}$,... erit etiam

$$[\theta_s] \geqq 1 + \frac{1}{-3} + \frac{1}{\theta_{s+2}} \geqq \frac{2}{3},$$

quia post quotientes $q_s = 1$, $q_{s+1} = -1$, necessario obtinet proximum locum quotiens negativus.

Sit $I\theta$ numerus integer qui omnium proxime accedit ad valorem ipsius θ; nanciscimur ex antedictis æquationes sequentes

$$q_1 = I\theta_1, \quad q_2 = I\theta_2, \ldots, \quad q_{i-1} = I\theta_{i-1}.$$

At vero est $q_i = \theta_i - \dfrac{1}{3\phi_i}$; quumque fieri possit ut sit $[\phi_i] < \frac{2}{3}$, æquatio $q_i = I\theta_i$ non semper valet. Valet tamen præterquam in eo casu, in quo est q_{i-1} unitati æqualis, $[\theta_i] < 2$, et in quo insuper alternantur signa quantitatum q_{i-2}, q_{i-1}, θ_i; quo in casu erit $[\phi_i] < \frac{2}{3}$, $I[\theta_i] = 2$, q_i unitas unitati q_{i-1} contraria. Cujus rei exemplum præbet æquatio $52 \times 7 = 3 \times 11^2 + 1$, ubi fit

$$\tfrac{52}{33} = [2, -3, 1, 3, -1, 6].$$

Hic enim est $\theta_3 = \frac{14}{9}$, ideoque $I\theta_3 = 2$, quum sit tamen $q_3 = 1$.

Quodsi in fractione continua ad postremum quotientem, qui est $3q_1$, accedat quævis ejusdem signi quantitas, valores θ_1, θ_2, ..., θ_i ipsos quidem aliquatenus turbatum iri manifestum est; sed nihilo secius permansuras æquationes $q_1 = I\theta_i$, $q_2 = I\theta_2$, ..., $q_{i-1} = I\theta_{i-1}$, atque determinationem quotientis q_i modo traditam. Quod patet ex ipsa demonstrationis forma.

4. Æquatio $P_1P_2 = 3R_1^2 - 1$ ita facillime tractatur si scribatur

$$P_1P_2 = -3R_1^2 + 1,$$

ut P_2 et R_1 numeri negativi evadant. Quo facto, determinentur numeri $P_3, P_4, \ldots, P_{i+1}$; $R_2, R_3, \ldots, R_i$; per æquationes hasce

$$R_2 = R_1 - q_1 P_2,$$
$$P_3 = \frac{-3R_2^2 + 1}{P_2}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(in quibus numeros R quam minimos fieri intelligendum est) ... donec perveniatur ad æquationem in qua sit $P_{i+1} = \epsilon$, $R_i = \epsilon q_i$, designante litera ϵ unitatem $(-1)^i$ ... Tum vero erit

$$\epsilon P_1 = (q_1, -3q_2, \ldots, q_2, -3q_1),$$
$$\epsilon P_2 = (q_2, -3q_3, \ldots, q_3, -3q_2),$$
$$\epsilon R = (q_1, -3q_2, \ldots, -3q_3, q_2);$$

unde apparet ipsos P_1 et P_2 fore formæ $y^2 - 3x^2$, aut formæ $3x^2 - y^2$, prout i par est aut impar.

Cœterum in æquatione penultima $R_i = R_{i-1} - q_{i-1} P_i$, si est $P_i = \pm 2$, oritur ambiguitas in determinatione quotientis q_{i-1}; quam ita tollimus ut quotientes q_{i-1}, q_i (e quibus hic certe unitas erit) afficiantur signis contrariis. Facile autem demonstratur in serie quotientium $q_1, q_2, \ldots, q_i$, si duæ eodem signo unitates juxta veniant, mutationem signi fieri in proximo quotiente. Hinc in fractione continua

$$-\frac{P_1}{3R_1} = [q_1, -3q_2, q_3, \ldots -3q_3, q_2, -3q_1]$$

si fiat deinceps

$$-\frac{P_1}{-3R_1} = \theta_1 = q_1 + \frac{1}{3\theta_2}, \quad \theta_2 = q_2 + \frac{1}{-3\theta_3}, \ldots,$$
$$\theta_i = q_i + \frac{1}{-3\phi_i}, \quad \phi_i = q_i + \frac{1}{-3\phi_{i-1}}, \ldots,$$

et sic porro, erit ut supra $[\phi_s] > \frac{1}{\sqrt{3}}$, $[\theta_s] > \frac{2}{3}$. Quotientes autem $q_1, q_2, \ldots$ determinantur per æquationes $q_1 = I\theta_1$, $q_2 = I\theta_2, \ldots$; inter quas etiam extrema illa $q_i = I\theta_i$ locum sibi vindicat. Nam, si $q_i = \pm 1$, (qui solus est casus de quo dubitatio oriri potest) erit necessario $P_i = \pm 2$, atque discrepabunt inter se signa quotientium q_{i-1}, et q_i; quo fit ut valor absolutus ipsius $\phi_i = q_i + \dfrac{1}{-3q_{i-1} + \dfrac{1}{\phi_{i-2}}}$ unitate major evadat.

Quum autem numerus ϵP_1 infinitis modis per formulam $y^2 - 3x^2$ repræsentari

queat, ostendere convenit eam repræsentationem quæ continetur formula

$$\epsilon P_1 = (q_1, -3q_2, \ldots, q_2, -3q_1)$$

per numeros minimos fieri. Et generaliter quidem, si est

$$y^2 - Ax^2 = N, \quad t^2 - Au^2 = 1,$$

inveniri possunt in una eademque repræsentationum familia totidem repræsentationes repræsentatione data magis simplices quot habentur numeri u fractione $\frac{2xy}{N}$ absolute minores. Etenim cum omnes indeterminati x valores in hac formula $tx - uy$ comprehendantur, si qua est repræsentatio repræsentatione data magis simplex, habetur inæqualitas $(tx-uy)^2 < x^2$; quæ, eliminato numero t, convenit cum inæqualitate $[u] < \left[\frac{2xy}{N}\right]$. Jam vero, si $i = 2n$, habemus æquationem

$$P_1 = y^2 - 3x^2 = (q_1, -3q_2, \ldots, q_{2n-1}, -3q_{2n})^2 - 3(q_1, -3q_2, \ldots, q_{2n-1})^2,$$

in qua erit

$$\left[\frac{y}{3x}\right] = [\phi_n] > 1;$$

ideoque

$$[y] < \sqrt{\frac{3P_1}{2}}, \quad [x] < \sqrt{\frac{P_1}{6}}.$$

Sin autem $i = 2n+1$, erit

$$P_1 = 3x^2 - y^2 = 3(q_1, -3q_2, \ldots, q_{2n+1})^2 - (q_1, -3q_2, \ldots, 3q_{2n})^2,$$

$$\left[\frac{x}{y}\right] = [\phi_n] > 1, \quad [x] < \sqrt{\frac{P_1}{2}}, \quad y < \sqrt{\frac{P_1}{2}}.$$

Hinc in utroque casu habetur $\left[\frac{2xy}{P_1}\right] < 1$; unde apparet numeri P_1 repræsentationem magis simplicem exhiberi non posse in ea saltem familia, in qua fit $\sqrt{3} \equiv \frac{1}{R_1}$, mod P_1.

Exemplum. Sit æquatio proposita

$$20306 \times -5197 = -3(-5931)^2 + 1;$$

erit $\epsilon = -1$,

$$\begin{aligned} 20306 &= -(1, 6, 1, -12, -1, 3, 4, -3, -2, -3) \\ &= 3(1, 6, 1, -12, -1)^2 - (1, 6, 1, -12)^2 \\ &= 3 \times 97^2 - 89^2; \\ 5197 &= (6, 1, -12, -1)^2 - 3(-2, -3, 4)^2 \\ &= 85^2 - 3 \times 26^2. \end{aligned}$$

5. Sit λ numerus primus formæ $4n+3$; atque in æquatione $\lambda\mu = 1+AB$, tribuantur literæ A singillatim valores 1, 2, 3, ..., $2n+1$: quo facto, erit B unus aliquis ex iisdem numeris vel positive vel negative sumptus. Hinc patet in hac æquationum caterva binas inter se invicem respondere; quarum numerus cum impar sit, una super erit, quæ necessario hanc formam induet $\lambda\mu = \pm 3A^2+1$. Hinc per exclusionem nanciscimur demonstrationem non inelegantem notissimi theorematis; primos formæ $12n+7$ esse etiam formæ x^2+3y^2; primos autem formæ $12n+11$ habere formam $3x^2-y^2$. Ac simili fere modo demonstrari posset utramque formam x^2+3y^2, x^2-3y^2, primis $12n+1$ competere; neutram vero primis $12n+5$. Sed primos formæ $4n+1$ in præsens missos facimus. Nam repræsentationes primorum $4n+3$ per formas x^2+3y^2, $3x^2-y^2$ indagari possunt per evolutiones radicum quadratarum in fractiones continuas; quæ proprietas ad primos formæ $4n+1$ nequaquam pertinere videtur. Quo autem facilius intelligantur rationes evolutionum, quibus in hac quæstione utimur, pauca, licet aliunde nota, præmittenda sunt.

6. Sit N integer non quadratus; α integer ipso $\surd N$ proxime minor; quantitas $\alpha+\surd N$ in fractionem continuam conversa dabit quotientium periodum ad hoc exemplar

$$2\alpha, \mu_1, \mu_2, \ldots, \mu_n, \beta, \mu_n, \mu_{n-1}, \ldots, \mu_1;$$

ubi primus quotiens quasi singularis est; cœteri ordine symmetrico circa quotientem medium β hinc et hinc disponuntur. Cujus symmetriæ eo causa est, quod æquatio

$$\alpha^2 - N - 2\alpha x + x^2 = 0, \quad \ldots\ldots\ldots \quad (A)$$

a qua periodus originem trahit, *anceps* est, cum tertius in ipsa coefficiens secundum metiatur. Quod si evolutionem a quotiente medio incipimus, habebimus hanc periodi descriptionem,

$$\beta, \mu_n, \mu_{n-1}, \ldots, \mu_1, 2\alpha, \mu_1, \mu_2, \ldots, \mu_n,$$

quæ est plane ejusdem formæ cum periodo data. Hinc patet inter æquationes periodicas, præter primam illam, mediam quoque ancipitem esse. Cujus æquationis cognitio magnum momentum habet ad explorandam totius periodi naturam; sunt autem certi casus in quibus etiam sine evolutione innotescit. Sit, exempli causa, $N = \lambda_1 \times \lambda_2$, designantibus literis λ_1 et λ_2 numeros primos formæ $4n+3$; sit etiam $\lambda_1 > \lambda_2$; b integer surdo $\surd\frac{\lambda_1}{\lambda_2}$ proxime minor. His positis, erit $2b$ quotiens medius in evolutione radicis $\surd\lambda_1\times\lambda_2$, et æquatio media hanc formam habebit

$$\lambda_2 b^2 - \lambda_1 - 2b\lambda_2 x + \lambda_2 x^2 = 0. \quad \ldots\ldots\ldots \quad (B)$$

Cum enim $\lambda_1 \times \lambda_2$ summa duorum quadratorum esse nequeat, duæ illæ æquationes ancipites, quæ in periodo inveniuntur, diversæ inter se esse debent; at præter duas æquationes (A) et (B) nulla alia existere potest, quæ et anceps sit, et characterem æquationes periodicæ præ se ferat. Unde videmus evolutiones radicum $\sqrt{\lambda \times \lambda_2}$ et $\sqrt{\frac{\lambda_1}{\lambda_2}}$ una eademque opera exhiberi; idque fieri per hujusmodi æquationes

$$a + \sqrt{\lambda_1 \times \lambda_2} = \left[2a, \mu_1, \mu_2, \ldots, \mu_n, b + \sqrt{\frac{\lambda_1}{\lambda_2}}\right]$$

$$b + \sqrt{\frac{\lambda_1}{\lambda_2}} = [2b, \mu_n, \mu_{n-1}, \ldots, \mu_1, a + \sqrt{\lambda_1 \times \lambda_2}];$$

e quarum utravis concludimus fore

$$\left.\begin{aligned} \lambda_2 \times (\mu_1, \mu_2, \ldots, \mu_n, b) &= (a, \mu_1, \mu_2, \ldots, \mu_n), \\ \lambda_1 \times (\mu_1, \mu_2, \ldots, \mu_n) &= (a, \mu_1, \mu_2, \ldots, \mu_n, b), \end{aligned}\right\} \quad \ldots\ldots \quad (C)$$

ita ut fiat

$$\lambda_1 \times (\mu_1, \mu_2, \ldots, \mu_n)^2 - \lambda_2 \times (\mu_1, \mu_2, \ldots, \mu_n, b)^2 = (-1)^n.$$

7. Sit *primo* $-\lambda_2$ ipsius λ_1 residuum quadraticum; erit $(-1)^n = +1$, atque ideo e numeris $(\mu_1, \mu_2, \ldots, \mu_n)$ et $(\mu_1, \mu_2, \ldots, \mu_n, b)$ hic impar, ille par erit. Hinc sequitur, æquationem $4\lambda_1 x^2 - \lambda_2 y^2 = 1$ resolubilem esse; et, designantibus a' et b' numeros ipsis $2\sqrt{\lambda_1 \times \lambda_2}$ et $2\sqrt{\frac{\lambda_1}{\lambda_2}}$ proxime minores, æquationem

$$\lambda_2 x^2 - 2b'\lambda_2 x + \lambda_2 b'^2 - 4\lambda_1 = 0$$

occupare medium locum in periodo æquationis

$$x^2 - 2a'x + a'^2 - 4N = 0;$$

hoc est, evolutiones quantitatum $a' + 2\sqrt{\lambda_1 \times \lambda_2}$, $b' + 2\sqrt{\frac{\lambda_1}{\lambda_2}}$ eodem modo alteram ab altera pendere, quo evolutiones quantitatum $a + \sqrt{\lambda_1 \times \lambda_2}$, $b + \sqrt{\frac{\lambda_1}{\lambda_2}}$.

Evoluta autem quantitate $a' + 2\sqrt{\lambda_1 \times \lambda_2}$ sit

$$2a', \nu_1, \nu_2, \ldots, \nu_j, 2b', \nu_j, \ldots, \nu_2, \nu_1$$

quotientium periodus; habebimus æquationes sequentes antecedentibus (C) consimiles

$$\left.\begin{aligned} \lambda_2 \times (\nu_1, \nu_2, \ldots, \nu_j, b') &= (a', \nu_1, \nu_2, \ldots, \nu_j), \\ 4\lambda_1 \times (\nu_1, \nu_2, \ldots, \nu_j) &= (a', \nu_1, \nu_2, \ldots, \nu_j, b'); \end{aligned}\right\} \quad \ldots\ldots \quad (D)$$

habebimus etiam solutionem æquationis $4\lambda_1 x^2 - \lambda_2 y^2 = 1$, scilicet,

$$4\lambda_1 (\nu_1, \nu_2, \ldots, \nu_j)^2 - \lambda_2 (\nu_1, \nu_2, \ldots, \nu_j, b') = 1;$$

quæ quum convenire debeat cum solutione superius inventa (utraque enim minimos numeros exhibet qui æquationi satisfacere possunt) colligimus æquationes sequentes

$$(\mu_1, \mu_2, \ldots, \mu_i) = 2(\nu_1, \nu_2, \ldots, \nu_j),$$

$$(a, \mu_1, \mu_2, \ldots, \mu_i) = (a', \nu_1, \nu_2, \ldots, \nu_j),$$

$$(\mu_1, \mu_2, \ldots, \mu_i, b) = (\nu_1, \nu_2, \ldots, \nu_j, b'),$$

$$(a, \mu_1, \mu_2, \ldots, \mu_i, b) = \tfrac{1}{2}(a', \nu_1, \nu_2, \ldots, \nu_j, b');$$

e quibus patet quomodo, ex evolutione quantitatis $\sqrt{\lambda_1 \times \lambda_2}$, evolutiones quantitatum $2\sqrt{\lambda_1 \times \lambda_2}$ et $2\sqrt{\frac{\lambda_1}{\lambda_2}}$ derivari possint per conversionem fractionis vulgaris in fractionem continuam; si enim est

$$\frac{A}{B} = [a, \mu_1, \mu_2, \ldots, \mu_n, b],$$

erit etiam

$$\frac{2A}{B} = [a', \nu_1, \nu_2, \ldots, \nu_j, b'].$$

Sit, *secundo*, λ_2 ipsius λ_1 residuum quadraticum; erit $(-1)^n = -1$, $(\mu_1, \mu_2, \ldots, \mu_n, b)$ par, $(\mu_1, \mu_2, \ldots, \mu_n)$ impar; atque habebitur solutio æquationis $\lambda_1 x^2 - 4\lambda_2 y^2 = -1$. Unde derivari poterunt evolutiones quantitatum $\frac{1}{2}\sqrt{\lambda_1 \times \lambda_2}$, et $\frac{1}{2}\sqrt{\frac{\lambda_1}{\lambda_2}}$, quibus tamen in præsens opus non est.

8. Jam vero sit $\lambda_2 = 3$, $\lambda_1 = \lambda$ numerus primus formæ $12n+7$; erit -3 ipsius λ residuum quadraticum; atque habebitur ex antedictis æquatio

$$(a, \mu_1, \mu_2, \ldots, \mu_n, b)(\mu_1, \mu_2, \ldots, \mu_n) = 3 \times (\mu_1, \mu_2, \ldots, \mu_n, b)^2 + 1;$$

quæ tamen, quia $(\mu_1, \mu_2, \ldots, \mu_n)$ numerus est impariter par, non potest commode tractari per methodum supra (Art. 2) traditam. Utendum est igitur evolutione quantitatis $2\sqrt{\frac{\lambda}{3}}$ quæ si statuitur esse

$$2b', \nu_j, \nu_{j-1}, \ldots, \nu_1, \quad 2a', \nu_1, \ldots, \nu_{j-1}, \nu_j,$$

suppeditabit æquationem

$$(a', \nu_1, \nu_2, \ldots, \nu_j, b')(\nu_1, \nu_2, \ldots, \nu_j) = 3(\nu_1, \nu_2, \ldots, \nu_j, b')^2 + 1,$$

in qua $(\nu_1, \nu_2, \ldots, \nu_j)$ impar erit, $(a', \nu_1, \nu_2, \ldots, \nu_j, b')$ pariter par. Itaque si in hac æquatione fiat

$$(a', \nu_1, \nu_2, \ldots, \nu_j, b') = (k_0, 3k_1, k_2, \ldots, 3k_2, k_1, 3k_0),$$

$$(\nu_1, \nu_2, \ldots, \nu_j) = (3k_1, k_2, \ldots, 3k_2, k_1).$$

$$(\nu_1, \nu_2, \ldots, \nu_j, b') = (k_0, 3k_1, k_2, \ldots, 3k_2, k_1),$$

evolutio quantitatis $2\sqrt{\frac{\lambda}{3}}$ ad sequentem formam redigetur

$$k_0 + 2\sqrt{\frac{\lambda}{3}} = [2k_0, 3k_1, k_2, \ldots, 3k_2, k_1, 6k_0, k_1, 3k_2, \ldots, k_2, 3k_1].$$

Quod facile demonstratur, si in evolutione vulgari quantitatis $2\sqrt{\frac{\lambda}{3}}$ quotientes $2a'$, $2b'$, interpositis cifris ita discerpantur ut fiat

$$\sqrt{\frac{\lambda}{3}} = [a', \nu_j, \nu_{j-1}, \ldots, \nu_1, b', 0, b', \nu_1, \ldots, \nu_j, a', 0, a'];$$

et si præterea intelligatur, quod in fractione infinita licitum est, quotientium seriem ita terminari ut una ex cifris interpositis locum extremum obtineat.

Sit $-A - 2B\theta + C\theta^2 = 0$ æquatio quadratica; numeri ABC integri; determinans $B^2 + AC$ positivus. Sit item $q_1, q_2, \ldots, q_n$ quotientium periodus per quam æquatio ista in se ipsam transit; ita ut fiat

$$\theta = [q_1, q_2, \ldots, q_n, \theta];$$

habentur æquationes notissimæ

$$\frac{(q_2, q_3, \ldots, q_n)}{A} = \frac{(q_1, q_2, \ldots, q_n) - (q_2, \ldots, q_{n-1})}{B} = \frac{(q_1, q_2, \ldots, q_{n-1})}{C};$$

in quibus si forte fiat aut $(q_1, q_2, \ldots, q_{n-1}) = \mu(q_2, q_3, \ldots, q_n)$, aut

$$(q_2, q_3, \ldots, q_n) = \mu(q_1, q_2, \ldots, q_{n-1}),$$

determinans æquationis erit formæ $x^2 + \mu y^2$. Quodsi in evolutione quantitatis $k_0 + 2\sqrt{\frac{\lambda}{3}}$ periodum ab illo quotiente inchoamus qui est aut triplex, aut tertia pars quotientis proxime antecedentis, habetur series aut hujus formæ

$$3k_i, k_{i-1}, \ldots, 3k_2, k_1, 6k_0, k_1, 3k_2, \ldots, k_2, 3k_1, 2k_0, 3k_1, k_2, \ldots, 3k_{i-1}, k_i,$$

aut hujus

$$k_i, 3k_{i-1}, \ldots, 3k_2, k_1, 6k_0, k_1, 3k_2, \ldots, k_2, 3k_1, 2k_0, 3k_1, k_2, \ldots, k_{i-1}, 3k_i;$$

prout numerus i par est aut impar. Unde patet in æquatione periodica $P_{i+1} + 2Q_{i+1}x + P_{i-2}x^2 = 0$, e qua istam quotientium seriem originem trahere statuimus, fore vel $3P_{i+1} + P_{i+2} = 0$, vel $P_{i+1} + 3P_{i+2} = 0$, ita ut habeatur partitio

numeri 12λ, ac proinde ipsius λ, in quadratum et triplicem quadrati. Quo autem in æquatione $\lambda = A^2 + 3B^2$ valores numerorum A et B facilius investigentur, aperienda est via ad evolutionem quantitatis $k_0 + 2\sqrt{\frac{\lambda}{3}}$ sine operoso calculo inveniendam. Nimis enim molestum esset eam evolutionem a vulgari evolutione radicis $2\sqrt{\frac{\lambda}{3}}$ per transformationem supra traditam elicere.

9. Et primum quidem observamus in evolutione quantitatis $k_0 + 2\sqrt{\frac{\lambda}{3}}$, si alterni quotientes per ternarium dividantur, haberi periodum ejusdem plane formæ cum periodis primorum $4n+1$. Patet autem ex iis quæ antea (Art. 3) disputavimus quotientes $k_0, k_1, \ldots, k_{i-1}$ deinceps determinari posse eadem prope calculi forma quæ vulgo usitata est ad radices quadraticas evolvendas. Habemus enim schema hujusmodi, si incipimus ab æquatione

$$P_0 + 2Q_0 x + P_1 x^2 = 0,$$

in qua est $P_0 = 3k_0^2 - 4\lambda$, $Q_0 = -3k_0$, $P_1 = 3$, $k_0 = I2\sqrt{\frac{\lambda}{3}}$;

$$Q_1 = 2k_0 P_1 + Q_0, \quad P_2 = \frac{Q_1^2 - 12\lambda}{P_1}, \quad k_1 = I\frac{-Q_1 - 2\sqrt{3\lambda}}{3P_2},$$

$$Q_2 = 3k_1 P_2 + Q_1, \quad P_3 = \frac{Q_2^2 - 12\lambda}{P_2}, \quad k_2 = I\frac{-Q_2 + 2\sqrt{3\lambda}}{P_3},$$

$$Q_3 = k_2 P_3 + Q_2, \quad P_4 = \frac{Q_3^2 - 12\lambda}{P_3}, \quad k_3 = I\frac{-Q_3 - 2\sqrt{3\lambda}}{3P_4},$$

$$Q_4 = 3k_3 P_4 + Q_3, \quad P_5 = \frac{Q_4^2 - 12\lambda}{P_4}, \quad k_4 = I\frac{-Q_4 + 2\sqrt{3\lambda}}{P_5},$$

. . . . . .

in quo videmus unumquemque numerorum Q esse ternarii multiplum, sed contra alternos tantum e numeris P. Si ergo scribatur $Q_s = 3b_s$, $P_{2s} = a_{2s}$, $P_{2s+1} = 3a_{2s+1}$, habetur calculi forma paullo simplicior,

$$b_1 = 2k_0 a_1 + b_0, \quad a_2 = \frac{3b_1^2 - 4\lambda}{a_1}, \quad k_1 = I\frac{-b_1 - 2\sqrt{\frac{\lambda}{3}}}{a_2},$$

$$b_2 = k_1 a_2 + b_1, \quad a_3 = \frac{3b_2^2 - 4\lambda}{a_2}, \quad k_2 = I\frac{-b_2 + 2\sqrt{\frac{\lambda}{3}}}{a_3},$$

. . . . . .

$$b_s = k_{s-1}a_s + b_{s-1}, \quad a_{s-1} = \frac{3b_3^2 - 4\lambda}{a_s}, \quad k_s = I\,\frac{-b_s + (-1)^s 2\sqrt{\frac{\lambda}{3}}}{a_{s+1}},$$

. . . . . .

vel, si notationem Gaussianam sequi placet

$$b_1 \equiv b_0, \mod a_1; \quad a_2 = \frac{3b_1^2 - 4\lambda}{a_1};$$

$$b_2 \equiv b_1, \mod a_2; \quad a_3 = \frac{3b_2^2 - 4\lambda}{a_2};$$

. . . . . .

$$b_s \equiv b_{s-1}, \mod a_{s-1}; \quad a_s = \frac{3b_{s-1}^2 - 4\lambda}{a_{s-1}}$$

. . . . . .

ubi numeros b_s, $k_{s-1} = \frac{b_s - b_{s-1}}{a_{s-1}}$, e congruentiis $b_s \equiv b_{s-1}$, mod a_s, ita determinari oportet ut b_s quam proxime accedat ad valorem radicis $(-1)^{s+1} 2\sqrt{\frac{\lambda}{3}}$.

His in schematis determinationes numerorum

$$k_2 = I\theta_1, \quad k_2 = I\theta_2, \ \ldots, \ k_{i-1} = I\theta_{i-1}$$

ex antedictis (Art. 3) accuratas esse apparet; sed contra determinationem quotientis $k_i = I\theta_i$ falsam fore, si

$$\phi_i = \left[\frac{-b_{i+1} + (-1)^{i+2} 2\sqrt{\frac{\lambda}{3}}}{a_{i+1}}\right] < \tfrac{2}{3};$$

hoc est, si in æquatione

$$4\lambda = a_{i+1}^2 + 3b_{i+1}^2$$

fiat $[b_{i+1}] < \frac{1}{12}[a_{i+1}]$. Quocirca calculus eo usque producendus est donec perveniatur ad æquationem in qua aut $a_{i+1} + a_{i+2} = 0$, aut $\frac{1}{2}a_i \pm 3b_i + 2a_{i+1} = 0$. Nam, si $[\phi_i] > \frac{2}{3}$, æquatio quæ conditioni $a_{i+1} + a_{i+2} = 0$ satisfacit quæque ideo in toto schemate ultima est, calculi normam sequenti suo loco se offeret. Si vero est $[\phi_i] < \frac{2}{3}$, ultima ista æquatio quodammodo extra ordinem calculi erit, quia fallit determinatio $k_i = I\theta_i$. Sed hoc in casu est necessario $k_i = \pm 1$, unde sequitur penultimam æquationem talem fore ut sit

$$\tfrac{1}{2}a_i \pm 3b_i + 2a_{i+1} = 0;$$

cujusmodi æquatio si quando se obtulerit, ipsa certe est proxima ante ultimam, atque, ut ad ultimam perveniatur, quotienti k_i tribuendus est valor ± 1; idque observandum est etiam si forte fiat $[\theta_i] > \frac{3}{2}$. Calculum autem numerorum a et b

ulterius extendere omnino opus non est; cum in horum periodis semissis prior hanc habeat formam

$$a_1;\ a_2, a_3, \ldots, a_i,\ a_{i+1}, -a_{i+1}, -a_i, \ldots, -a_2;$$
$$b_1,\ b_2, b_3, \ldots, b_i;\ b_{i+1};\ b_i,\qquad b_{i-1}, \ldots, b_1;$$

altera autem semissis eosdem numeros signis mutatis eodem ordine repræsentet. Quæ omnia animadvertere operæ pretium est propter similitudinem primorum formæ $4n+1$.

Exemplum I. Sit $\lambda = 199$; erit

$$I2\sqrt{\frac{\lambda}{3}} = 16, \quad a_0 = -28, \quad b_0 = -16, \quad a_1 = 1.$$

Æquationes autem $P_s + 2Q_s x + P_{s+1}x^2 = 0$ brevitatis causa per notas sic scribimus (a_s, b_s, a_{s+1}); quo pacto habebimus ex calculi ordine has æquationes in primo periodi quadrante

$$(-28, -16, 1),\ (1, 16, -28),\ (-28, -12, 13),$$
$$(13, 14, -16),\ (-16, -18, -11),\ (-11, 15, 11):$$

unde fit $4 \times 199 = 11^2 + 3 \cdot 15^2$;

$$16 + 2\sqrt{\tfrac{199}{3}} = [3\dot{2}, 3, 2, 6, -3, -9, 2, 6, 1, 66, 1, 6, 2, -9, -3, 6, 2, \dot{3}].$$

Exemplum II. Sit $\lambda = 607$; erit

$$I2\sqrt{\frac{\lambda}{3}} = 28, \quad a_0 = -76, \quad b_0 = -28, \quad a_1 = 1.$$

Hinc oritur periodus

$$(-76, -28, 1),\ (1, 28, -76),\ (-76, -48, -59),$$
$$(-59, 11, 35),\ (35, -24, -20),\ (-20, 36, -73),$$
$$(-73, -37, -23),\ (-23, 32, -28),\ (-28, -24, 25),$$
$$(25, -26, -16),\ (-16, -22, 61),\ (61, 39, 35),$$
$$(35, -31, 13),\ (13, 34, 80),\ (80, -46, 49),$$
$$(49, 3, -49), \ldots$$

unde fit $4 \times 607 = 49^2 + 3 \cdot 3^2$;

$$28 + 2\sqrt{\tfrac{607}{3}} = [56, 3, -1, -3, -3, 3, -3, 6, -2, 9, 1, -6, 5, -3,$$
$$+1, 3, -1, 15, -2, 3, 3, -6, 2, -9, -1, -9, -1, -3,$$
$$1, 168, 1, -3, -1, -9, 1, -9, 2, -6, 3, 3, -2, 15, -1,$$
$$3, 1, -3, 5, -6, 1, 9, -2, 6, -3, 3, -3, -3, -1, 3].$$

Observandum autem est hic in penultima æquatione $(80, -46, 49)$, sive (a_{14}, b_{14}, a_{15}), haberi $40 - 3 \times 46 + 2 \times 49 = 0$, unde sequitur esse $k_{14} = 1$, quamvis sit $\frac{46+28}{49} > \frac{3}{2}$.

10. Quum autem numerus 4λ per formam $x^2 + 3y^2$ trifariam repræsentari possit, illa quidem repræsentatio, quæ numeros x et y pares habet, cum nostra convenire non potest. Ex duabus, quæ reliquæ sunt, una sola est in qua $x \equiv \pm 1$, mod 12; atque hæc illa est quam præbet æquatio $4\lambda = a_{i+1}^2 + 3b_{i+1}^2$. Si enim i impar est, habemus

$$3(k_0,\ 3k_1,\ \ldots,\ 3k_i)^2 - 4\lambda(3k_1,\ k_2,\ \ldots,\ 3k_i)^2 = P_{i+2} = 3a_{i+2}$$

sive

$$(k_0,\ 3k_1,\ \ldots,\ 3k_i)^2 - 12\lambda(k_1,\ 3k_2,\ \ldots,\ k_i)^2 = a_{i+2}.$$

Unde fit $a_{i+2} \equiv 1$, mod 12. Ac similiter, si numerus i par est, erit $a_{i+1} \equiv 1$, mod 12.

Cæterum, ut ex inventa æquatione $4\lambda = a_{i+1}^2 + 3b_{i+1}^2$, ipsius λ repræsentatio habeatur, fiat $a_{i+1} \pm b_{i+1}$, $\equiv 0$, mod 4: quo pacto erit

$$\lambda = \left(\frac{a_{i+1} \mp 3b_{i+1}}{4}\right)^2 + 3\left(\frac{a_{i+1} \pm b_{i+1}}{4}\right)^2,$$

contra, si data fuerit æquatio $\lambda = A^2 + 3B^2$, fiat $A = 6\mu + 2\epsilon_1$, $B = 2\nu + \epsilon_2$, designantibus literis ϵ_1, ϵ_2 unitates, quarum altera ipsa æquatione definitur, altera ita capiatur ut $\mu + \nu$ par evadat. Quibus rite animadversis, habebuntur æquationes

$$4\lambda = (3\epsilon_2 B - \epsilon_1 A)^2 + 3(\epsilon_1 A + \epsilon_2 B)^2,$$

$$3\epsilon_2 B_1 - \epsilon_1 A = (-1)^i a_{i+1} = (-1)^{i+1} a_{i+2},\ \equiv 1, \text{ mod } 12$$

$$[\epsilon_1 A + \epsilon_2 B] = (-1)^{i+1} b_{i+1};$$

nam $(-1)^{i+1} b_{i+1}$ certo numerus positivus est, cum debeat esse

$$[\phi_i] = \left[\frac{-b_{i+1} + (-1)^{i+1} 2\sqrt{\frac{\lambda}{3}}}{a_{i+2}}\right] > \frac{1}{\sqrt{3}}.$$

Hinc nanciscimur theorema:

'Designante $\lambda = A^2 + 3B^2$ numerum primum formæ $12n + 7$, resolubilia sunt hæc tria æquationum paria

$$x^2 - 3\lambda y^2 = 3\epsilon_2 B - \epsilon_1 A;\ = 2\epsilon_1 A;\ = -3\epsilon_2 B - \epsilon_1 A;$$

$$\lambda x^2 - 3y^2 = 3\epsilon_2 B - \epsilon_1 A;\ = 2\epsilon_1 A;\ = -3\epsilon_2 B - \epsilon_1 A;$$

cujus veritas facile colligitur ex æquivalentia formarum $(-4\lambda, 0, 3)$, $(-12\lambda, 0, 1)$, $(P_{i+1}, Q_{i+1}, P_{i+2})$.

Tres isti numeri $[3\epsilon_2 B - \epsilon_1 A]$, $[2\epsilon_1 A]$, $[3\epsilon_2 B + \epsilon_1 A]$, qui repræsentant totidem valores indeterminati x in æquatione

$$4\lambda = x^2 + 3y^2,$$

eo etiam nomine nobis memorandi sunt, quod, evoluta radice $\sqrt{3\lambda}$ in fractionem continuam vulgarem, inter denominatores quotientium completorum uno tractu veniunt. Atque, si est $A > 3B$, erit $2A$ medius, si vero $A < 3B$, erit idem vel primus vel postremus. Nam, si $A > 3B$, colligimus æquationes

$$3B - A - 2(3B + A)x + 2Ax^2 = 0,$$
$$2A - 2(3B - A)x - (3B + A)x^2 = 0$$

alteram alteram excipere in periodo æquationis $-3\lambda + x^2 = 0$; utramque enim æquationi $-3\lambda + x^2 = 0$ æquivalere per demonstrationem theorematis præcedentis evincitur; utraque autem characterem æquationis periodicæ habet propter inæqualitatem $A > 3B$. Similiter, si $A < 3B$, æquationes

$$-2A - 2(3B - A)x + (3B + A)x^2 = 0$$
$$(3B + A) + 4Ax - (3B - A)x^2 = 0$$

deinceps occurrunt in eodem periodo. Hinc nanciscimur methodum non inelegantem solvendi æquationem $\lambda = A^2 + 3B^2$. Namque in evolvenda radice $\sqrt{3\lambda}$, inveniemus tres juxta denominatores, quarum medius æquatur extremorum summæ, idemque exsuperat radicem $\sqrt{3\lambda}$; ex his unus erit $2A$, reliqui duo habebunt valores impares $[3B \pm A]$.

Exemplum. Sit $\lambda = 139 = 8^2 + 3 \cdot 5^2$;

$$4 \times 139 = 16^2 + 3 \cdot 10^2 = (-23)^2 + 3 \cdot 3^2 = 7^2 + 3 \cdot 13^2.$$

Hic est $\epsilon_1 A = 8$, $3\epsilon_2 B - \epsilon_1 A = -23$, $-3\epsilon_2 B - \epsilon_1 A = 7$. Atque erit ex evolutione nostra, cum sit $I2 \wedge \sqrt{\frac{\lambda}{3}} = 14$,

$$(32, -14, 1), (1, 14, 32), (32, -18, 13), (13, 8, -28),$$
$$(-28, -20, -23), (-23, 3, 23), \ldots$$

Unde fit $4 \times 139 = (-23)^2 + 3 \cdot 3^2$;

$$14 + 2\sqrt{\tfrac{139}{3}} = [\,28, -3, -2, 3, -1, -3, 1, -6, -1, 84,$$
$$-1, -6, 1, -3, -1, 3, -2, -3].$$

Contra ex evolutione vulgari æquationis

$$-17 - 40x + x^2 = 0,$$

orientur hujusmodi periodus

$$(-17, -20, 1),\quad (1, 20, -17),\quad (-17, -14, 13),$$
$$(13, 12, -21),\quad (-21, -9, 16),\quad (16, 7, -23),$$
$$(-23, -16, 7),\quad (7, 19, -8),\quad (-8, -13, 31),$$
$$(31, 18, -3),\quad (3, -18, 31),\ldots$$

in qua habemus

$$20+\sqrt{417}=[\dot{4}0, 2, 2, 1, 1, 1, 5, 4, 1, 12, 1, 4, 5, 1, 1, 1, 2, \dot{2}];$$

videmus autem numeros $16, -23, 7$, inter coefficientes æquationum periodicarum comparere, idque accidere fere vergente ad finem primo periodi quadrante.

11. Numeri primi formæ $12n+11$ similes aliquatenus proprietates habent. Namque ex evolutione vulgari

$$b+\sqrt{\frac{\lambda}{3}}=[\dot{2}b, \mu_j, \mu_{j-1}, \ldots, \mu_1, 2a, \mu_1, \ldots, \dot{\mu}_j], \quad . \quad . \quad . \quad . \quad . \quad (A)$$

elicimus æquationem

$$-(a, \mu_1, \ldots, \mu_j, b)\times(\mu_1, \ldots, \mu_j)=-3(\mu_1, \mu_2, \ldots, \mu_j, b^2+1).$$

Quæ cum æquatione $P_1P_2=-3R_1^2+1$ ita comparetur ut fiat

$$\begin{aligned}
P_1&=(a, \mu_1, \ldots, \mu_j, b) &&=\epsilon(k_0, -3k_1, \ldots, k_1, -3k_0),\\
P_2&=-(\mu_1, \ldots, \mu_j) &&=\epsilon(-3k_1, k_2, \ldots, -3k_2, k_1),\\
R_1&=-(\mu_1, \mu_2, \ldots, \mu_j, b) &&=\epsilon(k_0, 3k_1, \ldots, -3k_2, k_1),\\
-3R_1&=(a, \mu_1, \mu_2, \ldots, \mu_j) &&=\epsilon(-3k_1, k_2, \ldots, k_1, -3k_0).
\end{aligned}$$

Quibus positis, facile perspicitur in evolutione (A) quotientium seriem

$$b, \mu_j, \mu_{j-1}, \ldots, \mu_2, \mu_1, a$$

cum hac serie

$$k_0, -3k_1, k_2, \ldots, k_1, -3k_0$$

sine fraude commutari posse. Erit itaque

$$k_0+\sqrt{\frac{\lambda}{3}}=[2\dot{k}_0, -3k_1, k_2, \ldots, -3k_2, k_1, -6k_0, k_1, -3k_2, k_2, -3.k_1]\ . \quad (B)$$

Qua in evolutione altera ex æquationibus, quæ medium locum in prima periodi semisse tenent, dabit aut hanc æquationem

$$3\lambda=Q_{i+1}^2-3P_{i+2}^2,$$

aut hanc

$$3\lambda=Q_{i+1}^2, -P_{i+1}^2,$$

prout scilicet numerus i par est aut impar. Ponimus autem eodem fere quo supra modo

$$Q_s = 3b_s, \quad P_{2s} = -a_{2s}, \quad P_{2s+1} = 3a_{2s+1},$$

$$a_0 = \lambda - 3k_0^2, \quad b_0 = -k_0 = -I\sqrt{\frac{\lambda}{3}}, \quad a_1 = 1;$$

et calculum periodi instituimus secundum hoc schema

$$b_1 = 2k_0a_1 + b_0; \qquad a_2 = \frac{\lambda - 2b_1^2}{a_1}; \quad k_1 = I\frac{-b_1 - \sqrt{\frac{\lambda}{3}}}{a_2};$$

$$b_2 = k_1a_2 + b_1; \qquad a_3 = \frac{\lambda - 3b_2^2}{a_2}; \quad k_3 = I\frac{-b_2 + \sqrt{\frac{\lambda}{3}}}{a_3};$$

. . . . . .

$$b_s = k_{s-1}a_s + b_{s-1}; \quad a_{s+1} = \frac{\lambda - 3b_s^2}{a_s}; \quad k_0 = I\frac{-b_s + (-1)^s\sqrt{\frac{\lambda}{3}}}{a_{s+1}}$$

. . . . . .

vel ex notatione Gaussiana

$$b_1 \equiv b_0, \quad \mod a_1; \quad a_2 = \frac{\lambda - 3b_1^2}{a_1};$$

$$b_2 \equiv b_1, \quad \mod a_2; \quad a_3 = \frac{\lambda - 3b_2^2}{a_2};$$

. . . . . .

$$b_s \equiv b_{s-1}, \quad \mod a_s; \quad a_{s+1} = \frac{\lambda - 3b_s^2}{a_s};$$

. . . . . .

numeris b_s ita determinaris ut quam proxime ad valorem radicis $(-1)^s\sqrt{\frac{\lambda}{3}}$ accedant. Cui calculo finis faciendus est cum primum æquatio $a_{i+1} + a_{i+2} = 0$ sese obtulerit; eo enim usque valet formula $k_s = I\theta_s$, quæ ulterius progredientem destituit. Periodi autem numerorum a, b, k, eandem plane descriptionem habent, quam supra exemplis illustravimus.

Quum autem in æquatione $\lambda = 3b_{i+1}^2 - a_{i+1}^2$ numerus a_{i+1} par esse nequeat, evenire potest ut repræsentatio ipsius λ, quam ejus æquationis ope nanciscimur, non sit omnium simplicissima. Est tamen vel ipsa omnium simplicissima, vel proxima post simplicissimam. Habemus enim (Art. 4)

$$[\phi_i] = \left[\frac{-b_{i+1} + (-1)^{i+1}\sqrt{\frac{\lambda}{3}}}{a_{i+1}}\right] > 1;$$

hoc est,
$$\sqrt{\lambda + a_{i+1}^2} + \sqrt{\lambda} > [a_{i+1}\sqrt{3}],$$
ideoque etiam
$$[a_{i+1}] < \sqrt{3\lambda}, \quad [b_{i+1}] < 2\sqrt{\frac{\lambda}{3}} \quad [a_{i+1}b_{i+1}] < 2\lambda.$$

At in æquatione $t^2 - 3u^2 = 1$, valores numeri n sic procedunt, 1, 4, 15, ..., unde concludimus, si fuerit $[b_{i+1}] < [a_{i+1}]$, unam fore eamque solam repræsentationem quæ per numeros minores fiat; quæ si est $\lambda = 3x^2 - y^2$, erit y par, x impar. Itaque si A et B minimi numeri exstant, per quos æquationi $\lambda = 12B^2 - A^2$ satisfieri possit, æquationes

$$x^2 - 3\lambda y^2 = \epsilon A, \quad = 3\epsilon A, \quad = 4(\epsilon A \pm 3B),$$
$$3x^2 - \lambda y^2 = \epsilon A, \quad = 3\epsilon A, \quad = 4(\epsilon A \pm 3B),$$

resolubiles erunt, signo unitatis ϵ ita determinato ut fiat $\epsilon A \equiv 1$, mod 3. Facile autem perspicitur æquationi

$$-4(3B - A) - 2(6B - A)x + Ax^2 = 0$$

competere characterem æquationis periodicæ: cum sit

$$[A] < 3[B], \quad [B] > \tfrac{1}{2}\sqrt{\frac{\lambda}{3}}.$$

Quare vulgaris quoque evolutio radicis $\sqrt{3\lambda}$ suppeditat solutionem simplicissimam æquationis $\lambda = 12B^2 - A^2$; veniemus enim in ea evolutione ad æquationem $p_s + 2q_s x + p_{s+1}x^2 = 0$ in qua erit $\frac{1}{2}p_s = q_s + p_{s+1}$, ideoque etiam,

$$q_s \equiv p_{s+1}, \text{ mod } 6, \quad \lambda = 12\left(\frac{p_{s+1} - q_s}{6}\right)^2 - p_{s+1}^2.$$

Exemplum. Sit $\lambda = 167$; erit $I\sqrt{\frac{\lambda}{3}} = 7$; atque habebimus ex evolutione nostra hanc periodi formam

(−20, 7, 1), (1, 7, −20), (−20, −13, −17),
(−17, 4, 7), (7, −3, −20), (−20, 17, −35),
(−35, −18, −23), (−23, 5, 4), (4, −7, −5),
(5, 8, −5), ...

Erit itaque $167 = 3 \cdot 8^2 - 5^2$;

$$7 + \sqrt{\tfrac{167}{3}} = [\dot{1}4, -3, -1, 3, -1, -3, -1, 9, 3, -9, -3,$$
$$3, -1, 3, -1, 3, 1, -42, 1, 3, -1, 3, 1, 3,$$
$$-3, -9, 3, 9, -1, 3, -1, 3, -1, -\dot{3}].$$

Evoluta autem radice $\sqrt{501}$ secundum methodum vulgatam invenimus periodum æquationum hancce

$$\begin{array}{lll} (-17, -22, 1), & (1, 22, -17), & (-17, -12, 21), \\ (21, 9, -20), & (-20, -11, 19), & (19, 8, -23), \\ (-23, -15, 12), & (12, 21, -5), & (-5, -19, 28), \\ (28, 9, -15), & (-15, -21, 4), & (4, 19, 35), \\ (-35, -16, 7), & (7, 19, -20), & (-20, -21, 3), \\ (3, 21, -20)... & & \end{array}$$

e quibus nona suppeditat solutionem æquationis $167 = 3 \times 8^2 - 5^2$, cum sit $\frac{1}{2} \times 28 = 19 - 5$. Erit præterea $22 + \sqrt{501} =$

$$[\dot{44}, 2, 1, 1, 1, 1, 3, 8, 1, 2, 10, 1, 5, 2, 14, 2, 5, 1, 10, 2, 1, 8, 3, 1, 1, 1, 1, \dot{2}].$$

12. Æquationes $P_1 P_2 = 2R_1^2 + 1$, $P_1 P_2 = -2R_1^2 + 1$, et repræsentationes inde orientes numerorum primorum per formas $2x^2 + y^2$, $2x^2 - y^2$, a Goepelio in commentatione inaugurali* longe pulcherrima (ex qua disquisitionem nostram delibatam esse libenter fatemur) tanta felicitate ope fractionum continuarum illustratæ sunt, ut eam rem iterum attingere prope supervacaneum sit. Tamen ne quis analogiam determinantium 2 et 3 a nobis prætermissam desiderit, in vestigiis ejus viri paullisper immorari liceat. Et primum quidem adnotamus, ab æquatione $P_1 P_2 = 2R_1^2 + 1$, eodem fere quo antea modo, perveniri posse ad æquationem

$$P_1 = (k_1, 2k_2, k_3, \ldots, 2k_3, k_2, 2k_1);$$

eamque transformationem ita fieri posse ut quotientem unitati æqualem excipiat ejusdem signi quotiens. Ex quo apparet in fractione continua

$$\frac{P_1}{2R_1} = [k_1, 2k_2, \ldots, k_2, 2k_1];$$

sese ob[illegible]quationes $k_1 = I\theta$, $k_2 = I\theta_2$, ... eamque legem etiam in extremo quotiente destituit. [illegible] nisi forte fiant signa quotientium k_{i-1} et θ_i contraria, ac præterea habent, quam [illegible]n casu erit $I\theta_i = \pm 2$, $k_i = \pm 1$. Nec secus in fractione continua

Quum autem [illegible]quatione $P_1 P_2 = -2R_1^2 + 1$, erit

evenire potest ut [illegible]. $P_1 = (k_1, -2k_2, k_3, \ldots, -2k_3, k_2, -2k_1)$,

non sit omnium sim[illegible] proxima post simplicis- $\frac{P_1}{2R_1} = [k_1, -2k_2, k_3, \ldots, k_2, -2k_1]$,

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$[\phi_i] =$ [illegible]eine und angewandte Mathematik,' vol. xlv. pp. 1–13.

$k_1 = I\theta_1$, $k_2 = I\theta_2$, ... atque adeo $k_i = I\theta_i$, cum sit utique $\phi_i > \frac{1}{2}(2+\sqrt{2})$: quæ res item docet in neutra formarum æquivalentium $x^2 - 2y^2$, $2x^2 - y^2$, dari repræsentationem numeri P_1 repræsentatione nostra magis simplicem. Tum vero, designante λ numerum primum formæ $4n+3$, vel duplum talis numeri, atque evoluta radice $\sqrt{\lambda}$ secundum præcepta vulgata erit in media periodo hujusmodi æquatio anceps

$$-\tfrac{1}{2}(\lambda - b^2) - 2bx + 2x^2 = 0$$

(ea enim sola est quæ exsistere potest), ubi numerus b ita determinatur ut $\frac{1}{2}(\lambda - b^2)$ integer atque positivus, idemque quam minimus fiat. Sit a numerus integer surdo $\sqrt{\lambda}$ proxime minor; erit aut $a = b$, aut $a = b+1$. Atque si est $a = b$, habebimus hujusmodi evolutionem

$$\sqrt{\lambda} = [a, \mu_1, \mu_2, \ldots. \mu_n, \quad \tfrac{1}{2}a + \tfrac{1}{2}\sqrt{\lambda}], \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

unde nanciscimur æquationes

$$(\mu_1, \mu_2, \ldots, \mu_n, \tfrac{1}{2}a) = \tfrac{1}{2}(a, \mu_1, \mu_2, \ldots, \mu_n),$$
$$(a, \mu_1, \mu_2, \ldots, \mu_n, \tfrac{1}{2}a) = \tfrac{1}{2}\lambda(\mu_1, \mu_2, \ldots, \mu_n);$$

e quibus concludimus fore

$$(a, \mu_1, \mu_2, \ldots, \mu_n)^2 - \lambda(\mu_1, \mu_2, \ldots, \mu_n)^2 = (-1)^{n+1}2,$$
$$(\mu_1, \mu_2, \ldots, \mu_n)(\mu_2, \ldots, \mu_{n-1}) = 2(\mu_1, \mu_2, \ldots, \mu_{n-1})^2 + (-1)^n.$$

Itaque, si fiat $\epsilon = (-1)^n$, erit $\epsilon = +1$, vel $\epsilon = -1$, prout λ (vel $\frac{1}{2}\lambda$) est formæ $8n+3$ aut formæ $8n+7$; atque per transformationem jam sæpius in hac commentatione usitatam habebimus hujusmodi evolutionem

$$a + \sqrt{\lambda} = [2\dot{a}, k_1, 2\epsilon k_2, \ldots, k_2, 2\epsilon k_1, \epsilon a, 2\epsilon k_1, k_2, \ldots, 2\epsilon k_2, \dot{k}_1]$$

cujus veritas facili confirmatur, si formulam ex evolutione vulgari oriundam paullulum immutatam adhibemus,

$$a + \sqrt{\lambda} = [\dot{a}, 0, a, \mu_1, \mu_2, \ldots, \mu_n, \tfrac{1}{2}a, 0, \tfrac{1}{2}a, \mu_n, \mu_{n-2}, \ldots, \mu_2, \dot{\mu}_1].$$

Si autem est $a = b+1$, erit

$$\sqrt{\lambda} = [a, \mu_1, \mu_2, \ldots, \mu_n, \tfrac{1}{2}(a - 1 + \sqrt{\lambda})],$$

quæ æquatio, si sic scribatur,

$$\sqrt{\lambda} = [a+1, 0, -1, \mu_1, \ldots, \mu_n, -1, 0, \tfrac{1}{2}(a+1+\sqrt{\lambda})],$$

abit in formam precedenti (A) similem, atque eodem prorsus modo tractari potest. Itaque habebitur evolutio ad hanc normam concinnata

$$a+1+\sqrt{\lambda} = [2(\dot{a}+1), k_1, 2\epsilon k_2, \ldots, k_2, 2\epsilon k_1, \epsilon(a+1), 2\epsilon k_1, k_2, \ldots, 2\epsilon k_2, \dot{k}_1]$$

in qua observandum est quotientem k_1 negativum fore.

Schema autem calculi, quo ad has evolutiones utimus, hoc ferme est. Sit $a_1 = 1$, $2\epsilon a_0 = \beta_0^2 - \lambda$, numero negativo β_0 ita determinato ut a_0 integer atque idem quam minimus fiat. Tum sumpto ab æquatione

$$2\epsilon a_0 + 2\beta_0 x + a_1 x^2 = 0$$

initio cæteras hoc pacto derivamus

$$\beta_1 \equiv \beta_0, \mod 2a_1; \qquad a_2 = \frac{\lambda - \beta_1^2}{2\epsilon a_1};$$

$$\beta_2 \equiv \beta_1, \mod 2a_2; \qquad a_3 = \frac{\lambda - \beta_2}{2\epsilon a_2};$$

. . . . . .

$$\beta_s \equiv \beta_{s-1}, \mod 2a_s; \qquad a_{s+1} = \frac{\lambda - \beta_s^2}{2\epsilon a_s};$$

. . . . . .

ubi numeros impares β_s ad valorem radicis $(-1)^s\sqrt{\lambda}$ quam proxime accedere, quotientes autem k_s per æquationes $k_s = \frac{\beta_s - \beta_{s-1}}{2\epsilon a_s}$ determinari intelligendum est. Finis autem calculo faciendus est simul ac pervenerimus ad æquationem in qua est $a_{i+1} + a_{i+2} = 0$, quæque adeo suppeditat discerptionem quæsitam. Quando autem $\epsilon = +1$, hoc est quando λ, vel $\frac{1}{2}\lambda$, est numerus primus formæ $8n+3$, fieri potest ut regula generalis in ultimo quotiente k_i falsa sit; quæ exceptio tum locum habet cum in discerptione quæsita $\lambda = A^2 + 2B^2$ numerus B major est quam $2A$. Quo in casu erit in penultima æquatione $a_i \pm 2\beta_i + 3a_{i+1} = 0$; atque, ut habeatur ad extremum $a_{i+1} + a_{i+2} = 0$, quotienti k_i tribuendus erit valor ± 1, etiamsi fiat $I\theta_i = \pm 2$.

Exemplis brevitatis causa supersedemus; illud unum adjicimus, pro numeris primis formæ $8n+3$, eorumque duplis æquationes

$$x^2 - \lambda y^2 = A, \; = -2A, \; = -A \pm 2B,$$

esse resolubiles, si quidem ponatur $\lambda = 2A^2 + B^2$, et numerus A eo signo afficiatur, ut fiat $(-1)^{\frac{1}{2}(A-1)} = 1$, vel $(-1)^{\frac{1}{2}(A-1)+\frac{1}{8}(A^2-1)} = 1$, prout est ipse λ numerus primus, vel duplum numeri primi: quæ quidem conditiones ex ipsa æquationum forma oriuntur. Et similiter pro numeris primis formae $8n+7$, eorumque duplis, æquationes

$$x^2 - \lambda y^2 = A, \; = 2A_1 = 3A \pm 2B,$$

resolvi poterunt, designantibus literis A et B minimos numeros qui æquationi $\lambda = B^2 - 2A^2$ satisfaciunt, et determinato ut supra ipsius A signo.

13. Coronidis loco observamus eandem fere fractionum continuarum transformationem utilitate non carere in theoria numerorum complexorum. Sit enim

$$\mu = a + bi, \quad \mu' = a - bi, \quad \mu\mu' = a^2 + b^2 = N.\mu = N.\mu';$$

habebimus hujusmodi æquationem

$$\begin{aligned} K &= (\mu_1, \mu'_2, \mu_3, \ldots, \mu'_{2n}, \mu_{2n}, \ldots, \mu'_3, \mu_2, \mu'_1) \\ &= (\mu_1, \mu'_2, \ldots, \mu'_{2n}) \times (\mu_{2n}, \ldots, \mu_2, \mu'_1) \\ &\quad + (\mu_1, \mu'_2, \ldots, \mu'_{2n-1}) \times (\mu'_{2n-1}, \ldots, \mu_2, \mu'_1); \end{aligned}$$

unde sequitur determinantem K summam esse duarum, quas vocant, normarum, hoc est, quattuor quadratorum. Ponatur ergo

$$P_1 P_2 = 1 + R_1 R'_1;$$

designantibus literis P_1, P_2 numeros positivos reales, R_1, R'_1 numeros complexos conjugatos; atque in hac æquatione fiat

$$R_2 = R_1 - \mu_1 P_2, \quad P_3 = \frac{1 + R_2 R'_2}{P_2},$$

et sic porro, numeris R_s ita determinatis ut norma $R_s \times R'_s$ evadat quam minima. Hinc erit $P_1 > P_2 > P_3, \ldots$ atque ad extremum habebitur

$$P_1 = (\mu_1, \mu'_2, \mu_3, \ldots, \mu'_3, \mu_2, \mu'_1);$$

quæ formula, cum universi numeri primi formulam $1 + x^2 + y^2$ metiantur, continet demonstrationem theorematis Fermatiani omnes omnino numeros esse summas quattuor quadratorum.

Plane autem eodem modo evincitur omnes numeros habere hanc formam $x^2 + y^2 + 3(u^2 + v^2)$, atque præterea hanc etiam $x^2 + 2y^2 + 3u^2 + 6v^2$. Quarum enuntiationum altera intra veritatem cadit, cum omnes numeri in alterutra harum formularum

$$x^2 + y^2 + 3u^2, \quad x^2 + 3u^2 + 3v^2$$

comprehendantur; id quod a viro clarissimo Lejeune Dirichlet* jampridem demonstratum est: altera ea ipsa est quam olim summus in hac disciplina magister C. G. J. Jacobi† ex notissima formula elliptica originem trahere indicavit. Hoc autem loco satis erit demonstrare, proposita æquatione $\rho^2 + \rho + 1 = 0$, omnem numerum realem esse summam normæ simplicis et duplicis.

Sint igitur A, B, P numeri reales, $A - B\rho$ numerus complexus integer continens radicem æquationis $\rho^2 + \rho + 1 = 0$; erit

$$A^2 + AB + B^2 = N.(A - B\rho).$$

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* 'Journal für die reine und angewandte Mathematik,' vol. xl. pp. 231–232. † Ibid. vol. xxi.

Quod si ponatur $$A - B\rho = \mu P + a - b\rho,$$

numerus complexus μ ita determinari potest ut sit

$$N.(a - b\rho) < \tfrac{1}{2}(P^2 - 1).$$

Fiat enim, quod utique licet, $a \equiv A$, $b \equiv B$, mod P, $[a] \leqq \frac{1}{2}P$, $[b] \leqq \frac{1}{2}P$. Hic, si signa numerorum a et b contraria sunt, habebimus manifesto

$$N.(a - b\rho) < \tfrac{1}{4}P^2 < \tfrac{1}{2}(P^2 - 1).$$

Si vero hæc signa conveniunt inter se, atque est præterea uterque numerus a et b minor quam $\frac{P-1}{\sqrt{6}}$, erit iterum $N(a - b\rho) < \frac{1}{2}(P-1)^2 < \frac{1}{2}(P^2 - 1)$. Quod si alter eorum, velut a, eum limitem superaverit, faciendum erit $a' = a \pm P$, $[a'] < P$, ita ut habeatur, si quidem $P \geqq 5$,

$$N.(a' - b\rho) < \tfrac{1}{6}(P\sqrt{6} - P + 1)^2 < \tfrac{1}{2}(P^2 - 1);$$

experiendo autem invenitur eadem lege teneri numeros quinario minores.

Hinc patet calculum quo sæpius jam usi sumus accommodari posse ad æquationem

$$P_1 P_2 = 1 + 2(A_1^2 + A_1 B_1 + B_1^2);$$

in qua si fiat

$$A_2 - B_2\rho = A_1 - B_1\rho - \mu P_2,$$

$$P_3 = \frac{1 + 2(A_2^2 + A_2 B_2 + B_2^2)}{P_2},$$

et sic porro, ea lege ut normæ $A^2 + AB + B^2$ semper quam minimæ evadant, veniemus tandem ad æquationem

$$P_1 = (\mu_1, 2\mu_2', \mu_3', \ldots, 2\mu_3', \mu_2, 2\mu_1'),$$

quæ continet demonstrationem theorematis, cum omnis numerus impar metiatur formulam $1 + 2x^2 + 6y^2$, vel, si mavis, formulam

$$1 + 2(x^2 + xy + y^2).$$

Simile fere theorema oritur ab æquatione $\sigma^2 + \sigma + 2 = 0$. Universi enim numeri primi, præter septenarium, metiuntur formulam $1 + x^2 + 7y^2$, hoc est, formulam $1 + x^2 + xy + 2y^2$: ac præterea, posito

$$P_1 P_2 = 1 + A_1^2 + A_1 B_1 + 2B_1^2,$$

utique fieri potest

$$P_1^2 > A_1^2 + A_1 B_1 + 2B_1^2 > P_2^2.$$

Unde patet numerum quemcumque integrum summam esse duarum normarum; vel, quod idem est, in hac formula

$$x^2 + xy + 2y^2 + u^2 + uv + 2v^2$$

comprehendi. Omnes itaque numeri pariter pares in hac forma continentur

$$x^2 + y^2 + 7u^2 + 7v^2;$$

impares autem, atque impariter pares aut in illa aut in hac certe

$$x^2 + 2y^2 + 7(u^2 + 2v^2).$$

XL.

ON SOME DISCONTINUOUS SERIES CONSIDERED BY RIEMANN.

[Messenger of Mathematics, Ser. ii. vol. xi. pp. 1–11 (May 1881)].

RIEMANN, in a fragment published after his death in his collected works*, has determined the value of certain q series in the limiting case in which the analytical modulus of q is unity. If $q=\rho e^{i\theta}$, the series considered by Riemann contain θ only, and converge and diverge for an infinite number of values of θ between any two limits however near to one another; they appear, indeed, to have attracted Riemann's attention by this peculiarity. Two of these series, which may be regarded as examples of the rest, are considered in the present note, and some details relating to them are supplied which are omitted in Riemann's brief record of his results.

Let
$$S(q)=4\Sigma\frac{(-1)^n}{n}\frac{q^n}{1+q^n}, \quad \ldots\ldots\ldots \quad \text{(i)}$$

n denoting any positive integral number from $+1$ to ∞, so that ('Fundamenta Nova,' p. 103)

$$S(q)=\log\frac{k}{4\sqrt{q}}.$$

Put $q=\rho e^{i\theta}$, multiply each side of (i) by $\frac{dq}{q}=\frac{d\rho}{\rho}$, and take the rectilineal integral from 0 to $\rho e^{i\theta}$ along the vector $e^{i\theta}$; we have

$$\int_0^{\rho e^{i\theta}} S(q)\frac{dq}{q}=4\Sigma\frac{(-1)^n}{n^2}\log(1+\rho^n e^{ni\theta}), \quad \ldots\ldots\ldots \quad \text{(ii)}$$

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* 'Mathematische Werke,' p. 427.

the logarithms on the right being so taken as to vanish with ρ; or, which is the same thing,

$$\int_0^{\rho e^{i\theta}} S(q)\frac{dq}{q} = 4\Sigma\frac{(-1)^n \log \Delta_n}{n^2} + 4i\Sigma\frac{(-1)^n}{n^2}\tan^{-1}\left(\frac{\rho^n \sin n\theta}{1+\rho^n \cos n\theta}\right), \quad \text{(iii)}$$

where Δ_n is the arithmetical square root of

$$1 + 2\rho^n \cos n\theta + \rho^{2n},$$

$\log \Delta_n$ is a real logarithm, and $\tan^{-1}\left(\frac{\rho^n \sin n\theta}{1+\rho^n \cos n\theta}\right)$ is an angle included between the limits $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. So long as $\rho < 1$, the series $S(q)$, and the integrated series (ii) or (iii) are absolutely convergent. When $\rho = 1$, the series (i) and (iii) respectively become

$$4\Sigma\frac{(-1)^n}{n}\frac{e^{ni\theta}}{1+e^{ni\theta}} = -2\log 2 - 2\Sigma\frac{(-1)^n}{n} + 4\Sigma\frac{(-1)^n}{n}\frac{e^{\frac{1}{2}ni\theta}}{e^{\frac{1}{2}ni\theta}+e^{-\frac{1}{2}ni\theta}}$$

$$= -2\log 2 + 2i\Sigma\frac{(-1)^n}{n}\tan\tfrac{1}{2}(n\theta), \quad \text{(iv)}$$

and

$$2\Sigma\frac{(-1)^n}{n^2}\log\{4\cos^2\tfrac{1}{2}(n\theta)\} + 4i\Sigma\frac{(-1)^n}{n^2}[\tfrac{1}{2}(n\theta)]; \quad \text{(v)}$$

where by $[\frac{1}{2}(n\theta)]$ we understand the absolutely least angle which has the same tangent as $[\frac{1}{2}(n\theta)]$, except when $[\frac{1}{2}(n\theta)]$ is an uneven multiple of $\frac{1}{2}\pi$, in which case $[\frac{1}{2}(n\theta)] = 0$. The imaginary part of (v) is always convergent; the convergence or divergence of (iv) and of the real part of (v) depends on the nature of the value assigned to θ.

First, let $\frac{\theta}{\pi}$ be incommensurable. In this case the series (iv) is always divergent. For if $\frac{a'}{b'}, \frac{a}{b}$ are two consecutive convergents to $\frac{\theta}{\pi}$ of which the latter has an uneven numerator a, and if ϕ is the complete quotient immediately succeeding $\frac{a}{b}$, we have

$$\tfrac{1}{2}b\theta = \tfrac{1}{2}a\pi \pm \frac{\frac{1}{2}\pi}{b\phi + b'};$$

whence, when b is very great, the absolute value of $\frac{1}{b}\tan\frac{1}{2}b\theta$ is approximately $\frac{\phi + \frac{b'}{b}}{\frac{1}{2}\pi}$. Thus the series (iv) contains an infinite number of terms of which the

absolute value is finite, and it is consequently divergent. Riemann says that when $\frac{\theta}{\pi}$ is incommensurable, the real part of the series (v) is also always divergent. But this appears to be an oversight; for the real part of (v) is in fact divergent or convergent according as the incommensurable quantity θ does, or does not, satisfy certain conditions; and, in particular, it can be shown that, whenever θ is the incommensurable root of an equation of a finite order having integral coefficients, the real part of (v) is convergent. For each successive value of n we form the equation

$$\tfrac{1}{2}n \times \frac{\theta}{\pi} - m = \tfrac{1}{2}\epsilon(1-\nu),$$

where m is an integral number, $\epsilon = \pm 1$, and $0 < \nu < 1$; it is evident that m, ϵ, and ν can always be determined in one way and in one way only. We then have

$$\log\{4\cos^2\tfrac{1}{2}(n\theta)\} = \log\{4\sin^2\tfrac{1}{2}(\nu\pi)\},$$

and, since

$$\frac{2}{\pi} < \frac{2\sin\frac{1}{2}(\nu\pi)}{\nu\pi} < 1,$$

the series

$$\Sigma \frac{(-1)^n}{n^2} \log \frac{4\sin^2\frac{1}{2}(\nu\pi)}{\nu^2\pi^2}$$

is absolutely convergent; *i.e.* the series

$$\Sigma \frac{(-1)^n}{n^2} \log\{4\cos^2\tfrac{1}{2}(n\theta)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

is convergent or divergent according as the series

$$\Sigma(-1)^n \frac{\log\nu}{n^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii)}$$

is convergent or divergent. Let $\frac{\theta}{\pi}$ be the root of an irreducible equation

$$f(x) = Ax^s + Bx^{s-1} + \ldots + Mx + N = 0;$$

of which the coefficients $A, B, \ldots$ are integral; it may be shown as follows by a method due to M. Liouville, that the series (vii) is convergent.

Let $x_1, x_2, \ldots$ be the real roots other than $\frac{\theta}{\pi}$ of $f(x)$, $y_1 \pm iz_1$, $y_2 \pm iz_2, \ldots$ its imaginary roots; and let $\delta - 1$ be a positive quantity greater than the greatest

of the differences $\frac{\theta}{\pi}-x, \frac{\theta}{\pi}-y$; we have

$$n^s f\left(\frac{2m+\epsilon}{n}\right) = A \times \frac{\epsilon\nu}{n} \times n^s \times \Pi\left(\frac{2m+\epsilon}{n}-x\right) \times \Pi\left\{\left(\frac{2m+\epsilon}{n}-y\right)^2+z^2\right\}.$$

But $n^s f\left(\frac{2m+\epsilon}{n}\right)$ is an integral number, which cannot be zero, because $f(x)$ is irreducible; hence

$$\nu > \frac{1}{A\,\delta^{s-1} n^{s-1}},$$

or $-\log\nu < (s-1)\log n + c$, where c is a finite constant; this shows that (vii) is convergent, because $\Sigma\frac{\log n}{n^2}$ is convergent.

The same reasoning would suffice to prove, that when $\frac{\theta}{\pi}$ is a root of any equation such as $f(x)$, the series

$$\Sigma\frac{\log\{\cos^2\frac{1}{2}(n\theta)\}}{\phi(n)}$$

always converges if $\Sigma\frac{\log n}{\phi(n)}$ is absolutely convergent.

On the other hand, between any two limits however near, there is always an infinite number of values of θ, for which the series (vi), and indeed any series of the form

$$\Sigma\frac{\log\{\cos^2\frac{1}{2}(n\theta)\}}{\phi(n)} \quad \ldots\ldots\ldots \text{(viii)}$$

becomes divergent, however rapidly the positive function $\phi(n)$ may increase with n. If, for example, in the development of $\frac{\theta}{\pi}$ in a continued fraction, it happens an infinite number of times that a convergent $\frac{a}{b}$, having an uneven numerator a, is immediately followed by a complete quotient which surpasses $\frac{c}{b}\epsilon^{\phi(b)}$, where c is any finite positive constant, the series (viii) contains an infinite number of terms of which the absolute value is finite, and is therefore divergent.

Secondly, let $\frac{\theta}{\pi}=\frac{a}{b}$, a rational fraction in its lowest terms. If a is uneven, the series (iv) and (vi) contain terms which are infinite. But when a is even the two series are convergent, and Riemann has succeeded in assigning their sums in the following manner.

Writing $2a$ for a, and observing that b is uneven, and designating the sum of the series (iv) by $-2\log 2+2iS$, we have

$$S=i\Sigma_{n=1}^{n=\infty}\frac{(-1)^n}{n}\tan\left(\frac{na}{b}\pi\right)=\Sigma_{s=1}^{s=2b}\Sigma_{t=0}^{t=\infty}\frac{(-1)^s}{2bt+s}\tan\left(\frac{sa}{b}\pi\right)$$

$$=i\Sigma_{s=1}^{s=2b}\tan\left(\frac{sa}{b}\pi\right)\times\int_0^1\frac{x^{s-1}dx}{1-x^{2b}}.$$

But if $\gamma=e^{\frac{i\pi}{b}}$,

$$i\tan\left(\frac{sa\pi}{b}\right)=\frac{\gamma^{2as}-1}{\gamma^{2as}+1}=\Sigma_{\sigma=1}^{\sigma=b-1}(-1)^{\sigma-1}\gamma^{2a\sigma s},$$

and

$$\int_0^1\frac{x^{s-1}dx}{1-x^{2b}}=\frac{1}{2b}\Sigma_{\tau=1}^{\tau=2b}\gamma^{-\tau s}\int_0^1\frac{\gamma^{\tau}dx}{1-\gamma^{\tau}x};$$

whence

$$S=\frac{1}{2b}\Sigma_{\sigma=1}^{\sigma=b-1}(-1)^{\sigma-1}\Sigma_{\tau=1}^{\tau=2b}\int_0^1\frac{\gamma^{\tau}dx}{1-\gamma^{\tau}x}\Sigma_{s=1}^{s=2b}(-1)^s\gamma^{s(2a\sigma-\tau)}.$$

But

$$\Sigma_{s=1}^{s=2b}(-1)^s\gamma^{s(2a\sigma-\tau)}=2b \text{ or } 0,$$

according as the congruence $\tau\equiv 2a\sigma+b$, mod $2b$, is or is not satisfied. We have therefore

$$S=\Sigma_{\sigma=1}^{\sigma=b-1}(-1)^{\sigma}\int_0^1\frac{\gamma^{2a\sigma}dx}{1+\gamma^{2a\sigma}x}$$

$$=\Sigma_{\sigma=1}^{\sigma=\frac{1}{2}(b-1)}(-1)^{\sigma}\int_0^1\left(\frac{\gamma^{2a\sigma}}{1+\gamma^{2a\sigma}x}-\frac{\gamma^{-2a\sigma}}{1+\gamma^{-2a\sigma}x}\right)dx.$$

But

$$\int_0^1\frac{\sin\theta\,dx}{1+2x\cos\theta+x^2}=\left[\tfrac{1}{2}\theta\right],$$

and hence, finally,

$$S=2i\pi\Sigma_{\sigma=1}^{\sigma=\frac{1}{2}(b-1)}(-1)^{\sigma}\left[\sigma\frac{a}{b}\right]$$

$$=i\pi\Sigma_{\sigma=1}^{\sigma=b-1}(-1)^{\sigma}\left[\sigma\frac{a}{b}\right],$$

the symbol $\left[\sigma\frac{a}{b}\right]$ denoting the excess (positive or negative) of $\sigma\frac{a}{b}$ above the integral number lying absolutely nearest to $\sigma\frac{a}{b}$.

The method employed by Riemann to effect the preceding summation is due to Lejeune Dirichlet, and was employed by him in his memoir on the Arithmetical Progression ('Transactions of the Academy of Berlin for 1837,' sections 4 and 10), and in his 'Recherches sur diverses applications de l'analyse infinitesimale à la théorie des nombres ('Crelle's Journal,' vols. xix. and xxi; see sections 1, 9 and 10 of the Memoir).

Again, denoting by S' the sum of the series

$$\Sigma_{n=1}^{n=\infty} \frac{(-1)^n}{n^2} \log \{4 \cos^2 \tfrac{1}{2}(n\theta)\},$$

we have, if $\theta = \frac{2a}{b}\pi$,

$$S' = 2 \log 2 \times \Sigma \frac{(-1)^n}{n^2} \times \frac{1}{b^2} + \Sigma_{s=1}^{s=b-1} (-1)^s \log \left(4 \cos^2 s \frac{a}{b} \pi\right) \times \Sigma_{t=0}^{t=\infty} \frac{(-1)^t}{(bt+s)^2}$$

$$= -\frac{\pi^2 \log 2}{6b^2} + \Sigma_{s=1}^{s=\frac{1}{2}(b-1)} (-1)^s \log \left(4 \cos^2 \sigma \frac{a}{b} \pi\right) \times \Sigma_{t=-\infty}^{t=+\infty} \frac{(-1)^t}{(bt+s)^2}.$$

But, by a formula due to Euler,

$$\Sigma_{t=-\infty}^{t=+\infty} \frac{(-1)^t}{(bt+s)^2} = \left[\frac{1}{s^2} + \frac{1}{(2b+s)^2} + \frac{1}{(2b-s)^2} + \frac{1}{(4b+s)^2} + \frac{1}{(4b-s)^2} + \ldots\right]$$

$$- \left[\frac{1}{(b+s)^2} + \frac{1}{(b-s)^2} + \frac{1}{(3b+s)^2} + \frac{1}{(3b-s)^2} + \ldots\right]$$

$$= \frac{\pi^2}{4b^2} \operatorname{cosec}^2 \left(\frac{s\pi}{2b}\right) - \frac{\pi^2}{4b^2} \sec^2 \left(\frac{s\pi}{2b}\right) = \frac{\pi^2}{b_2} \frac{\cos\left(\frac{s\pi}{b}\right)}{\sin^2\left(\frac{s\pi}{b}\right)};$$

whence, finally,

$$S' = -\frac{\pi^2}{6b^2} \log 2 + \frac{\pi^2}{b^2} \Sigma_{s=1}^{s=\frac{1}{2}(b-1)} (-1)^s \log \left(4 \cos^2 s \frac{a}{b} \pi\right) \times \frac{\cos\left(\frac{s\pi}{b}\right)}{\sin^2\left(\frac{s\pi}{b}\right)}$$

$$= -\frac{\pi^2}{6b^2} \log 2 + \frac{\pi^2}{2b^2} \Sigma_{s=1}^{s=b-1} (-1)^s \log \left(4 \cos^2 s \frac{a}{b} \pi\right) \times \frac{\cos\left(\frac{s\pi}{b}\right)}{\sin^2\left(\frac{s\pi}{b}\right)}.$$

When the series

$$\Sigma \frac{(-1)^n}{n^2} \log \{4 \cos^2 \tfrac{1}{2}(n\theta)\}$$

is divergent, we must regard the equation

$$4 \int_0^1 \Sigma \frac{(-1)^n}{n} \frac{\rho^n e^{ni\theta}}{1 + \rho^n e^{ni\theta}} \times \frac{d\rho}{\rho}$$

$$= 2\Sigma \frac{(-1)^n}{n^2} \log \{4 \cos^2 \tfrac{1}{2}(n\theta)\} + 4i \Sigma \frac{(-1)^n}{n^2} [\tfrac{1}{2}(n\theta)] \qquad \text{(ix)}$$

as having no assignable meaning. When that series is convergent, the equation

always has a meaning, but some explanation is required to show clearly what its meaning is. As to the signification of the left-hand member no question can arise. For just as by $\int_0^\infty \phi(x)\,dx$ we understand the limit of $\int_0^h \phi(x)\,dx$ when h is increased without limit, so by

$$\int_0^1 \Sigma \frac{(-1)^n}{n} \frac{\rho^n e^{ni\theta}}{1+\rho^n e^{ni\theta}} \times \frac{d\rho}{\rho}$$

we are to understand the limit of

$$\int_0^\rho \Sigma \frac{(-1)^n}{n} \frac{\rho^n e^{ni\theta}}{1+\rho^n e^{ni\theta}} \times \frac{d\rho}{\rho},$$

when $1-\rho$ is positive and decreases without limit. We thus have the equation

$$4\int_0^1 \Sigma \frac{(-1)^n}{n} \frac{\rho^n e^{ni\theta}}{1+\rho^n e^{ni\theta}} \times \frac{d\rho}{\rho} = 4 \times \text{limit of } \Sigma \frac{(-1)^n}{n^2} \log(1+\rho^n e^{ni\theta}),$$

when $\rho = 1$. But before we can pass from this equation to the equation (ix), we have to show that the limit of

$$\Sigma \frac{(-1)^n}{n^2} \log(1+\rho^n e^{ni\theta}),$$

when $\rho = 1$, is

$$\Sigma \frac{(-1)^n}{n^2} \log(1+e^{ni\theta}).$$

And although this is true, it cannot be assumed without proof; any more than we could assume that the limit of the series

$$\tfrac{1}{2}(\sin x) + \tfrac{1}{3}(\sin 3x) + \tfrac{1}{5}(\sin 5x) + \ldots,$$

when $x = 0$, is the same (which it certainly is not) as the value of that series when $x = 0$.

It would seem that Riemann in his fragmentary note intended to give the required proof; for he developes the expression

$$\Sigma \frac{(-1)^n}{n^2} \log(1+q^n)$$

in a series proceeding by powers of q (just as Jacobi has dealt with similar series in the 'Fundamenta'), and finds

$$\Sigma \frac{(-1)^n}{n^2} \log(1+q^n) = \Sigma \frac{1+(-1)^n 2}{n} \Phi(n)\, q^n,$$

$\Phi(n)$ denoting the sum of the *uneven* divisors of n. He also enunciates and

proves the following theorem (which is really due to Abel, Œuvres, vol. i, p. 69*):

'Si series

$$a_0 + a_1 + a_2 + \ldots,$$

eo quo scripsimus ordine summata summam habet convergentem, functio ipsius r hâc serie

$$a_0 + a_1 r + a_2 r^2 + \ldots$$

expressa, convergente r versus limitem 1, convergit versus valorem eundem.'

But the fragment breaks off abruptly with the words 'ex hoc theoremate ..... facile deducitur,' and it is not very easy to see how he proposed to complete the demonstration, because the series

$$\Sigma \frac{1+(-1)^n 2}{n^2} \Phi(n) q^n$$

is certainly not always convergent when $q = e^{i\theta}$. The following considerations may serve to replace Riemann's intended demonstration. Denoting $e^{i\theta}$ by q_0, we have, in fact, to show that $\lim \Sigma \frac{(-1)^n}{n^2} \times \log \frac{1+q^n}{1+q_0{}^n} = 0$, when ρ converges to unity, or $q = \rho e^{i\theta}$ to $q_0 = e^{i\theta}$. The imaginary part of this series is

$$i\Sigma \frac{(-1)^n}{n^2} \lim \left[\tan^{-1}\left(\frac{\rho^n \sin n\theta}{1+\rho^n \cos n\theta}\right) - \left[\tfrac{1}{2}(n\theta)\right] \right]; \quad \ldots \quad \text{(x)}$$

the real part is

$$\tfrac{1}{2} \lim \Sigma \frac{(-1)^n}{n^2} \log\left[\rho^n + \{\tfrac{1}{2}(1-\rho^n)\}^2 \sec^2 \tfrac{1}{2}(n\theta)\right]; \quad \ldots \quad \text{(xi)}$$

it will suffice to consider the latter only. Let h be any very great number; and let Δ be so great that $4\Delta h$ is greater than the greatest value of $\sec^2 \frac{1}{2}(n\theta)$ for any value of n not surpassing h (we observe that $\sec^2 \frac{1}{2}(n\theta)$ cannot be infinite, because, in the cases under consideration, $\frac{\theta}{\pi}$ is not a rational fraction having an uneven numerator); also let ρ be so near to $+1$ that $1-\rho^n < \frac{1}{\Delta h}$ for all values of n not surpassing h; then, for all such values of n, $\rho^n + \{\frac{1}{2}(1-\rho^n)\}^2 \sec^2 \frac{1}{2}(n\theta)$ lies between $1 - \frac{1}{\Delta h}$, and $1 + \frac{1}{\Delta h}$; and, consequently,

$$\Sigma_{n=1}^{n=h} \frac{(-1)^n}{n^2} \log\left[\rho^n + \{\tfrac{1}{2}(1-\rho^n)\}^2 \sec^2 \tfrac{1}{2}(n\theta)\right] \quad \ldots \quad \text{(xii)}$$

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* [The reference is to Holmboe's edition, 1839. In Sylow and Lie's edition (1881) the theorem occurs in vol. i, p. 223.]

is evanescent, being intermediate between the two values

$$\log\left(1 \pm \frac{1}{\Delta h}\right) \times \Sigma_{n=1}^{n=h} \frac{1}{n^2},$$

which are both evanescent because $\frac{1}{\Delta h}$ is as small as we please.

Again, the series

$$\Sigma_{n=h+1}^{n=\infty} \frac{(-1)^n}{n^2} \log\left[\rho^n + \{\tfrac{1}{2}(1-\rho^n)\}^2 \sec^2 \tfrac{1}{2}(n\theta)\right] \quad . \quad . \quad . \quad . \quad \text{(xiii)}$$

is evanescent, for $\rho^n + \{\frac{1}{2}(1-\rho^n)\}^2 \sec^2 \frac{1}{2}(n\theta)$ is certainly greater than $\frac{1}{4}$ and less than $\sec^2 \frac{1}{2}(n\theta)$; whence this series is less in absolute magnitude than the sum of the two series

$$\log 4 \times \Sigma_{n=h+1}^{n=\infty} \frac{1}{n^2}, \qquad \Sigma_{n=h+1}^{n=\infty} \frac{\log \sec^2 \frac{1}{2}(n\theta)}{n^2};$$

and of these the first is evidently evanescent when h is very great; the second is also evanescent because, the series $\Sigma \frac{\log \sec^2 \frac{1}{2}(n\theta)}{n^2}$ being convergent in the cases here considered, the remainder after h terms vanishes when h is very great. Hence, finally, the series (xi), which is the sum of (xii) and (xiii), is evanescent.

XLI.

NOTES ON THE THEORY OF ELLIPTIC TRANSFORMATION.

[Messenger of Mathematics, Ser. II. vol. xii. pp. 49–99 (August–November, 1882).]

I. *On the Formulæ of Transformation, with especial reference to the case in which the modular equation has equal roots.*

1. LET $y = F(x)$, where $F(x)$ is a rational fraction containing x^n, but no higher power of x, be an integral of the equation

$$\frac{dy^2}{(1-y^2)(1-\lambda^2 y^2)} = \frac{1}{M^2}\frac{dx^2}{(1-x^2)(1-k^2x^2)}; \quad \ldots\ldots (1)$$

so that, if $y_0 = F(x_0)$,

$$\int_{y_0}^{y} \frac{dy}{\sqrt{\{(1-y^2)(1-\lambda^2y^2)\}}} = \frac{1}{M}\int_{x_0}^{x} \frac{dx}{\sqrt{\{(1-x^2)(1-k^2y^2)\}}}, \quad \ldots (2)$$

the track of the integration and the initial sign of the radical being assumed arbitrarily on the right-hand side, and being determined on the left-hand side in accordance with these assumptions by the equation $y = F(x)$. If $(4K, 2iK')$, $(4\Lambda, 2i\Lambda')$ are pairs of conjugate periods of the integrals*

$$\int \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}} \text{ and } \int \frac{dy}{\sqrt{\{(1-y^2)(1-\lambda^2y^2)\}}}$$

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* Any value of one of these integrals extended over a closed track, at the end of which the radical has the same sign that it had at the beginning, is a period of the integral. The periods, and the pairs of conjugate periods, of the integral $\int \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}$ are, in fact, the periods, and the pairs of conjugate periods of the doubly periodic function $x = \operatorname{sn} u$ defined by the equation

$$u = \int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}; \quad \left(\frac{du}{dx}\right)_0 = +1.$$

respectively, the equation (2) implies the equations

$$\left.\begin{aligned} \frac{4K}{M} &= 4a\Lambda + 2bi\Lambda' \\ \frac{2iK'}{M} &= 4c\Lambda + 2di\Lambda' \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (3)$$

in which a, b, c, d are integral numbers; because whenever x describes a closed contour, at the end of which the radical under the integral sign has the same value that it had at the beginning, y describes a like contour, and consequently any period of the integral on the right-hand side of the equation (2), must be a sum of multiples of the periods of the integral on the left-hand side. The determinant $ad-bc=m$ is different from zero, because the quotient $\frac{iK'}{K}$ is not real; and is positive, if (as we may suppose to be the case) the coefficients of i in the complex quantities $\frac{iK'}{K}$, $\frac{i\Lambda'}{\Lambda}$ have the same sign.

Let us now assume that the fraction $F(x)$ satisfies the conditions

$$F(0)=0, \qquad F(1)=1, \qquad F(\infty)=\infty;$$

in this case, we have the equation

$$\int_0^y \frac{dy}{\sqrt{\{(1-y^2)\,(1-\lambda^2y^2)\}}} = \frac{1}{M}\int_0^x \frac{dx}{\sqrt{\{(1-x^2)\,(1-k^2x^2)\}}}, \quad \ldots \quad (4)$$

and as particular cases of it, the equations

$$\left.\begin{aligned} \int_0^1 \frac{dy}{\sqrt{\{(1-y^2)\,(1-\lambda^2y^2)\}}} &= \frac{1}{M}\int_0^1 \frac{dx}{\sqrt{\{(1-x^2)\,(1-k^2x^2)\}}} \\ \int_0^\infty \frac{dy}{\sqrt{\{(1-y^2)\,(1-\lambda^2y^2)\}}} &= \frac{1}{M}\int_0^\infty \frac{dx}{\sqrt{\{(1-x)\,(1-k^2x^2)\}}} \end{aligned}\right\} \ldots \quad (5)$$

The equation (4) implies that $F(x)$ is an uneven function of x, and also that the limit of $\frac{y}{x}$, when $x=0$, is $\pm\frac{1}{M}$; the condition $F(\infty)=\infty$ shows that the numerator of $F(x)$ is of higher dimensions than its denominator. Consequently n is an uneven number, and the equation $y=F(x)$ assumes the form

$$y = \frac{x}{M} \times \frac{1+a_1x^2+a_2x^4+\ldots+a_{\frac{1}{2}(n-1)}x^{n-1}}{1+b_1x^2+b_2x^4+\ldots+b_{\frac{1}{2}(n-1)}x^{n-1}}, \quad \ldots\ldots \quad (6)$$

the coefficients a and b being connected by the relation

$$1+\Sigma b = \frac{1+\Sigma a}{M},$$

and the initial values of the radicals in the equations (4) and (5) being $+1$.

We shall henceforward suppose that the pairs of conjugate periods $(4K, 2iK')$, $(4\Lambda, 2i\Lambda')$, employed in the equations (3), are *primary*.*

On this hypothesis, since K is one of the values of the integral

$$\int_0^1 \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},$$

it follows that $\frac{K}{M}$ is one of the values of the integral

$$\int_0^1 \frac{dy}{\sqrt{\{(1-y^2)(1-\lambda^2y^2)\}}}; \quad \ldots\ldots\ldots \quad (7)$$

similarly, iK' is one of the values of the integral

$$\int_0^\infty \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},$$

and $\frac{iK'}{M}$ is one of the values of the integral

$$\int_0^\infty \frac{dy}{\sqrt{\{(1-y^2)(1-\lambda^2y^2)\}}}. \quad \ldots\ldots\ldots \quad (7)$$

But all the values of the first of the integrals (7) are comprised in the formula $(4p+1)\Lambda + 2qi\Lambda'$, and all the values of the second of these integrals are comprised in the formula $2r\Lambda + (2s+1)i\Lambda'$, p, q, r, s being whole numbers; we have therefore

$$\frac{K}{M} = (4p+1)\Lambda + 2qi\Lambda',$$

$$\frac{iK'}{M} = 2r\Lambda + (2s+1)i\Lambda',$$

or, which is the same thing, the numbers a, b, d in the equations (3) satisfy the congruences

$$a \equiv 1,\ b \equiv 0,\ \text{mod } 4; \qquad d \equiv 1,\ \text{mod } 2,$$

so that $m = ad - bc$ is an uneven number.

Let u be one of the values of the integral

$$M \times \int_0^y \frac{dy}{\sqrt{\{(1-y^2)(1-\lambda^2y^2)\}}},$$

corresponding to a given value of y; then all the values of that integral, cor-

* The definition of primary pairs of conjugate periods, and a statement of the properties relating to them which are employed in the text, will be found in Note II.

responding to the same value of y, are comprehended in the formulæ

$$\left.\begin{aligned} u + M(4q\Lambda - 2pi\Lambda') \\ 2M\Lambda - u - M(4q\Lambda - 2pi\Lambda') \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (8)$$

where p and q are indeterminate integers. Writing $2mM\Lambda$ for $2M\Lambda$ in the second formula, which is admissible, because m is uneven, and replacing 4Λ and $2i\Lambda'$ by their values in terms of $4K$ and $2iK'$, we obtain the formulæ

$$\left.\begin{aligned} u + \frac{1}{m}[(pc+qd)\,4K - (pa+qb)\,2iK'] \\ 2K - u - \frac{1}{m}[(pc+qd)\,4K - (pa+qb)\,2iK'] \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (9)$$

in the second of which we have written $2K$ for $2dK - biK'$, as we may do, because d is uneven and b even, and because we may omit multiples of the periods $4K$ and $2iK'$; any such omission being in fact equivalent to a change in the indeterminates p and q. Thus all the values of the integral

$$\int_0^x \frac{dx}{\sqrt{\{(1-x^2)\,(1-k^2x^2)\}}}, \quad \ldots\ldots\ldots \quad (10)$$

which correspond in the equation (4) to the given value of y, are comprised in one or other of the two formulæ (9). But since the congruences

$$\left.\begin{aligned} pa + qb \equiv r \\ pc + qd \equiv s \end{aligned}\right\}, \text{ mod } m \quad \ldots\ldots\ldots \quad (11)$$

are resoluble for m and only m pairs of values of r, s, incongruous to one another for the modulus m, the values comprised in the formulæ (9) group themselves in $2m$ sets not reducible to one another by the addition of multiples of the periods $4K$ and $2iK'$. As representatives of these $2m$ sets we take the m pairs of values

$$\left.\begin{aligned} u + \frac{4sK - 2riK'}{m} \\ 2K - u - \frac{4sK - 2riK'}{m} \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (12)$$

and we observe that to values comprised in the same set, or in two sets of the same pair, there answers one and the same value of x. Hence in the equation (4), and consequently also in the equation (6), to a given value of y there answer m values of x. We conclude that $m = n$.

Writing $a = \alpha$, $b = 2\beta$, $\gamma = 2c$, $d = \delta$, we may enunciate the theorem :

'If $y = F(x)$ is a rational integral of the equation (1), satisfying the conditions

$$F(0) = 0, \quad F(1) = 1, \quad F(\infty) = \infty;$$

$F(x)$ is of the form (6), the exponent n of the highest power of x contained in it being uneven, and any two pairs of primary periods of the elliptic integrals are connected by the equations

$$\left.\begin{aligned} \frac{K}{M} &= \alpha\Lambda + \beta i\Lambda', \\ \frac{iK'}{M} &= \gamma\Lambda + \delta i\Lambda', \end{aligned}\right\}$$

where α, β, γ, δ are whole numbers satisfying the equation $\alpha\delta - \beta\gamma = n$, and where the matrix $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$ is *primary*; viz. it satisfies the congruences

$$\alpha \equiv 1, \text{ mod } 4; \quad \begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}, \text{ mod } 2.'$$

2. The foregoing theorem and its demonstration will be found in substance in the 'Traité des Fonctions Elliptiques' of MM. Briot and Bouquet. Both have been reproduced here, with some slight modifications, in order to establish with precision the relation between the rational transformations $y = F(x)$ of the elliptic integral (10), and the arithmetical transformations (3) of its pairs of primary periods. The following observations, though not necessary for the immediate purpose of this note, may be useful as further illustrating this relation.

(α) Every rational transformation must satisfy the condition $F(x) = 0$, as otherwise the equation $y = F(x)$ would indeed be a rational integral of the differential equation (1), but would not transform the integral (10) into another of the same form having the same lower limit. Since $\frac{2K}{M}$ must be equal to a sum of even multiples of Λ and $i\Lambda'$, this condition implies that b is even in the equations (3); or, which is the same thing, that every rational transformation $y = F(x)$ corresponds to an arithmetical transformation of the periods which may be expressed in the form

$$\left.\begin{aligned} \frac{K}{M} &= \alpha\Lambda + i\beta\Lambda' \\ \frac{iK'}{M} &= \gamma\Lambda + i\delta\Lambda' \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (13)$$

the coefficient γ being an even number.

Again, when the condition $F(0)=0$ is satisfied, a in the equations (3), or α in the equations (13) is uneven. For we can cause x to travel from x to $-x$ by a route such that the right-hand member of the equation (4) shall be increased by $\frac{2K}{M}$; y will travel at the same time from y to $-y$, and the left-hand member of the equation will be increased by 2Λ together with certain multiples of the periods 4Λ and $2i\Lambda'$; $\frac{2K}{M}$ is therefore equal to an uneven multiple of 2Λ, together with a multiple of $2i\Lambda'$.

The condition $F(0)=0$ being satisfied, $F(x)$ is (as we have already seen) a fraction of which the denominator is an even, and the numerator an uneven function of x. The order of the numerator may be either higher or lower by an unit than the order of the denominator; in the former case we have n uneven, and $F(\infty)=\infty$, in the latter, n even, and $F(\infty)=0$. Further, if $F(\infty)=\infty$, $\frac{iK'}{M}$ must be the sum of an even multiple of Λ and an uneven multiple of $i\Lambda'$; if $F(\infty)=0$, $\frac{iK'}{M}$ must be the sum of even multiples of both Λ and $i\Lambda'$; hence in the former case, d in the equations (3), or δ in the equations (13) is uneven; in the latter case, this number is even. Thus the matrix $\left|\begin{matrix}\alpha, & \beta\\ \gamma, & \delta\end{matrix}\right|$ is of one of the four types, mod 2,

m uneven, $\left|\begin{matrix}1, & 0\\ 0, & 1\end{matrix}\right|$, $\left|\begin{matrix}1, & 1\\ 0, & 1\end{matrix}\right|$;

m even, $\left|\begin{matrix}1, & 0\\ 0, & 0\end{matrix}\right|$, $\left|\begin{matrix}1, & 1\\ 0, & 0\end{matrix}\right|$.

(c) The demonstration that $m=n$, given in Art. 1, applies only to the case when m is uneven, and b even; but in all the cases alike the second of the formulæ (8) may be written in the form

$$-u-M[4(q-\tfrac{1}{2})\Lambda-2pi\Lambda'],$$

so that the second of the formulæ (9) becomes

$$-u-\frac{1}{m}[\{pc+(q-\tfrac{1}{2})d\}\,4K-\{pa+(q-\tfrac{1}{2})\,b\}2iK'].$$

In discussing this expression we have to consider the congruences

$$\left.\begin{aligned} pa+qb &\equiv \tfrac{1}{2}b+r\\ pc+qd &\equiv \tfrac{1}{2}d-\tfrac{1}{2}m+s\end{aligned}\right\}, \text{ mod } m,$$

which, like the congruences (11), are resoluble for m and only m pairs of values of r and s, and are moreover resoluble for the same pairs of values of r and s as those congruences, because the congruences

$$\left.\begin{aligned} pa+qb &\equiv \tfrac{1}{2}b \\ pc+qd &\equiv \tfrac{1}{2}d-\tfrac{1}{2}m \end{aligned}\right\}, \text{ mod } m,$$

are resoluble. Hence the values of the integral (10) answering in the equation (4) to a given value of y are comprised in $2m$ sets, which, as before, may be represented by the formulæ (12); we have therefore $m=n$ in all the four cases.

(d) The remaining condition, $F(1)=1$, renders $a\equiv 1$, $b\equiv 0$, mod 4, in the equations (3), or $a\equiv 1$, mod 4, $\beta\equiv 0$, mod 2, in the equations (13). Thus, finally, if $F(x)$ satisfies the two conditions $F(0)=0$, $F(1)=1$, we must have either

$$n \text{ uneven}, \quad F(\infty)=\infty, \quad \begin{vmatrix} a, & \beta \\ \gamma, & \delta \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}, \text{ mod } 2,$$

or

$$n \text{ even}, \quad F(\infty)=0, \quad \begin{vmatrix} a, & \beta \\ \gamma, & \delta \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 0 \end{vmatrix}, \text{ mod } 2.$$

When a or α is uneven, the condition that it is to be $\equiv 1$, mod 4, can always be satisfied by changing the sign of M.

It is important to observe that, if the pairs of primary periods have been chosen once for all in the equations (3), the rational transformation $y=F(x)$ can correspond to only one arithmetical transformation of the periods; viz. an equation of the form

$$a\Lambda+\beta i\Lambda' = a_1\Lambda+\beta_1 i\Lambda'$$

implies the equations $a=a_1$, $\beta=\beta_1$, because the ratio $\Lambda : i\Lambda'$ is always imaginary.

3. If in the equation (1) we regard k^2 as a given quantity having any value except one of the three 0, 1, ∞, and M, λ^2 as quantities to be determined, any rational integral $y=F(x)$ of that equation which satisfies the conditions

$$F(0)=0, \qquad F(1)=1, \qquad F(\infty)=\infty,$$

is termed a *primary* transformation of the elliptic integral (10)*. It results from

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* The word *primary* was employed by Gauss in the theory of complex integers of the form $a+bi$, to distinguish one of the four associated uneven numbers $i^\mu(a+bi)$, $\mu=0, 1, 2, 3$, from the other three; and the same expression has since been employed in other complex theories. The essential character of primary numbers in these theories is that they reproduce themselves by multiplication; viz. the product of two primary numbers is primary. It is natural to extend the use of the term to the

the theorem of Art. 1 that every primary transformation of this elliptic integral implies an equation of the form

$$\frac{iK'}{K} = \frac{\gamma + \delta \frac{i\Lambda'}{\Lambda}}{\alpha + \beta \frac{i\Lambda'}{\Lambda}}, \quad \ldots\ldots\ldots\ldots \quad (14)$$

where $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$ is a primary matrix. Conversely, it is known from the theory of the transformation of the Theta functions, that every equation of the form (14) supplies one, and only one, primary transformation of the elliptic integral; and further, that if two matrices $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$ are equivalent by post-multiplication with a primary unit matrix, the transformations arising from them are the same; that the transformations corresponding to matrices not so equivalent are always different follows from the concluding observation of Art. 2. Thus the number of different primary transformations of the order n is the same as the number of non-equivalent matrices* of determinant n; *i.e.* it is $\sigma(n)$, if $\sigma(n)$ is the sum of the divisors of n; and there is a correspondence one to one between the primary transformations and the non-equivalent primary matrices.

These $\sigma(n)$ transformations are, however, not all *primitive*;† viz, if $\alpha, \beta, \gamma, \delta$ have a greatest common divisor μ, the transformation corresponding to $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$ is not primitive, but is compounded of a multiplication of the argument by μ, and of a primary and primitive transformation of the order $\nu = \frac{n}{\mu^2}$, corresponding

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theory of matrices; and to characterise as primary, matrices of the type defined by the congruences

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}, \text{ mod } 2, \; a \equiv 1, \text{ mod } 4,$$

because this type reproduces itself in multiplication. Again, it is convenient to designate as primary those pairs of conjugate periods of an elliptic integral or function which are formed from the fundamental pair (see note II) with primary unit matrices. Lastly, the introduction of the phrase *primary transformation* is justified by the consideration that the primary periods of the elliptic integrals transformed into one another by a primary transformation are connected by a primary matrix; and that, as a consequence of this relation, the composition of two primary transformations gives a primary transformation.

* For brevity, matrices equivalent or non-equivalent by primary post-multiplication are here simply termed equivalent or non-equivalent.

† The word *primitive* was applied by Gauss to quadratic forms of which the coefficients are relatively prime; it is natural to extend the use of the term to matrices of which the elements have no common divisor other than unity; and consequently to the rational transformations appertaining to such matrices.

to the matrix $\left|\begin{matrix}\frac{\alpha}{\mu}, & \frac{\beta}{\mu}\\ \frac{\gamma}{\mu}, & \frac{\delta}{\mu}\end{matrix}\right|$; in fact, in this case the transformation (6) arises from the combination of two primary transformations

$$z = F_1(x), \quad y = F_2(z),$$

of the orders μ^2 and ν respectively, corresponding to the equations

$$\int_0^z \frac{dz}{\sqrt{\{(1-z^2)(1-k^2z^2)\}}} = \mu \int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},$$

$$\int_0^y \frac{dy}{\sqrt{\{(1-y^2)(1-\lambda^2y^2)\}}} = \frac{1}{\mu M}\int_0^z \frac{dz}{\sqrt{\{(1-z^2)(1-k^2z^2)\}}};$$

of which, however, the former represents a multiplication, and not a transformation properly so called, of the elliptic integral. The number of primitive transformations is of course the same as the number of non-equivalent primitive matrices $\left|\begin{matrix}\alpha, & \beta\\ \gamma, & \delta\end{matrix}\right|$, and is given by the formula

$$\sigma'(n) = n\Pi\left(1+\frac{1}{p}\right),$$

where the sign of multiplication extends to every prime divisor p of n. The primitive transformations are alone to be regarded as proper transformations of the order n, and in what follows we may confine our attention to them.

4. Let $\omega = \frac{iK'}{K}$, $\Omega = \frac{i\Lambda'}{\Lambda}$, so that $\omega = \frac{\gamma+\delta\Omega}{\alpha+\beta\Omega}$; and let $\phi(\theta)$ be the function defined by the equations

$$\phi(\theta) = \sqrt{2}e^{\frac{1}{4}i\pi\theta}\Pi_{m=1}^{m=\infty}\frac{1+e^{2mi\pi\theta}}{1+e^{(2m-1)i\pi\theta}}; \quad \ldots\ldots\ldots \quad (15)$$

this function is characterized by the two converse properties (i) that the equation

$$\theta_1 = \frac{C+D\theta}{A+B\theta}, \quad \ldots\ldots\ldots \quad (16)$$

where $\left|\begin{matrix}A, & B\\ C, & D\end{matrix}\right|$ is a primary unit matrix, involves the equation

$$\phi^8(\theta_1) = \phi^8(\theta), \quad \ldots\ldots\ldots \quad (17)$$

and (ii) that the equation (17) involves the existence of an equation of the form (16).

It is known, from the theory of the Theta functions, that

$$k^2 = \phi^8(\omega), \quad \lambda^2 = \phi^8(\Omega);$$

and that the $\sigma'(n)$ modules $\phi^8(\Omega)$, which present themselves in the different primary transformations of order n, are the roots of an equation $f(k^2, \lambda^2) = 0$, termed the modular equation, which is of the order $\sigma'(n)$, and of which the coefficients are rational and integral functions of $k^2 = \phi^8(\omega)$. In the general case, in which the roots of this equation are unequal, the $\sigma'(n)$ primary transformations of the elliptic integral and the transformed modules correspond to one another one to one. We shall now show that, when the modular equation has equal roots, the values of M^2 answering to the equal roots are different; so that, although in this special case two or more different primary transformations answer to one and the same transformed modulus, yet in all cases alike the $\sigma'(n)$ different primary transformations correspond one to one to $\sigma'(n)$ different pairs of values of λ^2 and M^2. To establish this assertion, the truth of which might be inferred from the concluding observation of Art. 2, we consider the equation of Jacobi

$$\frac{1}{M^2} = \frac{nk^2(1-k^2)}{\lambda^2(1-\lambda^2)} \times \frac{d.\lambda^2}{d.k^2}; \quad \ldots\ldots\ldots \quad (18)$$

from which it appears that if for the same value of k^2 we have simultaneously $\lambda_1^2 = \lambda_2^2$, $M_1^2 = M_2^2$, we must also have

$$\frac{d.\lambda_1^2}{d.k^2} = \frac{d.\lambda_2^2}{d.k^2}. \quad \ldots\ldots\ldots\ldots \quad (19)$$

But this equation is inadmissible. In a memoir 'On the Singularities of the Modular Equations and Curves,'* it has been shown that the modular curve represented by the equation

$$f(X, Y) = 0, \quad k^2 = X, \quad \lambda^2 = Y,$$

has no superlinear branches except at the points

$$(X = 0,\ Y = 0), \quad (X = 1,\ Y = 1), \quad (X = \infty,\ Y = \infty),$$

which may be left out of consideration; and that it has no tangents at a finite distance parallel to either of the axes of X or Y. Hence the only points which need to be examined are multiple points free from any superlinearity; and at these it has only to be shown that two branches cannot touch one another. Let

$$\phi^8(\Omega_1) = \phi^8(\Omega_2), \quad \omega = \frac{\gamma_1 + \delta_1 \Omega_1}{\alpha_1 + \beta_1 \Omega_1}, \quad \omega = \frac{\gamma_2 + \delta_2 \Omega_2}{\alpha_2 + \beta_2 \Omega_2},$$

* 'Proceedings of the London Mathematical Society,' vol. ix. p. 242 [vol. ii. p. 242].

the matrices $\left|\begin{matrix}\alpha_1, \beta_1\\ \gamma_1, \delta_1\end{matrix}\right|$ and $\left|\begin{matrix}\alpha_2, \beta_2\\ \gamma_2, \delta_2\end{matrix}\right|$ being non-equivalent primary matrices of determinant n. The equation $\lambda_1^2 = \lambda_2^2$, or $\phi^8(\Omega_1) = \phi^8(\Omega_2)$, implies, as we have seen, an equation of the form

$$\Omega_2 = \frac{C + D\Omega_1}{A + B\Omega_1},$$

where $\left|\begin{matrix}A, B\\ C, D\end{matrix}\right|$ is a primary unit matrix; if then

$$\left|\begin{matrix}\alpha, \beta\\ \gamma, \delta\end{matrix}\right| = \left|\begin{matrix}\alpha_2, \beta_2\\ \gamma_2, \delta_2\end{matrix}\right| \times \left|\begin{matrix}A, B\\ C, D\end{matrix}\right|,$$

we may replace the equation

$$\omega = \frac{\gamma_2 + \delta_2\Omega_2}{\alpha_2 + \beta_2\Omega_2}$$

by the equation

$$\omega = \frac{\gamma + \delta\Omega_1}{\alpha + \beta\Omega_1},$$

where $\left|\begin{matrix}\alpha, \beta\\ \gamma, \delta\end{matrix}\right|$ is a primary matrix of determinant n, equivalent to $\left|\begin{matrix}\alpha_2, \beta_2\\ \gamma_2, \delta_2\end{matrix}\right|$, and therefore non-equivalent to $\left|\begin{matrix}\alpha_1, \beta_1\\ \gamma_1, \delta_1\end{matrix}\right|$.

We now have

$$\lambda_1^2 = \lambda_2^2 = \phi^8(\Omega_1);$$

but to calculate $d.\lambda_1^2$ in the equation (19) we must employ the equation

$$\omega = \frac{\gamma_1 + \delta_1\Omega_1}{\alpha_1 + \beta_1\Omega_1},$$

and to calculate $d.\lambda_2^2$, the equation

$$\omega = \frac{\gamma + \delta\Omega_1}{\alpha + \beta\Omega_1}.$$

Observing that

$$\phi'(\omega) = -\frac{1}{2i\pi}\phi(\omega)\,[1 - \phi^8(\omega)]K^2(\omega),$$

$$\phi'(\Omega_1) = -\frac{1}{2i\pi}\phi(\Omega_1)\,[1 - \phi^8(\Omega_1)]\,K^2(\Omega_1),$$

where $K(\omega)$ is the one-valued function of $e^{i\pi\omega}$ defined by the equation

$$\sqrt{\frac{2K(\omega)}{\pi}} = 1 + 2e^{i\pi\omega} + 2e^{4i\pi\omega} + 2e^{9i\pi\omega} + \ldots,$$

and that these derived functions are neither zero nor infinite, we infer from the equation (19) the apparently but not really identical relation

$$\left(\frac{d\Omega_1}{d\omega}\right) = \left(\frac{d\Omega_1}{d\omega}\right),$$

of which the two members have different significations. This equation gives

$$\frac{\alpha_1+\beta_1\Omega_1}{\sqrt{n}} = \pm\frac{\alpha+\beta\Omega}{\sqrt{n}},$$

whence also

$$\frac{\gamma_1+\delta_1\Omega_1}{\sqrt{n}} = \pm\frac{\gamma+\delta\Omega_1}{\sqrt{n}},$$

and finally, $\alpha_1=\alpha, \quad \beta_1=\beta, \quad \gamma_1=\gamma, \quad \delta_1=\delta,$

contrary to the hypothesis that $\begin{vmatrix}\alpha_1, & \beta_1\\ \gamma_1, & \delta_1\end{vmatrix}$ and $\begin{vmatrix}\alpha_2, & \beta_2\\ \gamma_2, & \delta_2\end{vmatrix}$ are non-equivalent.

With a little modification this demonstration may be employed to show that no two modular curves can touch one another, except at the points $[0, 0]$, $[1, 1]$, $[\infty, \infty]$. Thus the $\sigma'(n)$ primitive transformations of order n are not only distinct from one another, but they are also distinct from any of the derived transformations of order n.

5. The result at which we have arrived, viz. that when two roots of the modular equation are equal the squares of the corresponding multipliers are always different, might seem to contradict the theorem, due to M. Kœnigsberger, that M is a rational function of k^2 and λ^2. But it is to be observed that the demonstrations of this theorem fail (as they ought to do) when the modular equation has equal roots. For example, M. Kœnigsberger has observed ('Theorie der Elliptische Functionen') that a rational expression for M may be obtained by combining the equation (18) written in the form

$$\frac{1}{M^2} = -\frac{nk^2(1-k^2)}{\lambda^2(1-\lambda^2)} \times \frac{\left(\frac{df}{d\,.\,k^2}\right)}{\left(\frac{df}{d\,.\,\lambda^2}\right)} \quad \ldots\ldots\ldots \quad (20)$$

with the equation of the multiplier; and this is of course in general true, but the right-hand side of the equation (20) assumes the form $\frac{0}{0}$ at a multiple point. Again, Prof. Cayley has shown ('On the Transformation of Elliptic Functions,' Phil, Trans., vol. clxiv, p. 423) how to form an expression for $\frac{1}{M}$, rational in k^2 and λ^2, by considering the symmetrical functions

$$\Sigma_s \frac{\lambda_s^{2r}}{M_s},$$

$$s=1, 2, \ldots\sigma'(n); \quad r=0, 1, 2,\ldots;$$

but the expression thus obtained must assume the form $\frac{0}{0}$ when two roots of the modular equation are equal, for its denominator is the discriminant of the modular equation, and the multiplier M is never infinite.

We may observe that the derived functions of λ^2 and M^2 with respect to k^2 are always finite, and rational in M^2, λ^2, and k^2; this may be inferred from the equation of Jacobi (18); because in the modular curve there is no superlinearity except at the excepted points, and because $\frac{dY}{dX}$ is never zero and never infinite; hence, even at a multiple point (X_0, Y_0), the development of each branch of the modular curve is of the form

$$Y = \mu_1 X + \frac{\mu_2}{1.2} X^2 + \ldots,$$

where μ_1 contains M^2 as well as X_0 and Y_0, but μ_2, μ_3,... can contain no new irrationality, because there is no contact of different branches.

It may also be noticed that M, which, according to the theorem of M. Kœnigsberger, is in general rational in k^2 and λ^2, is always rational in M^2, k^2, λ^2. This may be seen by evaluating according to the usual rule the expression for M, obtained by the method of Professor Cayley, and substituting for the derived functions of λ^2 with regard to k^2, the equivalent expressions, rational in M^2, k^2, λ^2, which we have shown to exist.

To obtain the values of M^2 at the multiple point (X_0, Y_0), we have only to form the equation, giving the directions of the tangents at that point. Substituting in this equation from the formula (18), we obtain an equation of the form $\chi\left(\frac{1}{M^2}, X_0, Y_0\right) = 0$. The coefficients of this equation are rational functions of X_0, Y_0, having rational numerical coefficients; and, as we shall presently see, χ can always be resolved into linear factors, rational in X_0, Y_0, but having coefficients involving an imaginary quadratic surd. The coordinates (X_0, Y_0) of the multiple point are themselves numerical quantities, which may be irrational, because in the most general case the multiple points of a modular curve present themselves in sets; the abscissas, or the ordinates (as the case may be) of the points of any one set, satisfying an irreducible equation.

6. In the general case in which the roots of the modular equation are unequal, the coefficients a_1, a_2,...b_1, b_2,... of the primary transformation are rational in k^2 and λ^2, because the primary transformations and the values of λ^2 correspond to one another one to one; and, when two or more roots are equal, these coeffi-

cients are at any rate rational in k^2, λ^2, M^2, because there is a similar correspondence between the primary transformations and the pairs of values of λ^2 and M^2. In fact the coefficients a and b, as well as the multiplier M, can always be expressed as rational functions of k^2 and λ^2; but in the general case the numerical coefficients entering into these expressions are rational; in the special case of equal roots the general expressions, or some of them, assume an indeterminate form, and new expressions present themselves, which may be deduced by evaluation from the general expressions, and in which the coefficients are no longer rational, but involve the same imaginary quadratic surd which enters into the expression of M. It is worth while to show how these assertions are consistent with the results of the algebraical theory of elliptic transformation.

It is shown in that theory (see the 'Fundamenta Nova,' or the Memoir of Professor Cayley already cited) (i) that any primary transformation $y=F(x)$ satisfies an equation of the form

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\times\left(\frac{P-Qx}{P+Qx}\right)^2, \quad \ldots\ldots\ldots \quad (21)$$

where $P=\alpha+\gamma x^2+\ldots$, $Q=\beta+\delta x^2+\ldots$, are functions of x^2, which are respectively of the orders

$$\tfrac{1}{4}(n-1),\ \tfrac{1}{4}(n-5);\quad \text{or } \tfrac{1}{4}(n-3),\ \tfrac{1}{4}(n-3),$$

according as $n\equiv 1$, or $n\equiv 3$, mod 4; the whole number of coefficients in P and Q being $\frac{1}{2}(n+1)$ in both cases alike; (ii) that the determination of the ratios $\alpha:\beta:\gamma\ldots$ depends on the $\frac{1}{2}(n+1)$ equations included in the formula

$$\left.\begin{aligned}(kx)^{n-1}\left[P^2\left(\frac{1}{k^2x^2}\right)+\frac{2}{k^2x^2}P\left(\frac{1}{k^2x^2}\right)Q\left(\frac{1}{k^2x^2}\right)+\frac{1}{k^2x^2}Q^2\left(\frac{1}{k^2x^2}\right)\right]\\ =\sqrt{\frac{\lambda}{k^n}}k^{n-1}[P^2+2PQ+Q^2x^2]\end{aligned}\right\}\ldots\ldots \quad (22)$$

which expresses the condition of Jacobi that the equation (6) or (21) remains unchanged when we write in it $\frac{1}{kx}$ for x, and $\frac{1}{\lambda y}$ for y. Putting U for $\sqrt{\frac{\lambda}{k^n}}$ we may write the $\frac{1}{2}(n+1)$ equations in the form

$$\phi_\mu=Uk^{n-1-2\mu}\psi_\mu, \quad \mu=0,\ 1,\ldots\tfrac{1}{2}(n-1), \ldots\ldots \quad (23)$$

where ϕ_μ and ψ_μ are homogenous quadratic functions of α, β, γ,.... The elimination of α, β, γ, ... from the equations (23) gives rise to an equation between U and k^2, of which the order in U, according to the general theory of elimination is $\frac{1}{2}(n+1)\times 2^{\frac{1}{2}(n-1)}=(n+1)2^{\frac{1}{2}(n-3)}$, but which must be capable of reduction to the

modular equation between U and k^2 by the exclusion of extraneous factors; of this reduction, however, no complete account has as yet been given.* If we assume that to each value of U satisfying that modular equation, there corresponds but one set of ratios $\alpha:\beta:\gamma\ldots$, we can infer that these ratios, and consequently the values of M and of the coefficients a and b, are rational functions of U and k^2 with rational numerical coefficients. And, in the general case the assumption is justified, not indeed by anything which has been proved concerning the algebraical nature of the system (23), but by the one to one correspondence established in Art. 3, which suffices to show (as we have already seen in Art. 4) that, when the roots of the modular equation between λ^2 and k^2 are unequal, only one primary transformation can correspond to given values of k^2 and λ^2, and, therefore, to given values of k^2 and U.

Again, to assert that M and the coefficients a and b are rational in U and k^2 is to assert that they are also rational in k^2 and λ^2, for, as we shall now show, U is itself rational in k^2 and λ^2, at least when λ^2 is not one of a group of equal roots in the modular equation between k^2 and λ^2. In fact, if, as usual, $u=\sqrt[4]{k}$, $v=\sqrt[4]{\lambda}$, it may be shown that $\frac{v}{u^n}$ and $\frac{u}{v^n}$ are rational in u^8 and v^8. For, if $u^r v^s$ be any term in the modular equation between u and v, we have, by a theorem due to Sohnke (see the Memoir 'Æquationes Modulares pro Transformatione Functionum Ellipticarum,' Crelle's Journal, vol. xvi. p. 97),

$$ns+r=n+1+8\mu,$$

where μ is an integer, so that

$$u^r v^s = u^{n+1}\left(\frac{v}{u^n}\right)^s \times u^{8\mu}.$$

Hence, dividing every term of the modular equation by u^{n+1}, we have a relation of the form

$$\frac{v}{u^n} f_1\left(\frac{v^8}{u^{2n}},\ u^8\right) = f_2\left(\frac{v^8}{u^{2n}},\ u^8\right),$$

where in general f_1 and f_2 cannot vanish simultaneously; viz. if f_1 and f_2 vanish for the values $u=u'$, $v=v'$, these functions also vanish for $u=u'$, $v=-v'$, and

* Professor Cayley in the memoir cited has shown how this reduction takes place in the case when $n=5$; and Mr. Ely has given the corresponding determination of the extraneous factors for $n=7$. ('Proceedings of the London Mathematical Society,' vol. xiii. p. 153; see also Prof. Cayley. 'Phil. Trans.' vol. clxix. p. 419.) The result in the case $n=7$ possesses considerable interest, because it enables us to conjecture in what way the reduction takes place for higher values of n: viz. by the exclusion of factors corresponding to the modular equations of lower orders.

the modular equation between λ^2 and k^2 has a pair of equal roots corresponding to $k^2 = u'^8$. Thus $\frac{v}{u^n}$ can be expressed rationally in terms of $\frac{v^2}{u^{2n}}$ and u^8; but if $u^2 = u_2$, $v^2 = v_2$, and $u_2^r v_2^s$ is any term in the modular equation between u_2 and v_2, we have

$$ns + r = n + 1 + 4\mu,$$

whence, as before, $\frac{v_2}{u_2^2}$ is rational in $\left(\frac{v_2}{u_2^n}\right)^2 = \frac{v^4}{u^{4n}}$ and $u_2^4 = u^8$. Lastly, if $k^r \lambda^s$ is any term in the equation between $u_2^2 = k$ and $v_2^2 = \lambda$, we have

$$ns + r = n + 1 + 2\mu,$$

whence, finally, $\frac{\lambda}{k^2} = \frac{v^4}{u^{4n}}$ is rational in $\frac{\lambda^2}{k^{2n}}$ and k^2; *i.e.* $\frac{v}{u^n}$ is rational in λ^2 and k^2.

Example. Let $n = 3$; the modular equations between (u, v), (u_2, v_2), (k, λ) may respectively be written in the forms

$$2\frac{v}{u^3} = \frac{\frac{\lambda}{k^3}k^2 - 1}{1 - \frac{v_2}{u_2^3}k^2},$$

$$4\frac{v_2}{u_2^3} = \frac{\frac{\lambda^2}{k^2} + 6\frac{k^3}{k^2}k^2 + 1}{1 + \frac{\lambda}{k^3}k^2},$$

$$4\frac{\lambda}{k^3} = \frac{\frac{\lambda^4}{k^4} + 6\frac{\lambda^2}{k^2} + 1}{4\lambda^2k^2 - 3\lambda^2 - 3k^2 + 4};$$

and give, by the elimination of $\frac{v_2^2}{u_2^3}$ and $\frac{\lambda}{k^3}$, the following expression for $\frac{v}{u^3}$,

$$\frac{v}{u^3} = \frac{1}{k^2} \times \frac{\lambda^4(16k^4 - 13k^2 + 1) + 34\lambda^2k^2(1 - k^2) - k^2(k^4 - 13k^2 + 16)}{32(1 - \lambda^2 - k^2) + 5(\lambda^4 + k^4) + 38\lambda^2k^2 - 8\lambda^2k^2(\lambda^2 + k^2)}.$$

Of course, this expression is only one of an infinite number of equivalent forms, reducible to one another by means of the modular equation between k^2 and λ^2; we have, for example,

$$\frac{u^3}{v} = \frac{\lambda^4(16k^4 - 13k^2 + 1) + 34\lambda^2k^2(1 - k^2) - k^2(k^4 - 13k^2 + 16)}{32\lambda^2k^2(\lambda^2 + k^2 - \lambda^2k^2) - 5(\lambda^4 + k^4) - 38\lambda^2k^2 + 8(\lambda^2 + k^2)}.$$

To obtain similar expressions for $\frac{u}{v^3}$, and $\frac{v^3}{u}$, we have only to interchange k^2 and λ^2 in these formulæ, and to change the sign.

The values of the ratios $\alpha : \beta : \gamma \ldots$, or of the coefficients a, b, have been expressed in terms of u and v by Jacobi (in the 'Fundamenta Nova') for $n = 3$, 5, and by Professor Cayley (in the memoir already cited, and in the addition to it, 'Phil. Trans.' vol. clxix., p. 419) for $n = 7$. It will be found that all these expressions contain only $\frac{v}{u^n}$ and u^8, or $\frac{u}{v^n}$ and v^8, and are consequently rational in k^2 and λ^2, in accordance with what has been said here.

Passing to the case in which the modular equation has equal roots, let (k^2, λ^2) be a point of multiplicity s on the modular curve; we have already seen that some at least of the expressions for the ratios $\alpha : \beta : \gamma \ldots$, and for the coefficients M, a, b, must assume indeterminate forms, and that the s different primary transformations may be elicited from these indeterminate forms by differentiating with regard to k^2, and by substituting for $\frac{d.\lambda^2}{d.k^2}$, and for the higher differential quotients of λ^2, their values corresponding to the s different branches of the curve. But instead of employing a process of evaluation, we may return to the equations (23), and, before proceeding to eliminate, we may attribute in them to k^2 and U the values which these quantities have at the multiple point. The system (23) will then admit of s different solutions; and will furnish an equation of order s for the determination of the ratio of $\beta : a$, or, which is the same thing, for the determination of $\frac{1}{M} = 1 + \frac{2\beta}{a}$. It must be remarked however that the modular equation between U and k^2 may have unequal roots, even in the case in which the modular equation between λ^2 and k^2 has equal roots; (because in this case it is possible that U may not be rational in k^2, λ^2, and that as many as four different values of U may answer to one value of λ^2); if this should happen, the equations (23) would enable us to express the coefficients a, b rationally in U and k^2; but these coefficients would not be rational in λ^2 and k^2.

Example. Let $n = 5$; and, for simplicity, let us consider, instead of the equation between k^2 and λ^2, the equation between u and v, viz.

$$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0.$$

The general formulæ for a primary transformation of the fifth order are

$$y = \frac{x}{M} \times \frac{1 + M\left\{\frac{1}{4}\left(\frac{1}{M}-1\right)^2 + \left(1+\frac{1}{M}\right)\frac{u^5}{v}\right\}x^2 + M\frac{u^{10}}{v^2}x^4}{1+\left\{\frac{1}{4}\left(\frac{1}{M}+1\right)^2 + \frac{2u^5}{v} - 1\right\}x^2 + \left(\frac{1}{M}+\frac{u^5}{v}-1\right)\frac{u^5}{v}x^4},$$

$$\frac{1}{M} = \frac{v-u^5}{v-v^4u}.$$

If θ is a primitive sixteenth root of unity, (θ, θ^5) is a double point on the modular curve, and the equation determining the values of $\frac{dv}{du}$ is

$$5\,du^2 + 6\theta^{-4}\,du\,dv - 5\,dv^2 = 0,$$

whence $\frac{dv}{du} = \frac{3\theta^{-4} \pm 4}{5}$. The expression for $\frac{1}{M}$ becomes indeterminate; evaluating, we find

$$\frac{1}{M} = -1 \pm 2\theta^4,$$

and we obtain two primary transformations of the integral $\int_0^x \frac{dx}{\sqrt{(1-x^4)}}$, included in the formula

$$\int_0^y \frac{dy}{\sqrt{(1-y^4)}} = (2i-1)\int_0^x \frac{dx}{\sqrt{(1-x^4)}},$$

$$y = x\frac{(2i-1)+x^4}{1+(2i-1)x^4},$$

in which either sign may be attributed to i.

Again, writing $k^2 = -1$, $U = 1$ in the equations (23), we have

$$\alpha^2 - \gamma^2 = 0, \quad 2\beta^2 + 2\beta\gamma + 4\alpha\gamma + 2\alpha\beta = 0, \quad 2\beta\gamma - 2\alpha\gamma = 0;$$

whence we infer

$$\gamma = \alpha, \quad 2\alpha^2 + 2\alpha\beta + \beta^2 = 0,$$

or

$$\frac{\beta}{\alpha} = -1 \pm \iota, \qquad \frac{\gamma}{\alpha} = 1.$$

The formula (21) becomes, when $n = 5$,

$$y = x\frac{(\alpha^2 + 2\alpha\beta) + (\beta^2 + 2\alpha\gamma + 2\beta\gamma)x^2 + y^2x^4}{\alpha^2 + (\beta^2 + 2\alpha\gamma + 2\alpha\beta)x^2 + (\gamma^2 + 2\beta\gamma)x^4},$$

and gives, on substituting for the ratios $\alpha : \beta : \gamma$ their values, the same result as before. It will be observed that, in accordance with the foregoing theory, the system (23) admits of two solutions, depending on a quadratic equation in $\beta : \alpha$ or $\frac{1}{M}$; it is a peculiarity, with which we are not here concerned, that the

coefficients M, a, b are rational in u and v, though irrational in k^2 and λ^2; the circumstance that $k^2 = \lambda^2$ is also immaterial.

The values of $\frac{1}{M^2}$ might of course be obtained immediately by combining the equation (18) of Jacobi with the equation which gives the values of $\frac{dv}{du}$ at the double point. In this way we should find

$$M^2 = -\frac{3 \pm 4i}{25}, \quad \frac{1}{M} = \pm 1 \pm 2i,$$

the extraction of the square root introducing a new ambiguous sign. This ambiguity may be removed by means of the equation of the multiplier, viz.

$$\frac{1}{M^6} - \frac{10}{M^5} + \frac{35}{M^4} - \frac{60}{M^3} + \frac{55}{M^2} - \frac{26 - 256k^2(1-k^2)}{M} + 5 = 0,$$

of which the left-hand member, when $k^2 = -1$, resolves itself into the product

$$\left[\left(\frac{1}{M} - 3\right)^4 - 80\right] \times \left[\frac{1}{M^2} + \frac{2}{M} + 5\right],$$

shewing that $\frac{1}{M} = -1 \pm 2i$, as we have already found.

7. The ratio of the two unequal multipliers, which correspond to two equal values of λ^2, is always a simple imaginary quadratic surd. For, as we have seen in Art. 4, the equality of two values of λ^2 implies an equation of the form

$$\omega = \frac{\gamma_1 + \delta_1 \Omega}{\alpha_1 + \beta_1 \Omega} = \frac{\gamma + \delta\Omega}{\alpha + \beta\Omega};$$

if therefore M_1 and M are the multipliers corresponding to these two transformations, we have

$$\left.\begin{aligned} \frac{K}{M_1} &= \alpha_1 \Lambda + i\beta_1 \Lambda', \quad \frac{iK'}{M_1} = \gamma_1 \Lambda + i\delta_1 \Lambda', \\ \frac{K}{M} &= \alpha \Lambda + i\beta \Lambda', \quad \frac{iK'}{M} = \gamma \Lambda + i\delta \Lambda', \end{aligned}\right\}$$

and consequently,

$$\left.\frac{M_1}{M} = \frac{\alpha + \beta\Omega}{\alpha_1 + \beta_1 \Omega} = \frac{\gamma + \delta\Omega}{\gamma_1 + \delta_1 \Omega}\right\} \quad \ldots\ldots\ldots \quad (24)$$

But Ω is a quadratic surd; viz. if

$$\left.\begin{aligned} 2\tau A &= \alpha\gamma_1 - \alpha_1\gamma, \\ 4\tau B &= \alpha\delta_1 - \alpha_1\delta + \beta\gamma_1 - \beta_1\gamma, \\ 2\tau C &= \beta\delta_1 - \beta_1\delta, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (25)$$

Ω is a root of the quadratic equation

$$A + 2B\Omega + C\Omega^2 = 0.$$

In the equations (25) it is convenient to suppose that 2τ is the greatest common divisor of

$$\tfrac{1}{2}(\alpha\gamma_1 - \alpha_1\gamma), \quad \tfrac{1}{4}(\alpha\delta_1 - \alpha_1\delta + \beta\gamma_1 - \beta_1\gamma), \quad \tfrac{1}{2}(\beta\delta_1 - \beta_1\delta),$$

which are whole numbers, because the matrices

$$\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \text{ and } \begin{vmatrix} \alpha_1, & \beta_1 \\ \gamma_1, & \delta_1 \end{vmatrix}$$

are primary, and which cannot vanish simultaneously because those two matrices are not identical; again, A and C cannot vanish, and $AC - B^2 = \Delta$ cannot vanish or be negative, because the imaginary part of Ω must be different from zero; thus we may attribute to τ the sign of $\alpha\gamma_1 - \alpha_1\gamma$ or $\beta\delta_1 - \beta_1\delta$, and may suppose that A, C and $\sqrt{\Delta}$ are positive; we then have without ambiguity

$$\Omega = \frac{-B + i\sqrt{\Delta}}{C} \quad \ldots\ldots\ldots\ldots \quad (26)$$

Let $2\sigma + 1$ denote the uneven number

$$\tfrac{1}{2}(\alpha\delta_1 + \alpha_1\delta - \beta\gamma_1 - \beta_1\gamma);$$

we find immediately

$$n^2 = (2\sigma + 1)^2 + 4\tau^2\Delta, \quad \ldots\ldots\ldots\ldots \quad (27)$$

and, substituting in (24) for Ω its value given by (26), we obtain for $\frac{M_1}{M}$ the expression

$$\frac{M_1}{M} = \frac{2\sigma + 1 + 2i\tau\sqrt{\Delta}}{n}, \quad \ldots\ldots\ldots\ldots \quad (28)$$

showing that the ratio of the two multipliers is a simple quadratic surd, having unity for its analytical modulus.

We might also arrive at the formula (28) by compounding the transformations

$$\Omega = \begin{vmatrix} \delta_1, & -\beta_1 \\ -\gamma_1, & \alpha_1 \end{vmatrix} \times \omega, \qquad \omega = \begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \times \Omega, \quad \ldots\ldots \quad (29)$$

and equating the product of their multipliers to the multiplier of the resulting transformation. The multipliers of the transformations (29) are $(nM_1)^{-1}$ and M respectively; the resulting transformation

$$\Omega = \begin{vmatrix} \alpha\delta_1 - \beta_1\gamma, & \beta\delta_1 - \beta_1\delta \\ \alpha_1\gamma - \alpha\gamma_1, & \alpha_1\delta - \beta\gamma_1 \end{vmatrix} \times \Omega,$$

which is in fact a complex multiplication of the argument, has for its multiplier

$$[2\sigma + 1 + 2i\tau\sqrt{\Delta}]^{-1}.$$

The equation (27) shows that, for a given value of n, the ratio of the multipliers corresponding to any two equal values of λ^2 can only be one out of a finite number of imaginary quadratic surds; for that equation can be satisfied only by a finite number of values of σ, τ, and Δ. We may add that, conversely, every set of numbers $2\sigma+1$, 2τ, Δ, which are all different from zero, and which satisfy the equation (27), may be employed in the formula (28), and will serve to express the ratio of at least one pair of multipliers corresponding to equal values of λ^2 in the modular equation of order n. The demonstration of this converse proposition would, however, require a more complete discussion of the multiple points of the modular curves than can be undertaken within the limits of this note. Such a discussion would also show, that if M, M_1, M_2, ... are the multipliers appertaining to as many equal values of λ^2, we may have an equation such as $\frac{M_1}{M}=\frac{M_3}{M_2}$, but not an equation such as $\frac{M_1}{M}=\frac{M_2}{M_1}$. For our present purpose it suffices to observe, that the ratio of two multipliers answering to equal values of λ^2, cannot be a root of unity, because the imaginary fourth and sixth roots of unity, which alone of all the roots of unity are simple quadratic surds, cannot be represented by the formula (28).

Let
$$\mu\left(\frac{1}{M}, k^2, \lambda^2\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$
be the equation, rational in k^2 and λ^2, which determines the values of M at the multiple point (k^2, λ^2). We may suppose that this equation is irreducible, *i.e.* that μ cannot be resolved into factors without the adjunction of some irrationality other than those contained in k^2 and λ^2; if $\mu=0$ is not irreducible, we must consider successively instead of μ, the irreducible factors of μ.

Let $M=\rho M_1$, M and M_1 being two of the roots of (30), and ρ being a quadratic surd of the form (28). Then the equation
$$\mu\left(\frac{\rho}{M}, k^2, \lambda^2\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$
will have some, but not all of its roots in common with the equation (30); viz. $\frac{1}{M}$ is certainly common to the two equations; if $\frac{1}{M_1}$ is also common to the two, we must have $M_1=\rho M_2$, $\frac{1}{M_2}$ being a root of (30), different from $\frac{1}{M}$ and $\frac{1}{M_1}$, because $M=\rho M_1=\rho^2 M_2$; if $\frac{1}{M_2}$ is common to (30) and (31), we must have $M_2=\rho M_3$, $\frac{1}{M_3}$ being itself a root of (30). Since ρ is not a root of unity, this

process cannot terminate until we arrive at a root of (30) which is not also a root of (31). Therefore $\frac{1}{M}$ is the root of an equation which is of lower dimensions than (30), but of which the coefficients contain an imaginary quadratic surd $i\sqrt{\Delta}$. If $\frac{1}{M'}$ is any second root of this equation, we shall have $M = \rho' M'$, ρ' being a quantity of the form (28), different from ρ, but containing the same surd $i\sqrt{\Delta}$. We can therefore again reduce the order of the equation of which $\frac{1}{M}$ is a root, without introducing any new irrationality, and we shall at last arrive at an expression of the type

$$\frac{1}{M} = \Psi(i\sqrt{\Delta}, k^2, \lambda^2), \quad \ldots\ldots\ldots \quad (32)$$

Ψ being a rational function, with rational coefficients, of $i\sqrt{\Delta}$, k^2, λ^2. It is proper however to observe, that each time that a new ratio ρ is employed, the radical $i\sqrt{\Delta}$ is introduced with an independent sign; so that the expression (32) (in which we regard k^2 and λ^2 as given) admits not of two values only but of 2^r, if r is the number of different ratios ρ employed in the reduction.

In the example, considered in Art. 6, the quotient of the two values of M is $\frac{1}{5}(-3 \mp 4i)$, in accordance with the foregoing theory. And in general, if p is a prime number of the form $4m+1$, so that we may write

$$p = a^2 + b^2; \; b \equiv 0, \mod 2; \; a \pm b \equiv 1, \mod 4;$$

the integral $\int_0^x \frac{dx}{\sqrt{(1-x^4)}}$ is transformed into itself by two primary transformations, answering to the two equations

$$1 + i = \begin{vmatrix} -a \mp b, & \pm b \\ a \mp b, & a \end{vmatrix} \times (1+i).$$

The corresponding multipliers are $a \pm bi$, of which the ratio is $\frac{a^2 - b^2 \pm 2abi}{p}$, in accordance with the formula (28).

II. *On the primary periods of the elliptic functions, and on the complete rectilinear integrals.*

1. Two periods of a doubly periodic function are said to be *conjugate*, when the vectors representing them are adjacent sides of an elementary parallelogram in the parallelogrammic system appertaining to the function. If (P, P')

is a pair of conjugate periods of the function, all the conjugate pairs are included in the formula

$$\left.\begin{aligned} Q &= \alpha P + \gamma P' \\ Q' &= \beta P + \delta P' \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots \quad (1)$$

when α, β, γ, δ are integral numbers, and $\alpha\delta - \beta\gamma = \pm 1$.

The theory of the reduction of binary quadratic forms of a negative determinant is applicable in the manner explained by Gauss, in his review of Seeber's 'Untersuchungen über die ternären quadratischen Formen,' to the parallelogrammic system appertaining to any doubly periodic function.* Thus there is always a reduced parallelism, one side of its elementary parallelogram being the vector which represents the absolutely least period of the function, and the adjacent side representing the least period which is not a numerical multiple of the absolutely least period. The sides of a reduced parallelogram cannot exceed its diagonals, and the included angles is less than 120° and not less than 60°.

If

$$\left.\begin{aligned} P &= r + is, \qquad P' = r' + is' \\ A &= r^2 + s^2, \quad B = rr' + ss', \quad C = r'^2 + s'^2 \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (2)$$

the quadratic form (A, B, C), or

$$Ax^2 + Bxy + Cy^2 \quad \ldots\ldots\ldots\ldots \quad (3)$$

represents the parallelism determined by the given pair of periods (P, P'); and if this form be reduced by the method of Gauss, the equivalent reduced form represents the reduced parallelism. The reduced parallelism as well as the reducing substitution is unique, except when (A, B, C) is equivalent to a multiple by any real quantity of

$$x^2 + y^2, \quad \text{or} \quad 2x^2 + xy + 2y^2.$$

2. It will be convenient to restate, in a form adapted to the purpose of this Note, some of the principal results included in the theory of the representation of parallelogrammic systems by binary quadratic forms of a negative determinant. This theory is well known; but, in order to apply it to the elliptic functions, we have to introduce two modifications into it; viz. (i) we have to regard the sides of the elementary parallelograms as vectors, (ii) we have to restrict the conception of equivalence, confining it to primary equivalence only; this restriction is introduced in Art. 3; in this article we adhere to the Gaussian definition of equivalence.

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* Gauss, 'Werke,' vol. ii. p. 188. See also Dirichlet in 'Crelle's Journal,' vol. xl. p. 209; and the 'Report of the Theory of Numbers,' Part v. Art. 120 ('Report of the British Association' for 1863) [vol. i. p. 263].

(α) When we are given the quadratic form (3), we are given the absolute magnitudes $\surd A$ and $\surd C$ of the vectors P and P'; we are also given the quotient of these vectors, but with an ambiguity of sign; viz. if

$$\Omega = \frac{P'}{P}, \quad \Delta = AC - B^2,$$

we have

$$C - 2A\Omega + B\Omega^2 = 0, \quad \Omega = \frac{B \pm i\surd\Delta}{A}.$$

To remove this ambiguity, we adopt the convention that the direction of rotation from the first vector P to the second P' is to be positive*; in accordance with this convention the amplitude of Ω is positive, and if we denote by $\surd\Delta$ the positive square root of Δ we have, without ambiguity,

$$\Omega = \frac{B + i\surd\Delta}{A}. \quad \ldots\ldots\ldots\ldots \quad (4)$$

The vector defined by this equation may be termed the vector associated to the form (A, B, C).

Thus, when the quadratic form is given, the parallelism represented by it in any plane in which the positive direction of rotation has been assigned is given in species, but not in position; for, if we draw one of the two vectors in any arbitrary direction from the zero point, and set the other at the proper inclination to it, we obtain a parallelism which is represented by (A, B, C), and which by turning it about in its own plane may be made to coincide with any other parallelism represented in that plane by (A, B, C).

If (A, B, C) is transformed into (A', B', C') by an unit matrix $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$, the vectors of the new parallelism are

$$Q = \alpha P + \gamma P', \quad Q' = \beta P + \delta P',$$

and, if $\Omega' = \frac{Q'}{Q}$, we have

$$C' - 2B'\Omega' + A'\Omega'^2 = 0, \quad \Omega' = \frac{B' + i\surd\Delta}{C'},$$

$$\Omega = \frac{-\beta + \alpha\Omega'}{\delta - \gamma\Omega'}. \quad \ldots\ldots\ldots\ldots \quad (5)$$

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* If $Re^{i\theta} = X + iY$ is any complex quantity of which R is the analytical modulus and θ the amplitude, we shall suppose throughout this note that $-\pi < \theta \leqq \pi$; so that the amplitude always has the same sign as Y. The angle of rotation from one vector P to another P' is the amplitude of the quotient $\frac{P'}{P}$; thus the angle of rotation is always a positive or negative Euclidean angle, or in the limiting case a positive angle of 180°; and the direction of rotation from P to P' is positive or negative, according as the amplitude of $\frac{P'}{P}$ is positive or negative.

By the convention which we have adopted improper equivalence is excluded; for, the amplitudes of Ω and Ω' being both positive, the equation (5) implies that $\alpha\delta-\beta\gamma$ is positive. Two improperly equivalent forms, such, for example, as the opposite forms (A, B, C) and $(A, -B, C)$ represent parallelogrammic systems, which are symmetric to one another, but which, except in the cases of ambiguity to be noticed presently, cannot be made to coincide with one another by turning either of them about in the plane in which they lie.

In the general theory of periodic functions there does not appear to be any reason for attending to the distinction between the positive and negative directions of rotation, or, which comes to the same thing, to the distinction between proper and improper equivalence. But, as we shall presently see, this distinction becomes of importance in the theory of the elliptic functions, and especially in the connexion of that theory with the theory of the Theta functions.

(*b*) When the form (A, B, C) is reduced, the vector $\Omega=X+iY$ is said to be reduced, and *vice versâ*. The conditions that the form (A, B, C) should be reduced are ('Disq. Arith.' Arts. 171, *sqq.*)

$$\left.\begin{array}{lll} \text{(i)} & A\leqq C, & [B]\leqq A^*, \\ \text{(ii)} & \text{If } A=C, & B\geqq 0, \\ & [B]=A, & B>0, \end{array}\right\}; \quad \ldots\ldots\ldots \quad (6)$$

hence, when (A, B, C) is reduced, Ω satisfies the inequalities

$$\left.\begin{array}{lll} \text{(i)} & X^2+Y^2\geqq 1, & -\frac{1}{2}<X\leqq\frac{1}{2}, \\ \text{(ii)} & \text{If } X^2+Y^2=1, & X\geqq 0, \end{array}\right\}, \quad \ldots\ldots\ldots \quad (7)$$

or, which is the same thing, the extremity of the vector Ω falls within the 'reduced space,' *i.e.* the space which lies above the real axis, between the lines $X\pm\frac{1}{2}=0$, and outside the circle $X^2+Y^2=1$; the bounding line $X+\frac{1}{2}=0$, and the bounding arc, from $\frac{1}{2}(-1+i\sqrt{3})$ inclusively to i exclusively, are not considered to belong to the reduced space.

When (A, B, C) is reduced, the imaginary part of Ω has a greater absolute value than the imaginary part of any equivalent vector; for $Y=\frac{\sqrt{\Delta}}{A}$, and $\sqrt{\Delta}$ is invariant, while A in the reduced form has the least value possible.

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* Square brackets are here used to denote the absolute value of the quantity included in them; the same notation may be conveniently employed in the case of complex quantities, so that $[Re^{i\theta}]=R$.

When the extremity of Ω falls on one of the boundaries of the reduced space, this statement requires an unimportant modification, viz. in this case the two vectors $\pm X+iY$, of which one only is reduced, are equivalent. The value of Y in a reduced vector can never be less than $\frac{1}{2}\sqrt{3}$.

(*c*) A parallelogrammic system is *ambiguous*, when it can be brought into coincidence with itself by rotation through an angle of 180° round one of the lines of the system ; if (A, B, C) represent the reduced parallelism of the system, this takes place in the following three cases :

(1) $B=0$; the reduced parallelogram is a rectangle ; the extremity of Ω falls on the line $Y=0$.

(2) $A=C$; the reduced parallelogram is a rhombus ; the extremity of Ω falls on the boundary $X^2+Y^2=1$.

(3) $A=B$; the reduced parallelogram has a side and a diagonal perpendicular to one another ; the extremity of Ω lies on the boundary $X-\frac{1}{2}=0$.

In case (1) the axes of symmetry are the sides of the reduced parallelogram ; in case (2) they are its diagonals ; in case (3) they are its least side and the line of the system perpendicular to that side. The axes of symmetry are conjugate lines of the system in case (1) only ; in the other two cases the vectors lying in the axes of symmetry contain a parallelogram which is double of an elementary parallelogram. There are two pairs of axes of symmetry, when the system consists of squares, and three pairs when it consists of equilateral triangles ; in all other cases of ambiguity there is one pair only.

(*d*) A parallelogrammic system can always be brought into coincidence with itself by a rotation in its own plane through an angle of 180° ; this rotation corresponds to the automorphic $\begin{vmatrix} -1, & 0 \\ 0, & -1 \end{vmatrix}$ of the quadratic form. When the system consists of squares, it can be brought into coincidence with itself by a rotation through an angle of 90°, and when it consists of equilateral triangles by a rotation through an angle of 60° ; to these rotations there correspond the two pairs of opposite automorphics appertaining to the system

$$(A, B, C)=(1, 0, 1), \quad \Omega=i,$$

and the three pairs appertaining to the system

$$(A, B, C)=(2, 1, 2), \quad \Omega=\tfrac{1}{2}(1+i\sqrt{3}) ;$$

the identical pair $\pm\begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}$ counting in each case as a pair.

The parallelogrammic system of $\sin^2 \mathrm{am}\, u$ consists of squares when $k^2 = \frac{1}{2}$, and of equilateral triangles when k^2 is an imaginary cube root of -1.

3. We now restrict the application of the term equivalence to primary equivalence, and henceforward when we say that two forms or two vectors are equivalent, we shall understand that they can be transformed into one another by a primary unit matrix; *i.e.* by a matrix $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$, satisfying the equation $\alpha\delta - \beta\gamma = 1$, and the congruences $\alpha \equiv \delta \equiv 1$, mod 4; $\beta \equiv \gamma \equiv 0$, mod 2. When this limitation is not intended, we shall say that the two forms or vectors are *absolutely* equivalent.

In the general theory of doubly periodic functions we have only to deal with absolute equivalence; but in the case of the elliptic and modular functions, the consideration of primary equivalence is indispensable. It will be noticed that the two *opposite* parallelisms (P, P') and $(-P, -P')$, though represented by the same quadratic form, are not, strictly speaking, equivalent; for $\begin{vmatrix} -1, & 0 \\ 0, & -1 \end{vmatrix}$ is not a primary unit matrix. In the reduced parallelisms which we shall have occasion to consider in the case of the elliptic functions, the real part of the first vector is always positive, and the opposite parallelism need not be considered. The restriction in the definition of equivalence introduces certain modifications into the theory of reduction, which we shall now briefly indicate.

A quadratic form (A, B, C) is said to be (primarily) reduced, when its coefficients satisfy the inequalities

$$\left.\begin{array}{ll} \text{(i)} & [B] \geqq A, \qquad [B] \leqq C \\ \text{(ii)} & \text{If } [B] = A, \quad \text{or } [B] = C, \quad B > 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The associated vector Ω is reduced when the quadratic form is reduced, and *vice versâ*. The 'reduced space' lies above the real axis, between the lines $X \pm 1 = 0$ and outside the semicircles $X^2 + Y^2 \pm X = 0$; the line $X + 1 = 0$, and the semicircle $X^2 + Y^2 + X = 0$ are regarded as not appertaining to the reduced space. The extremity of a reduced vector falls within the reduced space, so that the vector satisfies the inequalities

$$-1 < X \leqq 1; \quad -X < X^2 + Y^2 \geqq X. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Theorem. 'Every quadratic form is equivalent to one, and only one, reduced form, and by one, and only one, reducing transformation.'

(1) Let α be the least number represented by (A, B, C) with values of the indeterminates which satisfy the congruences $(x, y) \equiv (1, 0)$, mod 2; and, among

all the forms equivalent to (A, B, C), which have a for their first coefficient, let (a, b, c) be that which has the least third coefficient; if the two opposite forms (a, b, c) and $(a, -b, c)$ are equivalent, we take for (a, b, c) that one of the two in which the second coefficient is positive; we then have

$$a - 4[b] + 4c \geqq a, \quad 4a - 4[b] + c \geqq c,$$

whence it follows that (a, b, c) is reduced; *i.e.* a reduced form always exists equivalent to a given form (A, B, C).

(2) Again, to show that only one reduced form can be equivalent to a given form, it suffices to establish the following lemma, which is in substance due to Legendre, and which serves to show that two reduced forms cannot be equivalent without being identical.

'If (a, b, c) is a reduced form, and if the numbers primitively represented by (a, b, c), or by any equivalent form, are divided into three series according as the indeterminates satisfy the congruences

$$(x, y) \equiv (1, 0), \quad (x, y) \equiv (1, 1), \quad (x, y) \equiv (0, 1),$$

the least numbers in these series are respectively

$$a, \quad a - 2[b] + c, \quad c.'$$

Let $$f(x, y) = ax^2 - 2[b]xy + cy^2,$$

and let us suppose that x and y are positive; the identities

$$f(x-2, y) = f(x, y) - 4a(x-1) + 4[b]y,$$
$$f(x, y-2) = f(x, y) - 4c(y-1) + 4[b]x,$$

combined with the conditions $[b] \leqq a$, $[b] \leqq c$, give rise to the inequalities

$$f(x-2, y) \leqq f(x, y), \text{ if } x > y,$$
$$f(x, y-2) \leqq f(x, y), \text{ if } y > x,$$

which show immediately that $f(1, 0)$, $f(1, 1)$, $f(0, 1)$, or, which is the same thing, a, $a - 2[b] + c$, c are the least numbers in the three series respectively. We may add that in the second series the least number but one is $a + 2[b] + c$.

(3) Lastly, it is readily seen that a reduced form cannot be transformed into itself by any primary substitution other than $\begin{vmatrix} \pm 1, & 0 \\ 0, & \pm 1 \end{vmatrix}$; hence, the reducing substitution is always unique.

To obtain a reduced form equivalent to a given form, we may employ, with

a slight modification, the algorithm of Gauss.* Let (a_0, b_0, c) be the given form; if $[b_0] > a_0$, let $b_0 + 2\mu_1 a_0 = b_1$, $[b_1] \leqq a_0$, and let $\begin{vmatrix} 1, & 2\mu_1 \\ 0, & 1 \end{vmatrix}$ transform (a_0, b_0, c_0) into (a_0, b_1, c_1); if $[b_1] > c_1$, let $b_1 + 2\mu_2 c_1 = b_2$, $[b_2] \leqq c_1$, and let $\begin{vmatrix} 1, & 0 \\ 2\mu_2, & 1 \end{vmatrix}$ transform (a_0, b_1, c_1) into (a_1, b_2, c_1); if $[b_2] > a_1$, the process can be recommenced, and as long as it continues, we shall have $a_0 > c_1 > a_1 > c_2 > \ldots$. But a quadratic form of negative determinant can represent only a finite number of quantities less than any given quantity. Hence, we must at last arrive at a form $(a_\mu, b_{2\mu}, c_\mu)$, in which $[b_\mu] \leqq a_\mu$, $[b_\mu] \leqq c_\mu$, and which, if not reduced itself, has for its opposite an equivalent and reduced form.

The coefficient of i in a reduced vector $\Omega = X + iY$ is greater than in any equivalent vector; for the first coefficient of a reduced form is less than the first coefficient of any equivalent form; the enunciation requires an unimportant correction when the extremity of the reduced vertex lies on a boundary of the reduced space. There is no inferior limit to the value of Y, which cannot vanish, but may be any positive quantity however small.

From the inequalities (8), which are satisfied by the coefficients of a reduced form (a, b, c), it appears that in the corresponding parallelism the triangle of which the vertices are

$$(0, 0),\ (1, 0),\ (0, \pm 1), \quad \ldots\ldots\ldots \quad (10)$$

is acute-angled; one of the acute angles becoming a right angle in the limiting cases $b = 0$, $b = a$, $b = c$; in the symbol $(0, \pm 1)$, and throughout the rest of this article, the upper or lower sign is to be taken according as b is positive or negative. The triangle (10), which we shall designate by UVW, and which may be termed a reduced triangle of the system, is in fact a triangle contained by two adjacent sides and the lesser diagonal of an absolutely reduced elementary parallelogram. Every elementary triangle of the system, which is not obtuse angled, is a reduced triangle; for if UV, UW, are the least sides of such a triangle, the elementary parallelogram contained by UV, UW, is an absolutely reduced parallelogram, because its sides cannot exceed its diagonals; all the reduced triangles of the system are equal to one another in all respects. The three sides of a reduced triangle, taken positively and negatively, can be paired in twenty-four different ways; in twelve of these the direction of rotation from the first side to the second is positive; we thus obtain twelve parallelisms

* 'Disq. Arith.,' Art. 171. See also the account of the theory of reduction in the 'Report on the Theory of Numbers,' Art. 92 ('Report of the British Association' for 1861) [vol. i. p. 182].

absolutely but not primarily equivalent to one another, if, however, we confine our attention to one of each couple of opposite parallelisms, such as P, P') and $(-P, -P')$, it will suffice to consider the six exhibited in the following scheme:

$$\left.\begin{array}{lll} (UV, \pm UW), & (VW, \pm VU), & (WU, \pm WV) \\ (\pm UW, -UV), & (\pm VU, -VW), & (\pm WV, -WU) \end{array}\right\} \quad . \quad . \quad . \quad (11)$$

If $$\rho_1 = \begin{vmatrix} -1, & \mp 1 \\ \pm 1, & 0 \end{vmatrix}, \quad \rho_2 = \begin{vmatrix} -1, & \pm 1 \\ \mp 1, & 0 \end{vmatrix}, \quad \psi = \begin{vmatrix} 0, & -1 \\ 1, & 0 \end{vmatrix},$$

so that $$\rho_1^3 = \rho_2^3 = -\psi^2 = \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix} = 1,$$

the three parallelisms in the upper row of the scheme (11) are transformed into one another in cyclic order by the matrices $1, \rho_1, \rho_1^2$; the three in the lower row by $1, \rho_2^2, \rho_2$, and any two in the same column by $\phi^{\pm 1}$. The six parallelisms are all primarily reduced; they are in general different in species from one another; they are, however, of the same species in sets of two, if UVW is isosceles and right-angled, and in sets of three if UVW is equilateral. The system has a pair of axes of symmetry if UVW is either rectangular or isosceles; two pair, if it is both, and three pair if it is equilateral. But the quasi-symmetrical pairs of parallelisms are equivalent, only when UVW is right-angled, so that in this case only is the system ambiguous in the sense of primary equivalence.

4. Let $(4L, 2L')$ be a pair of conjugate periods of the elliptic function $\sin \mathrm{am}(u, k^2) = \lambda(u)$, which we here regard as defined by the equations

$$x = \lambda(u), \quad u = \int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the initial value of the radical being $+1$, and k^2 being any complex quantity whatever, other than $0, 1, \infty$.

It is important in the theory of the function $\lambda(u)$ to consider specially the pairs of conjugate periods, infinite in number, which satisfy the equation

$$\lambda(L) = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and in addition the condition that the direction of rotation from L to L' is to be positive. Such pairs of conjugate periods of $\lambda(u)$ we shall term *primary*.*

* There is a slight difference between the definition of the primary periods considered in the text and the definition of the elliptic periods of MM. Briot and Bouquet; viz. the pair $[4K, 2iK']$ is an elliptic, as well as a primary, pair of periods of $\lambda(u)$; but $[4Q, 2Q']$ is an elliptic pair of periods, only when β and γ are evenly even in the equations (15); these authors, however, in the first edition of their classical work (Paris, 1859) had given a slightly wider definition of elliptic periods, requiring only that β should be evenly even.

One pair of primary periods of $\lambda(u)$ can always be assigned. Let

$$\left.\begin{aligned} K &= \int_0^1 \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}} \\ iK' &= \int_{\frac{1}{k}}^1 \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}} \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (14)$$

the integrations being rectilinear. The initial sign of the radical in the former integral is taken to be positive; the sign of the radical in the second integral is so taken as to render positive the amplitude of the quotient $\frac{iK'}{K}$. The more accurate determination of these integrals will occupy us in Art. 5; for the present either sign may be attributed to $\frac{1}{k}$ in the lower limit of the second integral. By reducing to elementary contours the different tracks along which x may travel from the lower to the upper limit in the integral (12), it may be shown that $[4K, 2iK']$ is a pair of conjugate periods of the integral, or, which is the same thing, of the function $\lambda(u)$. But we have evidently $\lambda(K) = 1$, and the amplitude of $\frac{iK'}{K}$ is by hypothesis positive; therefore $[4K, 2iK']$ is a pair of primary periods of $\lambda(u)$.

Theorem I. 'If $[4L, 2L']$ is any given primary pair of periods of $\lambda(u)$, all the primary pairs $[4Q, 2Q']$ are exhibited by the formula

$$\left.\begin{aligned} Q &= \alpha L + \gamma L' \\ Q' &= \beta L + \delta L' \end{aligned}\right\},$$

in which $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$ is any primary unit matrix.'

It will be sufficient to demonstrate this theorem on the supposition that $L = K$, $L' = iK'$.

(1) Let $[4Q, 2Q']$ be a primary pair of periods of $\lambda(u)$. If $\lambda(v) = \lambda(u)$, we have, by the known properties of the function $\lambda(u)$, either $v = u$, or $v = 2K - u$, omitting multiples of the periods of $\lambda(u)$ in either case. Hence the equation $\lambda(Q) = 1 = \lambda(K)$ implies an equation of the form

$$Q = (4r+1)K + 2siK'.$$

Again, $2Q'$, which is a period of $\lambda(u)$, must be a sum of multiples of the conjugate periods $4K$ and $2iK'$; we have therefore

$$Q' = 2r'K + s'iK'.$$

Here s' is uneven, because, $[4Q, 2Q']$ being a conjugate pair, we must have $(4r+1)\,s'-4r's=\pm 1$; the upper sign is to be taken because the amplitudes of $\frac{Q'}{Q}$ and $\frac{iK'}{K}$ are both positive. Writing

$$a=4r+1,\ \beta=2r',\ \gamma=2s,\ \delta=s',$$

the matrix $\left|\begin{matrix}a, \beta\\ \gamma, \delta\end{matrix}\right|$ is primary, and the equations

$$\left.\begin{aligned} Q &= aK+\gamma iK'\\ Q' &= \beta K+\delta iK'\end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (15)$$

are satisfied.

(2) Conversely, if $[Q, Q']$ is a pair of quantities satisfying the equations (15), in which the matrix $\left|\begin{matrix}a, \beta\\ \gamma, \delta\end{matrix}\right|$ is primary, we find immediately $\lambda(Q)=\lambda(L)=1$; also the amplitude of $\frac{Q'}{Q}$ is positive; *i.e.* $[4Q, 2Q']$ is a primary pair of periods of $\lambda(u)$.

If $[4Q, 2Q']$ is any primary pair of periods of $\lambda(u)=\sin\operatorname{am} u$, the pairs

$$[Q, Q'];\ [2Q, 2Q'];\ [2Q-2Q', 2Q+2Q'];\ [2Q, 4Q'];$$

are pairs of conjugate periods of $\lambda^2(2u)$, of $\lambda^2(u)$, of $\cos\operatorname{am} u$, and of $\Delta\operatorname{am} u$ respectively, and may be termed primary pairs of periods of those functions. Every primary pair $[Q, Q']$, in addition to the equation (13), satisfies the equations

$$\left.\begin{aligned} &\lambda(2Q)=0,\quad \lambda(\pm Q')=\lambda(2Q\pm Q')=\infty\\ &\lambda(Q\pm Q')=\pm\frac{1}{k}\end{aligned}\right\}, \quad \ldots\ldots \quad (16)$$

of which the first two result from the known formulæ (see Art. 7)

$$\lambda(2K)=0,\quad \lambda(\pm iK')=\lambda(2K\pm iK')=\infty,$$

the last is an immediate consequence of the definitions of K and iK'. If the sign of k be determined as in Art. 5, the upper or lower sign has to be taken in the right-hand number of the last equation, accordingly as β is evenly or unevenly even in the equations (15).

The following theorem, in the enunciation of which $\phi(\Omega)$ is the function defined by the equations,

$$\phi(\Omega)=\sqrt{2}\,q^{\frac{1}{8}}\,\Pi_{m=1}^{m=\infty}\frac{1+q^{2m}}{1+q^{2m-1}},\quad q=e^{i\pi\Omega}, \quad \ldots\ldots \quad (17)$$

may serve to show the importance of considering the primary periods of elliptic functions.

Theorem II. 'If $\Omega = \frac{Q'}{Q}$, $[2Q, 2Q']$ being a primary pair of periods of $\lambda^2(u, k^2)$, Ω satisfies the equation

$$\phi^8(\Omega) = k^2,$$

and, conversely, if this equation be satisfied by a complex quantity Ω of positive amplitude, a pair of primary periods $[2Q, 2Q']$ of $\lambda^2(u, k^2)$ can be assigned, such that $\Omega = \frac{Q'}{Q}$.'

Demonstration (1). If $\Omega = \frac{Q'}{Q}$, $[2Q, 2Q']$ being a primary pair of periods of $\lambda^2(u, k^2)$, let

$$q = e^{i\pi\Omega}, \quad h = \tfrac{1}{2}\pi\left[\Sigma_{m=-\infty}^{m=+\infty} q^{m^2}\right]^2, \quad \mu = \frac{h}{Q}, \quad h' = \Omega h = \mu Q'.$$

It is known from the theory of the Theta functions that the function $\lambda[u, \phi^8(\Omega)]$ is identical with the function

$$\frac{1}{\phi^2(\Omega)} \times \frac{\vartheta_1\left(\frac{\pi u}{2h}, q\right)}{\vartheta_0\left(\frac{\pi u}{2h}, q\right)}, \quad \ldots\ldots\ldots \quad (18)$$

and has, therefore, $[4h, 2h']$, or $[4\mu Q, 2\mu Q']$ for a pair of primary periods. The two functions $\lambda[\mu u, \phi^8(\Omega)]$ and $\lambda[u, k^2]$ consequently have the same zero points $2mQ + 2nQ'$, and the same infinite points $2mQ + (2n+1)Q'$; that is, they differ only by a constant factor, which may be determined by observing that $\lim \frac{\lambda(u)}{u} = 1$, when $u = 0$. We thus obtain the equation $\lambda[\mu u, \phi^8(\Omega)] = \mu \times \lambda[u, k^2]$. But $\lambda[Q, k^2] = 1$, $\lambda[h, \phi^8(\Omega)] = 1$; we have therefore

$$\mu = 1, \quad \lambda[u, k^2] = \lambda^8[u, \phi(\Omega)],$$

and, finally, differentiating with regard to u, $\phi^8(\Omega) = k^2$.

(2) Conversely, if $\phi^8(\Omega) = k^2$, Ω being a complex quantity of positive amplitude, the function $\lambda^2(u, k^2)$ is identical with the square of the function (18), and has $[2h, 2h']$ for a pair of primary periods, of which the quotient is Ω.*

The primary pairs of periods of $\lambda^2(2u)$ are represented by a parallelogrammic system, which we may reduce by a primary transformation. If $[M, M']$ is the reduced primary pair, the least primary pairs of $\sin \mathrm{am}\, u$, $\cos \mathrm{am}\, u$, $\Delta \mathrm{am}\, u$, are

$$[4M, 2M'], \quad [2M - 2M', 2M + 2M'], \quad [2M, 4M'],$$

---

* The properties of the function $\phi(\Omega)$ employed in Note I, equations (16) and (17), are immediate consequences of the theorems of this article.

respectively. For $\sin \operatorname{am} u$ and $\Delta \operatorname{am} u$ this is evident, and it may be proved for $\cos \operatorname{am} u$ by observing that $M \pm M'$ are the two least values included in the formula

$$(2r+1)M+(2s+1)M$$

(see Art. 3). We shall presently show that the rectilineal integrals K, iK' are the reduced primary pair of periods of $\lambda^2(2u, k^2)$. In order, however, to give a precise signification to this assertion, we have in the first place to remove the ambiguity which attaches to the definition of iK', and in certain cases to the definition of K. We shall give in the next article the requisite determinations relating to these, and to some other elementary integrals which present themselves in the theory.

5. By the first of the equations (14), the integral K is completely determined, except when k^2 is real, positive, and greater than unity. In this case, if k is the positive square root of k^2, the radical vanishes when we come to the point $x=\frac{1}{k}$; after that point its value is a pure imaginary and its sign can only be fixed by a new convention. The convention which we adopt is that it has a negative amplitude $-\frac{1}{2}\pi$. This is the same thing as to suppose that x, when it arrives close to the point $\frac{1}{k}$, describes an infinitesimal semicircle round that point in the negative direction, *i.e.* above the real axis. We, in fact, regard k^2 as having an evanescent positive amplitude, so that $\frac{1}{k}$ is a point lying just below the real axis, and infinitely near to it. The effect of the convention, in the case to which it applies, is to render positive the amplitude of every element of K, and therefore the amplitude of K itself; the real part of K is in all cases positive.

To complete in the second equation (14) the determination of the integral iK', we understand by k that square root of k^2, of which the real part is positive, or, if k^2 is real and negative, that square root of k^2 of which the amplitude is the positive angle $\frac{1}{2}\pi$. The sign of the radical is so taken that the real part of the expression

$$\frac{i}{1-\frac{1}{k}} \times \surd\{(1-x^2)(1-k^2x^2)\}$$

is positive; this real part is nowhere equal to zero in the course of the integration except at the two limits; for if $x=\frac{1}{k}+\theta\left(1-\frac{1}{k}\right)$, θ being a real variable,

of which the limits are 0 and 1, the radical $\surd\{(1-x^2)(1-k^2x^2)\}$ may be written in the form

$$\frac{1}{i}\left(1-\frac{1}{k}\right)\surd\{\theta(1-\theta^2)(2-\theta)\}\times\sqrt{\left\{\left(1+\frac{1-\theta}{1+\theta}k\right)\left(1+\frac{\theta}{2-\theta}k\right)\right\}},$$

where the first radical factor is real, and is to be taken with a positive sign, and where it is readily seen that the real part of the second radical factor never vanishes, because the quantity under the radical sign never becomes real and negative. We thus have

$$K'=\int_0^1\frac{d\theta}{\sqrt{\left\{\theta(1-\theta^2)(2-\theta)\left(1+\frac{1-\theta}{1+\theta}k\right)\left(1+\frac{\theta}{2-\theta}k\right)\right\}}},$$

the real part of the radical under the integral sign being positive; whence it follows, that the real part of every element of K', and therefore the real part of K' itself, is positive.

The convention by which we have fixed the sign of the radical in the expression for iK' comes to saying that at every point P in the rectilinear track from $\frac{1}{k}$ to 1 the direction of rotation from the vector $1-\frac{1}{k}$ to the vector

$$\surd\{(1-x^2)(1-k^2x^2)\}$$

is to be negative. The rule thus indicated is equivalent to the following:

'Let a simple closed contour be drawn passing through 0 and P, and including $+1$, but not including -1, $+\frac{1}{k}$, $-\frac{1}{k}$; and let x, setting out from 0, describe this contour in the positive direction round $+1$; the radical, setting out with the initial value $+1$, will arrive at P with the sign which it is to have in the integral expression for iK'.'

To verify the coincidence of the two rules, it is sufficient to show that they agree at any one point of the rectilinear track. Let x travel from 0 along the real axis until it arrives at a point Q close to $+1$, and let it then describe in the positive direction a circular arc of infinitesimal radius round $+1$ as centre, until it arrives at a point P on the rectilinear track. Let α be the amplitude of $1-\frac{1}{k}$, β the amplitude of $k'=\surd(1-k^2)$, the real part of this radical being supposed to be positive; it will be seen that α and β are of opposite signs; viz. the amplitudes of k^2 and $1-\frac{1}{k}$ have the same sign, and the amplitudes of k^2 and k' have opposite signs. When x arrives at Q the amplitude of the radical

is approximately β; when x arrives at P, this amplitude is $\beta+\frac{1}{2}\alpha$, or $\pi+\beta+\frac{1}{2}\alpha$, according as the amplitude of k^2 is positive or negative; *i.e.* according as α is positive and β negative, or α negative and β positive. Subtracting the amplitude of $1-\frac{1}{k}$, we find that the difference $\beta-\frac{1}{2}\alpha$ lies between $-\pi$ and 0 and the difference $\pi+\beta-\frac{1}{2}\alpha$ between π and 2π, because $\frac{1}{2}\alpha$ and β are each in absolute magnitude less than $\frac{1}{2}\pi$. The two rules are therefore in agreement with one another. This conclusion holds even in the cases in which k^2 is real, and the demonstration may be applied to them, if we regard k^2 as a quantity of positive amplitude, evanescent if k^2 is positive, and differing from π by an infinitesimal if k^2 is negative. We may observe that when the amplitude of k^2 is positive the radical arrives at the point P with its proper sign, if x travels in a straight line from 0 to P; the reverse is the case when the amplitude of k^2 is negative.

6. The preceding determinations give immediately

$$\int_0^{\frac{1}{k}} \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}} = K \mp iK', \quad \ldots\ldots\ldots \quad (19)$$

the integration being rectilinear, and the initial value of the radical being $+1$; the upper or lower sign is to be taken according as the amplitude of k^2 is positive or negative.*

Again, writing $x=\frac{1}{ky}$ in the integral (19), we find

$$\int_0^{\frac{1}{k}} \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}} = \int_1^{\infty} \frac{dx}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}},$$

y being a real variable, and the real part of the radical under the integral sign on the right-hand side being positive; this determination is obtained by comparing corresponding elements, near to 0 and near to ∞ in the two integrals respectively; two such elements must have the same amplitude, viz. that of

* Instead of employing the second of the equations (14), we might define $2iK'$ as the elliptic integral $\int_0^0 \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}$, extended in the negative direction over a simple closed contour including $\frac{1}{k}$ and 1, but not $-\frac{1}{k}$ or -1. We should thus arrive immediately at the equation (19); and to show that in the equation (22) *infra* the upper sign has always to be taken, we might employ the principle of continuity; viz., when k^2 is real, positive, and less than unity, the integrals in the equation (22) are each of them real and positive, and as they do not become infinite or zero for any value of k^2, other than 0, 1, ∞, the sign cannot change when k^2 is made to pass from any one value to any other.

$\frac{1}{k}$ approximately. The sign thus attributed to the radical on the right-hand side is that which it would acquire if y were to travel from 0 along the real axis and were to describe a semicircle round $+1$ in the positive or negative direction, according as ik' or $-ik'$ has a positive real part, *i.e.* according as the amplitude of k^2 is positive or negative. Hence, combining the equation (19) with the first of the equations (14), we find

$$\left.\begin{aligned} iK' &= 2K - \int_0^\infty \frac{dy}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}} \\ iK' &= \quad -\int_0^\infty \frac{dy}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}} \end{aligned}\right\}, \qquad \ldots\ldots \quad (20)$$

according as the amplitude of k^2 is positive or negative, the integrals being rectilinear and the variable y being supposed to describe an infinitesimal semicircle in the positive direction round $+1$.

We can now assign the values of all the different *complete* rectilineal integrals; *i.e.* of all the rectilinear integrals included in the formulæ

$$(a, b) = \int_a^b \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},$$

$$(a, \Upsilon) = \int_a^\Upsilon \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},$$

when a, b are any two of the points $0, \pm 1, \pm\frac{1}{k}$, and Υ is a point at an infinite distance.

The following is a list of the complete integrals; the twenty integrals (a, b) are reduced to six by means of the formula

$$(a, b) = -(-a, -b) = -(b, a).$$

The upper or lower sign is to be taken throughout according as the amplitude of k^2 is positive or negative. In the symbol (a, Υ_s) we understand by $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$ points lying at an infinite distance in the angles respectively contained by the vectors drawn from a to the pairs of points

$$\left[\frac{1}{k}, 1\right], \quad \left[1, -\frac{1}{k}\right], \quad \left[-\frac{1}{k}, -1\right], \quad \left[-1, \frac{1}{k}\right];$$

the vector drawn from a to a is to be interpreted as the vector from 0 to a produced. If P is a point infinitely near to a on the track ab, or $a\Upsilon$, the initial sign of the radical in the integral (a, b), or (a, Υ), is the sign with which the radical arrives at P, when x travels in a straight line from 0 to P, and when the radical sets out with the value $+1$.

A.

$$(0, 1) = K, \qquad \left(0, \frac{1}{k}\right) = K \mp iK';$$

$$\left(1, \frac{1}{k}\right) = \mp iK', \quad (1, -1) = -2K, \quad \left(1, -\frac{1}{k}\right) = -2K \pm iK';$$

$$\left(\frac{1}{k}, -\frac{1}{k}\right) = -2K \pm 2iK'.$$

B.

$$\begin{aligned}
(0, \Upsilon_1) &= 2K \mp iK', & (0, \Upsilon_2) &= \pm iK';\\
(0, \Upsilon_3) &= -2K \pm iK', & (0, \Upsilon_4) &= \mp iK';\\
(1, \Upsilon_1) &= K \mp iK', & (1, \Upsilon_2) &= -K \pm iK';\\
(1, \Upsilon_3) &= -3K \pm iK', & (1, \Upsilon_4) &= -K \mp iK';\\
(-1, \Upsilon_1) &= 3K \mp iK', & (-1, \Upsilon_2) &= K \pm iK';\\
(-1, \Upsilon_3) &= -K \pm iK', & (-1, \Upsilon_4) &= K \mp iK';\\
\left(\frac{1}{k}, \Upsilon_1\right) &= K, & \left(\frac{1}{k}, \Upsilon_2\right) &= -K \pm 2iK';\\
\left(\frac{1}{k}, \Upsilon_3\right) &= -3K \pm 2iK', & \left(\frac{1}{k}, \Upsilon_4\right) &= -K;\\
\left(-\frac{1}{k}, \Upsilon_1\right) &= 3K \mp 2iK', & \left(-\frac{1}{k}, \Upsilon_2\right) &= K;\\
\left(-\frac{1}{k}, \Upsilon_3\right) &= -K, & \left(-\frac{1}{k}, \Upsilon_4\right) &= K \mp 2iK'.
\end{aligned}$$

In verifying the formulæ (*B*), it is useful to notice (i) that Υ_1, Υ_3 are *opposite* points in the formulæ (α, Υ_1), $(-\alpha, \Upsilon_3)$, and Υ_2, Υ_4 in the formulæ (α, Υ_2), $(-\alpha, \Upsilon_4)$; (ii) that if the tracks $\alpha\Upsilon_r$, $\alpha\Upsilon_s$ are separated by the track ab, and by that track only, we have $(\alpha, \Upsilon_r) + (\alpha, \Upsilon_s) = 2(\alpha, b)$.

The formulæ of this article have been constructed with reference to the general case in which k is any complex quantity whatever; but they retain their validity when k is real; we have only in this case to regard k^2 as a quantity of evanescent positive amplitude, and to interpret accordingly the tracks ab, $\alpha\Upsilon$, and the rule by which the sign of the radical has been determined in each integral. For example, if k^2 be real and less than unity, we have

$$\left(-\frac{1}{k}, \Upsilon_1\right) = \int_{-\frac{1}{k}}^{\infty} \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}} = 3K - 2iK',$$

the track of the integration lying along the real axis, above the points -1

and $\frac{1}{k}$, but below the point $+1$; and the initial sign of the radical being such that it acquires the value $+1$, when x arrives at 0.

7. We shall now establish the equation

$$K=\int_0^1 \frac{dy}{\sqrt{\{(1-y^2)(1-k'^2y^2)\}}}, \quad \ldots\ldots\ldots \quad (21)$$

the integration being rectilinear, and the initial value of the radical being $+1$; if k^2 is real and negative, so that $k'=\sqrt{(1-k^2)}$ is positive and greater than unity, the radical vanishes in the course of the integration; in this case we again suppose that k^2 is a complex quantity, of which the amplitude is positive, and differs from π by an infinitesimal; or, which comes to the same thing, we suppose that, after we have passed the point $x=\frac{1}{k}$, the amplitude of the radical, which has now become a pure imaginary, is positive. The effect of this convention, in the case to which it applies, is to render negative the amplitude of the integral

$$\int_0^1 \frac{dy}{\sqrt{\{(1-y^2)(1-k'^2y^2)\}}};$$

the real part of this integral is in all cases positive.

The substitution

$$x^2=\frac{1-k'^2y^2}{k^2}$$

transforms any given value of the integral

$$\frac{1}{i}\int_{\frac{1}{k}}^1 \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}$$

into a corresponding value of the integral

$$\int_0^1 \frac{dx}{\sqrt{\{(1-y^2)(1-k'^2y^2)\}}}.$$

To ascertain what value of the second integral is equal to K', we distinguish three cases.

First, let k^2 not be real, so that $\frac{1}{k}$ has an imaginary part different from zero, and a real part different from zero and positive. Let $\frac{1}{k}=\rho e^{i\theta}$, $x=X+iY$. As y pursues the rectilineal track from 0 to 1, $X+iY$ describes the finite arc of the hyperbola

$$X^2-Y^2+2\frac{1-\rho^2\cos 2\theta}{\rho^2\sin 2\theta}XY=1$$

from $\frac{1}{k}$ or $(\rho\cos\theta, \rho\sin\theta)$, to $+1$, or $(1, 0)$; it will be observed that the points $\frac{1}{k}$ and $+1$ lie on the same branch of the hyperbola, because the real part of $\frac{1}{k}$ is positive. But the curvilinear integral along the arc of the hyperbola is equal to the rectilinear integral along the chord $\left(\frac{1}{k}, +1\right)$, because neither of the two remaining discriminantal points -1 and $-\frac{1}{k}$ can lie in the segmental area included between the curve and the chord. Hence

$$K' = \pm\int_0^1 \frac{dy}{\sqrt{\{(1-y^2)(1-k'^2y^2)\}}}, \quad \ldots\ldots\ldots \quad (22)$$

and the upper sign has to be taken because the real part of K', as well as the real part of the integral on the right-hand side, is positive.

Secondly, let k^2 be real and negative; as y passes from 0 to $\frac{1}{k'}$, $X+iY$ describes the axis of Y from $\frac{1}{k}$ to 0, and as y passes from $\frac{1}{k'}$ to $+1$, $X+iY$ describes the axis of X from 0 to $+1$. But the integral along the broken line $\left(\frac{1}{k}, 0, +1\right)$ is equal to the rectilinear integral from $\frac{1}{k}$ to $+1$. Hence the equation (21) subsists in this case also, the sign being determined as before by the consideration of the real parts of the integrals. It will be noticed that the convention of this article by which we have fixed, between the points $\frac{1}{k'}$ and $+1$, the sign of the radical in the integral expression on the right-hand side of (21) agrees with the convention of Art. 5, according to which $-ik$ is positive; viz. these conventions render simultaneously negative the amplitudes of the two integrals in the equation (21).

The remaining case, when k^2 is real and positive, presents no difficulty, as the integrals in the equation (21) are both real and positive.

We may, henceforward, define K and K' as the rectilineal integrals

$$K = \int_0^{\frac{1}{2}\pi} \frac{dv}{\sqrt{(1-k^2\sin^2 v)}}, \qquad K' = \int_0^{\frac{1}{2}\pi} \frac{dv}{\sqrt{(1-k'^2\sin^2 v)}}, \quad \ldots \quad (23)$$

the initial values of the radicals being $+1$, and the same conventions as before being retained when k^2 is real and greater than $+1$, and when k^2 is real and negative.

In connexion with these integrals we shall consider the equation

$$K \mp iK' = \int_0^{\frac{1}{2}\pi} \frac{dv}{\sqrt{(k^2 - \sin^2 v)}}, \quad \ldots\ldots\ldots \quad (24)$$

which is obtained from the equation (19) by writing $x = \frac{\sin v}{k}$. The initial value of the radical is k, and the upper or lower sign is taken according as the amplitude of k^2 is positive or negative. When k^2 is real, positive, and less than unity, $\sin v$ must be supposed to describe a semicircle in the positive direction round k; but we shall not have occasion in what follows to employ this determination.

8. *Theorem* I. 'The vectors K and K' make acute angles with the vector $+1$ on opposite sides of it; and the angle between K and K' is an acute angle.'

In this enunciation we understand by an acute angle an angle lying between the limits 0° inclusively and 90° exclusively.

When k^2 is real, positive, and less than unity, the vectors K and K' are real and positive; when k^2 has any other real value one of them is real and positive, and the real part of the other is different from zero and positive; in these cases the theorem needs no demonstration. When k^2 is not real, the vectors $1 - k^2$ and k^2 lie on opposite sides of the real axis; and if we produce the vector k^2 indefinitely both ways, the vectors $1 - k^2$ and $+1$ lie on the same side of this indefinite line; hence the vectors k and k' (of which the real parts are positive) lie on opposite sides of $+1$, making acute angles with it and with one another; the same things are consequently true for the vectors $\frac{1}{k}$ and $\frac{1}{k'}$. But for every value of v the amplitude of the vector $\frac{1}{\sqrt{(1 - k^2 \sin^2 v)}}$ is less than the amplitude of $\frac{1}{k'}$, and is of the same sign; hence, the amplitude of every element of K, and therefore of K itself, is less than the amplitude of $\frac{1}{k'}$, and is of the same sign. Similarly, the amplitude of K' is less than the amplitude of $\frac{1}{k}$, and is of the same sign; *i.e.* the amplitude of K and K' are of opposite signs, and each of these amplitudes, as well as the amplitude of the quotient $\frac{K'}{K}$, is less in absolute magnitude than 90°.

Cor. The amplitude of the quotient $\frac{iK'}{K}$ lies between 0° and 180°, exclusively of both limits; it is less or greater than 90° according as the amplitude of k^2 is positive or negative.

Thus the ratio of the two periods of the elliptic integral

$$\int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}$$

is always imaginary. It is of some importance in the theory of the function $\lambda(u)$ to establish this elementary proposition by the direct consideration of the integral expressions for the periods themselves. It is indeed readily inferred from the equation of definition of $\lambda(u)$, that $\lambda(u)$ is a one-valued function of u, having the character of a rational fraction throughout the whole plane; and this character is incompatible with the existence of a pair of periods having a real incommensurable ratio; but is not incompatible with the existence of two periods having a commensurable ratio, and in fact reducible to a single period. Thus, in order to show that for all values of k^2, other than $0, 1, \infty$, $\lambda(u)$ is a doubly periodic function of u, it has to be shown that the ratio of the two periods is not commensurable. And this may perhaps be most simply done by showing, as has here been shown, that this ratio is always imaginary.

Theorem II. 'The triangles formed by the vectors

$$K, \quad -K+iK', \quad -iK', \quad \ldots\ldots\ldots\ldots \quad (a)$$

$$K, \quad -K-iK', \quad iK', \quad \ldots\ldots\ldots\ldots \quad (b)$$

are both right-angled when k^2 is real; when k^2 is not real, the triangle (a) is acute-angled, and the triangle (b) obtuse-angled, or *vice versâ*, according as the amplitude of k^2 is positive or negative.'

Demonstration (1). If k^2 is real, positive, and less than unity, the angle between the vectors K and iK' is a right angle. If k^2 is real, positive, and greater than unity, the integral (24) is real, and the angle between the vectors $K-iK'$ is a right angle; if k^2 is real and negative, the integral (24) is a pure imaginary, and the angle between $K-iK'$ and K is a right angle.

(2) Let the amplitude of k^2 be positive, *i.e.* greater than 0° and less than 180°; in this case the amplitudes of $\frac{1}{k}$ and $\frac{1}{ik'}$ are negative, the former amplitude being absolutely less than the latter, and each being absolutely less than 90°. But the vector $K-iK'$ lies between the two vectors $\frac{1}{k}$ and $\frac{1}{ik'}$, because $\sqrt{(k^2-\sin^2 v)}$ always lies between k and ik'. Since K lies between $+1$ and $\frac{1}{k'}$, the angle from $K-iK'$ to K is acute and positive; and since iK' lies between $\frac{i}{k}$ and i, the angle from iK' to $-K+iK'$ is also acute and positive.

(3) Let the amplitude of k^2 be negative. The vector $K+iK'$ may be shown precisely as in the former case to lie between the vectors $\frac{1}{k}$ and $\frac{i}{k'}$, each of which has a positive amplitude less than 90°, that of $\frac{i}{k'}$ being the greater. Since K lies between $\frac{1}{k'}$ and $+1$, the angle from K to $K+iK'$ is acute and positive, and since iK' lies between i and $\frac{i}{k}$, the angle from $K+iK'$ to iK is also acute and positive.

That the angle from K to iK' in case (2), and from $-iK'$ to K in case (3), is acute and positive has been shown in the demonstration of Theorem I.

Cor. The angle opposite to K in the acute-angled triangle is always absolutely greater than the amplitude of k', and the angle opposite to iK' is absolutely greater than the amplitude of k.

Theorem III. 'When k^2 is real, the right-angled triangles (a) and (b) are reduced triangles of the parallelogrammic system $[K, iK']$; when k^2 is not real, the acute-angled triangle is a reduced triangle of the system.'

Theorem IV. 'The pairs of periods

$$[2K, \quad 2iK'], \quad [-2K \pm 2iK', \mp 2K], \quad [\mp 2iK', \pm 2K - 2iK'],$$
$$[2iK', -2K], \quad [\mp 2K, \quad 2K \mp 2iK'], \quad [\pm 2K - 2iK', \pm 2iK'],$$

are respectively the least pairs of primary periods of the functions

$$\lambda^2(u, k^2), \quad \lambda^2\left(ik'u, \frac{1}{k'^2}\right), \quad \lambda^2\left(iku, -\frac{k'^2}{k^2}\right),$$
$$\lambda^2(iu, k'^2), \quad \lambda^2\left(k'u, -\frac{k^2}{k'^2}\right), \quad \lambda^2\left(ku, \frac{1}{k^2}\right),$$

or, which is the same thing, they are respectively the least pairs of periods of $\lambda^2(u, k^2)$ which satisfy the equations

$$\phi^8(\Omega) = k^2, \quad \phi^8(\Omega) = \frac{1}{1-k^2}, \quad \phi^8(\Omega) = \frac{k^2-1}{k^2},$$
$$\phi^8(\Omega) = 1-k^2, \quad \phi^8(\Omega) = \frac{k^2}{k^2-1}, \quad \phi^8(\Omega) = \frac{1}{k^2},$$

Ω representing the quotient obtained by dividing the second period of any pair by the first; the six quotients Ω are all primarily reduced, and one of them is absolutely reduced.'

The pairs of periods, and the linear transformations of $\lambda^2(u)$, which appear in the enunciation of the theorem IV, correspond to the elementary matrices

of Art. 3, which serve to pair in every admissible manner the sides of the reduced triangle.

It will be observed, that the parallelogrammic system of $\lambda^2(u)$ is primarily ambiguous, only and always when k^2 is real; and that the angle between K and iK' is a right angle, only and always when k^2 is real, positive, and less than unity.

9. *Theorem* I. 'Let Ω be any primarily reduced complex quantity of which the amplitude is positive; let also, as in Art. 4,

$$\left.\begin{aligned} q &= e^{i\pi\Omega}, \quad h(\Omega) = \tfrac{1}{2}\pi \times \left[\Sigma_{m=-\infty}^{m=+\infty} q^{m^2}\right]^2 \\ k^2 &= \phi^8(\Omega), \quad h'(\Omega) = \Omega \times h(\Omega) \end{aligned}\right\}; \quad \ldots\ldots \quad (25)$$

the rectilinear integrals K and K' defined by the equations (23) are then expressed in terms of Ω by means of the equations

$$K = h(\Omega), \quad iK' = h'(\Omega).'$$

For $2h(\Omega)$ and $2h'(\Omega)$ are a pair of primary periods of $\lambda^2(u, k^2)$; so also are $2K$ and $2iK'$; we have therefore (Art. 4)

$$h(\Omega) = \alpha K + \gamma i K',$$
$$h'(\Omega) = \beta K + \delta i K',$$

or, if
$$\omega = \frac{iK'}{K},$$

then
$$\omega = \frac{-\beta + \alpha\Omega}{\delta - \gamma\Omega};$$

the unit matrix $\left|\begin{matrix} \alpha, & \beta \\ \gamma, & \delta \end{matrix}\right|$ being primary.

By the theorem IV of Art. 8, ω is primarily reduced; Ω is so by hypothesis; and by the theorem of Art. 3 two primarily reduced quantities cannot be equivalent without being identical; we have therefore $\omega = \Omega$, $\alpha = \delta = 1$, $\beta = \gamma = 0$; or finally,

$$K = h(\Omega), \quad iK' = h'(\Omega).$$

Theorem II. 'Let Ω be any complex quantity whatever of which the amplitude is positive; and let $\left|\begin{matrix} \alpha, & \beta \\ \gamma, & \delta \end{matrix}\right|$ be the primary reducing substitution of the quadratic form to which Ω is associated, so that the vector

$$\Omega' = \frac{-\beta + \alpha\Omega}{\delta - \gamma\Omega}$$

is primarily reduced; then, the same notations being retained,

$$h(\Omega) = \alpha K + \gamma i K',$$
$$h'(\Omega) = \beta K + \delta i K'.'$$

For, as in the last demonstration, both $[2h(\Omega), 2h'(\Omega)]$ and $[2K, 2iK']$ are primary pairs of periods of $\lambda^2(u)$; we have therefore

$$\left.\begin{aligned} h(\Omega) &= AK + CiK' \\ h'(\Omega) &= BK + DiK' \\ \frac{iK'}{K} &= \frac{-B + A\Omega}{D - C\Omega} \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (26)$$

the matrix $\left|\begin{matrix} A, & B \\ C, & D \end{matrix}\right|$ being a primary unit matrix. Thus Ω is equivalent to each of two primarily reduced quantities $\frac{iK'}{K}$ and Ω'; these two reduced quantities, and the two reducing substitutions, are therefore identical; *i.e.* the equations (26) coincide with the equations which it is required to prove.

The theorems I and II of this article may be demonstrated in a different manner.*

When k^2 is real, positive, and less than unity, so that the quotient $\Omega = \frac{iK'}{K}$ is a pure imaginary, and $q = e^{i\pi\Omega}$ is a real positive quantity less than unity, the equations

$$K = h(\Omega), \quad iK' = \Omega \times h(\Omega), \quad \ldots\ldots\ldots \quad (27)$$

as has been shown by Jacobi, certainly subsist. But these equations cannot become untrue so long as the extremity of the vector Ω lies within the primarily reduced space, because within that space the rectilinear integrals K and K' are continuous and one-valued functions of k^2, which is itself a continuous and one-valued function of Ω. When k^2 is real and negative, there is, as we have seen, an ambiguity in the determination of the rectilineal integral K', and its differential coefficients with regard to k^2 become infinite; the same things are true for the rectilineal integral K, when k^2 is positive and greater than unity. But k^2 cannot become real and negative at any point of the reduced space unless Ω falls on the boundary $X = 1$; and k^2 cannot become positive and greater than unity, unless Ω falls on the boundary $X^2 + Y^2 - X = 0$; because, as we have already seen, if k^2 is real and negative, K is real, and $K - iK'$ is a pure imaginary

* See a note 'Sur les Intégrales Elliptiques complètes' printed in the 'Transunti della R. Accademia de Lincei,' vol. i. (3rd series) [vol. ii. p. 221].

of negative amplitude; and, if k^2 is positive and greater than unity, iK' is a pure imaginary of positive amplitude, and $K-iK'$ is real and positive. Thus the equations (27) must continue to subsist at every point in the interior of the reduced space; and they do not become untrue at the boundaries of that space; for the conventions by which we have removed the ambiguities in the determination of K and iK' are so arranged as to correspond with the convention by which the boundaries $X+1=0$, $X^2+Y^2+X=0$ are excluded from, and the boundaries $X-1=0$, $X^2+Y^2-X=0$ are included in, the reduced space; viz. the values assigned to the integrals (23) in the cases of ambiguity are such as to make the equations (27) continue true at the included boundaries, while becoming untrue at the excluded boundaries. A complete demonstration is thus obtained of the theorem I of this article; the theorem II, and the theorems of Art. 8, may be regarded as corollaries from it. This mode of demonstrating the propositions of Art. 8 has some advantage in respect of simplicity; but the method adopted in this note is perhaps more direct and natural, as it depends only on the elementary properties of the rectilinear integrals themselves.

The theorem that the quotient $\frac{iK'}{K}$ is always a primarily reduced complex quantity is of some importance in the theory of the connexion of the elliptic functions with the Theta functions.* If k^2 is given, the elliptic function $\lambda(u, k^2)$ is given; but if we wish to express $\lambda(u, k^2)$ as a quotient of Theta functions by means of the formula

$$\frac{1}{\phi^2(\Omega)} \frac{\vartheta\left(\frac{u}{h(\Omega)}, e^{i\pi\Omega}\right)}{\vartheta_0\left(\frac{u}{h(\Omega)}, e^{i\pi\Omega}\right)} = \lambda(u, k^2),$$

we have, first of all, to express the element Ω of the Theta functions in terms of k^2. The problem, as is well known, and as we have already seen in Art. 4, is indeterminate; viz. if Ω is any admissible value of the element, $\Omega' = \frac{-\beta+\alpha\Omega}{\delta-\gamma\Omega}$, the matrix being primary, is also an admissible value; or, which is the same thing, we may take for Ω the quotient obtained by dividing the second period

* See a note by Professor Weierstrass in the first volume of Jacobi's Collected Works (Berlin, 1881), p. 545.

by the first in any primary pair of periods of $\lambda^2(u)$. Since $[2K, 2iK']$ is a primary pair, the quotient $\frac{iK'}{K}$ is an admissible value of Ω; but it is also the simplest of all the admissible values, for it renders the analytical modulus of $q = e^{i\pi\Omega}$ the least possible, and the Theta series the most rapidly convergent.

XLII.

NOTES ON THE THEORY OF ELLIPTIC TRANSFORMATION.*

[Messenger of Mathematics, Ser. II. vol. xiii, pp. 1–54 (May–August, 1883).]

III.—*On the Functions $Q(\omega)$ and $Q'(\omega)$.*

LET $Q(\omega)$, $Q'(\omega)$ be two functions of ω defined by the equations

$$\left.\begin{aligned} iQ(\omega) &= -\tfrac{1}{2}\pi \frac{d}{d\omega}\left(\frac{1}{K}\right) = \frac{\pi}{2K^2}\frac{dK}{d\omega} \\ iQ'(\omega) &= \tfrac{1}{2}\pi \frac{d}{d\left(-\frac{1}{\omega}\right)}\left(\frac{1}{K'}\right) = \frac{\pi}{2K^2}\frac{dK'}{d\omega} \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (1)$$

which imply the equality

$$Q'(\omega) = -Q\left(-\frac{1}{\omega}\right), \quad \ldots\ldots\ldots \quad (2)$$

because $K'(\omega) = K\left(-\frac{1}{\omega}\right)$; and which may be replaced by either of the following pairs:

$$Q = 2k^2k'^2\frac{dK}{d.k^2}, \qquad Q' = 2k^2k'^2\frac{dK'}{d.k^2}, \quad \ldots\ldots\ldots \quad (3)$$

$$Q = k^2K - J, \qquad Q' = k^2K' - J', \quad \ldots\ldots\ldots \quad (4)$$

of which (3) was obtained by writing in (1) for $d\omega$ its value $-\frac{i\pi d.k^2}{4k^2k'^2K^2}$, and (4) by substituting in (3) the values of $\frac{dK}{d.k^2}$ and $\frac{dK'}{d.k^2}$ taken from the equations (i) and (v) of Art. 12.†

[* These Notes contain the fragments, relating to the continuation of the preceding Notes, No. XLI, which were found among Professor Smith's papers after his death. It is to be understood that they are only unfinished work which he would have greatly altered and extended. The Notes have been placed in what appears to be the most convenient order, and headings have been supplied.]

[† These references are to the Memoir on the 'Theta and Omega Functions,' No. XLIII. of the present Volume, p. 415.]

The values of $\frac{dJ}{d.k^2}$ and $\frac{dJ'}{d.k^2}$ taken from the same equations give

$$\frac{dQ}{d.k^2} - \tfrac{1}{2}K = 0, \qquad \frac{dQ'}{d.k^2} - \tfrac{1}{2}K' = 0; \quad \ldots\ldots\ldots \quad (5)$$

and eliminating between these equations and the equation (3), we have [writing x for k^2]

$$\frac{d^2Q}{dx^2} = \frac{Q}{4x(1-x)}, \quad \ldots\ldots\ldots \quad (6)$$

the complete solution being $CQ + C'Q'$.

The equations (1) and (5) respectively give

$$\frac{Q'}{Q} = \frac{dK'}{dK}, \quad \frac{dQ'}{dQ} = \frac{K'}{K}, \quad \ldots\ldots\ldots \quad (7)$$

of which either is a consequence of the other, because

$$K'Q - KQ' = KJ' - K'J = \tfrac{1}{2}\pi. \quad \ldots\ldots\ldots \quad (8)$$

We have also

$$\frac{1}{K}\frac{dQ}{d\omega} = \frac{1}{K'}\frac{dQ'}{d\omega} = -\frac{2k^2k'^2K^2}{i\pi} = \tfrac{1}{2}\frac{d.k^2}{d\omega},$$

$$\frac{1}{Q}\frac{dK}{d\omega} = \frac{1}{Q'}\frac{dK'}{d\omega} = -\frac{2}{i\pi}K^2,$$

$$\frac{Q'dQ - QdQ'}{d\omega} = \tfrac{1}{4}\pi\frac{d.k^2}{d\omega}.$$

Lastly, combining the equations (4) with the equations (vi) of Art. 10, we have

$$\left.\begin{aligned} Q &= \int_0^K k^2\cos^2\mathrm{am}\,x\,dx, \\ Q' &= -i\int_K^{K+iK'} k^2\cos^2\mathrm{am}\,x\,dx; \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (9)$$

thus the functions Q and Q' do not differ essentially from the functions

$$J = k^2K - Q, \quad J' = k^2K' - Q'$$

of M. Weierstrass, nor from the functions

$$E = K - J = k'^2K + Q, \quad E' = J' = k^2K' - Q'$$

of Legendre; their introduction serves to simplify the formula relating to the transformation of the complete functions of the second species.

If $\omega = \begin{vmatrix} a, & \beta \\ \gamma, & \delta \end{vmatrix} \times \Omega$ is any transformation whatever of determinant n, that is,

if $\omega = \dfrac{\gamma + \delta\Omega}{\alpha + \beta\Omega}$, we have

$$\frac{K(\omega)}{M} = \alpha K(\Omega) + i\beta K'(\Omega),$$

$$i\frac{K'(\omega)}{M} = \gamma K(\Omega) + i\delta K'(\Omega).$$

Differentiating with regard to k^2, multiplying by $-2k^2k'^2M$, and observing that

$$d.\lambda^2 = \frac{1}{nM^2} \times \frac{\lambda^2\lambda'^2}{k^2k'^2} d.k^2,$$

$$Q(\Omega) = 2\lambda^2\lambda'^2 \frac{d.K(\Omega)}{d.\lambda^2}, \quad Q'(\Omega) = 2\lambda^2\lambda'^2 \frac{d.K'(\Omega)}{d.\lambda^2},$$

we find

$$\frac{1}{nM}[\alpha Q(\Omega) + i\beta Q'(\Omega)] = Q(\omega) - 2k^2k'^2K(\omega)\frac{d.\log M}{d.k^2},$$

$$\frac{1}{nM}[\gamma Q(\Omega) + i\delta Q'(\Omega)] = iQ'(\omega) - 2k^2k'^2iK'(\omega)\frac{d.\log M}{d.k^2}.$$

In these formulæ, which serve to express the transformed functions $Q(\Omega)$, $Q'(\Omega)$ in terms of the given functions $Q(\omega)$, $Q'(\omega)$, $K(\omega)$, $K'(\omega)$, the differential coefficient $\dfrac{1}{M}\dfrac{dM}{d.k^2}$ is to be found from the equation of the multiplier.

For example, if $\omega = \begin{vmatrix} 3, & 0 \\ 0, & 1 \end{vmatrix} \times \Omega = 3\Omega$, then, from the equation

$$\frac{1}{M^4} - \frac{6}{M^2} + \frac{8(1-2k^2)}{M} - 3 = 0$$

of the multiplier, we find without difficulty

$$\frac{dM}{d.k^2} = \frac{-\frac{16}{3}M^4}{(M^2-1)^2} = \frac{-\frac{4}{3}M^2}{(M-1)^2 + 4k^2M},$$

and the formulæ then become

$$3Q(3\omega) = 3MQ(\omega) + \frac{8k^2k'^2M^2}{(M-1)^2 + 4k^2M} K(\omega),$$

$$3Q'(3\omega) = 3MQ'(\omega) + \frac{8k^2k'^2M^2}{(M-1)^2 + 4k^2M} K'(\omega),$$

where, if ω be a pure imaginary, M is the negative root of the foregoing equation*.

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* Cayley, 'On the Transformation of Elliptic Functions,' Phil. Trans., vol. clxiv. p. 421.

Some considerations of interest present themselves in connexion with the last-mentioned general formulæ.

(1) The two equations are not independent: viz. multiplying the first by $iK'(\omega)$, the second by $K(\omega)$, subtracting and attending to the relations

$$Q(\omega)K'(\omega) - Q'(\omega)K(\omega) = \tfrac{1}{2}\pi,$$
$$nMK(\Omega) = \delta K(\omega) - i\beta K'(\omega),$$
$$inMK'(\Omega) = -\gamma K(\omega) + i\alpha K'(\omega),$$

we obtain an identical result.

(2) In the case of Jacobi's first transformation

$$\omega = \begin{vmatrix} n, & 0 \\ 0, & 1 \end{vmatrix} \times \Omega = \frac{\Omega}{n},$$

the equations become

$$\frac{Q(\Omega)}{M} = Q(\omega) - \frac{2k^2(1-k^2)}{M} K(\omega) \frac{dM}{d.k^2},$$
$$\frac{Q'(\Omega)}{nM} = Q'(\omega) - \frac{2k^2(1-k^2)}{M} K'(\omega) \frac{dM}{d.k^2},$$

of which the first coincides with a relation considered by Prof. Cayley ('Elliptic Functions,' Art. 305), and expressed by him in the form

$$\frac{1}{nM^2}\left[\lambda'^2 - \frac{G}{\Lambda} - \frac{\lambda\lambda'^2}{M}\frac{dM}{d\lambda}\right] = k'^2 - \frac{E}{K}, \quad \ldots\ldots \text{(i)}$$

where

$$\Lambda = K(\Omega), \quad E = \int_0^K \Delta^2 \operatorname{am}(u, k^2)\, du, \quad G = \int_0^\Lambda \Delta^2 \operatorname{am}(u, \lambda^2)\, du,$$

so that

$$E = k'^2 K(\omega) + Q(\omega), \quad G = \lambda'^2 K(\Omega) + Q(\Omega),$$

and the equation of Professor Cayley becomes

$$\frac{Q(\omega)}{K(\omega)} = \frac{1}{nM^2}\left[\frac{Q(\Omega)}{K(\Omega)} + \frac{\lambda\lambda'^2}{M}\frac{dM}{d\lambda}\right], \quad \ldots\ldots \text{(ii)}$$

which is the same as the first of the foregoing equations, because

$$\frac{K(\omega)}{K(\Omega)} = nM, \quad \frac{\lambda\lambda'^2}{nM^2}\frac{d}{d\lambda} = kk'^2\frac{d}{dk}.$$

If instead of the first transformation we consider any reduced transformation

of the type $\begin{vmatrix} g, & 0 \\ -l, & \frac{n}{g} \end{vmatrix}$, the first equation becomes

$$\frac{g}{nM} Q(\Omega) = Q(\omega) - 2k^2(1-k^2)\frac{K(\omega)}{M}\frac{dM}{d.k^2},$$

which, on eliminating g by the equation $g = \frac{K(\omega)}{MK(\Omega)}$ and substituting for $2k^2k'^2\frac{d}{d.k^2}$ its equivalent $\frac{\lambda\lambda'^2}{nM^2}\frac{d}{d\lambda}$, coincides with equation (ii), *i.e.* with equation (i). Thus the equation of Prof. Cayley holds for all *reduced* transformations, *i.e.* for all transformations of which the second element of the matrix of transformation is $=0$. This remark is of some importance, as it enables us to show that the equation

$$\frac{d^2\Sigma}{dx^2} + 2n\frac{Q(\omega)}{K(\omega)}x\frac{d\Sigma}{dx} + 4nk^2(1-k^2)\frac{d\Sigma}{d.k^2} = 0,$$

which is satisfied by $\Sigma = \vartheta_0\left[\frac{\pi x}{2MK(\Omega)}, \Omega\right]$ when $\omega = \frac{\Omega}{n}$, *i.e.* when the transformation is the first transformation of Jacobi, is also satisfied by the same function when the transformation considered is any reduced transformation whatever.

By combining the foregoing results we can express the elements of the matrix of transformation in terms of the multiplier and of the integrals K, K', Q, Q'; we thus find

$$-\tfrac{1}{2}i\pi\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} = \left.\begin{vmatrix} \frac{1}{M}K(\omega), & nMQ(\omega) - 2nk^2(1-k^2)\frac{dM}{d.k^2}K(\omega) \\ \frac{1}{M}iK'(\omega), & nMiQ'(\omega) - 2nk^2(1-k^2)\frac{dM}{d.k^2}iK'(\omega) \end{vmatrix} \times \begin{vmatrix} iQ'(\Omega), & -Q(\Omega) \\ -iK'(\Omega), & K(\Omega) \end{vmatrix}\right\}, \quad \text{(iii)}$$

and the equation of Prof. Cayley is then found by putting $\beta = 0$ in the formula.

If $\gamma = 0$, we have

$$\frac{Q'(\omega)}{K'(\omega)} = \frac{1}{nM'}\left[\frac{Q'(\Omega)}{K'(\Omega)} + \frac{\lambda\lambda'^2}{M}\frac{dM}{d\lambda}\right]. \quad \text{(iv)}$$

When β and γ are each $=0$, *i.e.* in the real transformation $\omega = \begin{vmatrix} \alpha, & 0 \\ 0, & \delta \end{vmatrix} \times \Omega$, a combination of the foregoing equations gives $K(\omega)K'\omega = nM^2K(\Omega)K'(\Omega)$, which is true because

$$\frac{K(\omega)}{M} = \alpha K(\Omega), \quad \frac{iK'(\omega)}{M} = i\delta K'(\Omega).$$

IV.—*Employment of the Transformation of the Function of the second species to obtain expressions rational in k^2 and λ^2 for the coefficients in the formula of Transformation of the n^{th} order.*

The analytical theory of transformation supplies a direct proof that the coefficients a, b in the developments

$$1+a_1x^2+a_2x^4\ldots+a_{\frac{1}{2}(n-1)}x^{n-1}=\Pi_j\left[1-\frac{x^2}{\sin^2\operatorname{am}4j\eta}\right]$$

and $$1+b_1x^2+b_2x^4\ldots+b_{\frac{1}{2}(n-1)}x^{n-1}=\Pi_j[1-k^2x^2\sin^2\operatorname{am}4j\eta]$$

of the numerator and denominator functions in the formula of transformation of the n^{th} order are rational in k^2, λ^2, M, and $\frac{dM}{dk^2}$; that is in k^2, λ^2 and M, for $\frac{dM}{dk^2}$ is always rational in k^2, λ^2 and M: this follows from the equation of Jacobi combined with the theorem that at a multiple point on the modular curve the branches are all linear, and the tangents all different; so that even in the case where the modular equation has equal roots the coefficients contain no irrationalities other than those involved in the expression of M.

To obtain the proof it is convenient to consider the function of the second species defined by the equation

$$Z(u)=\int_0^u k^2\sin^2\operatorname{am}u\,du,$$

and expressed by Jacobi in the form

$$Z(u)=\frac{J}{K}u-\frac{d}{du}\log\vartheta_0\left(\frac{\pi u}{2K}\right),$$

where $J=Z(K)=\int_0^K k^2\sin^2\operatorname{am}u\,du$. Employing the usual notation $\omega=\frac{iK'}{K}$, $q=e^{i\pi\omega}$, and regarding $\sqrt[4]{k}=\psi(\omega)$, $K=K(\omega)$, $J=J(\omega)$ as one-valued functions of ω defined by the equations

$$\left.\begin{aligned}\sqrt[4]{k}&=\sqrt{2}q^{\frac{1}{8}}\Pi\frac{1+q^{2m}}{1+q^{2m-1}}, & \sqrt[4]{k'}&=\Pi\frac{1-q^{2m-1}}{1+q^{2m-1}},\\ \sqrt{\frac{2K}{\pi}}&=1+2q+2q^4+2q^9+\ldots, & J&=k^2K-\tfrac{1}{2}i\pi\frac{d}{d\omega}\frac{1}{K};\end{aligned}\right\}\quad\ldots\quad(1)$$

we consider the transformation

$$\omega=\left|\begin{matrix}\alpha,&\beta\\ \gamma,&\delta\end{matrix}\right|\times\Omega=\frac{\gamma+\delta\Omega}{\alpha+\beta\Omega},$$

where $$\left|\begin{matrix}\alpha,&\beta\\ \gamma,&\delta\end{matrix}\right|\equiv\left|\begin{matrix}1,&0\\ 0,&1\end{matrix}\right|,\ \text{mod}\,2,\ \text{and}\ \ \alpha\delta-\beta\gamma=n.$$

Defining the multiplier M by the equations

$$\left.\begin{aligned}\frac{1}{M}\,K(\omega) &= aK(\Omega)+i\beta K'(\Omega),\\ \frac{1}{M}\,iK'(\omega) &= \gamma K(\Omega)+i\delta K'(\Omega),\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2)$$

we write in the equation of Jacobi $\frac{u}{M}$ for u and Ω for ω. Dividing by M we obtain

$$\begin{aligned}\frac{1}{M}Z\left(\frac{u}{M},\lambda^2\right) &= \frac{J(\Omega)}{K(\Omega)}\frac{u}{M^2}-\frac{d}{du}\log\vartheta_0\left[\frac{\pi u}{2MK(\Omega)},\Omega\right]\\ &= \frac{J(\Omega)}{K(\Omega)}\frac{u}{M^2}-\frac{d}{du}\log\vartheta_0\left[\frac{\pi u(a+\beta\Omega)}{2K(\omega)},\Omega\right], \quad \ldots\ldots \quad (3)\end{aligned}$$

where $\lambda^2=\phi^8(\Omega)$. With this equation we combine a known formula for the transformation of the Theta functions, viz.

$$\begin{aligned}\vartheta_0\left[\frac{(a+\beta\Omega)\pi u}{2K(\omega)},\Omega\right] &= C\times\vartheta_0^n\left[\frac{\pi u}{2K(\omega)},\omega\right]\\ &\times e^{-i\pi\beta(a+\beta\Omega)\frac{u^2}{4K^2}}\times\Pi_{j=1}^{j=\frac{1}{2}(n-1)}[1-k^2\sin^2\mathrm{am}\,4j\eta\sin^2\mathrm{am}\,u], \quad \ldots\ldots \quad (4)\end{aligned}$$

in which $C=\vartheta_0[\Omega]\div\vartheta_0^n[\omega]$, and $\eta=\dfrac{\mu K+i\nu K'}{n}$, ν being the greatest common measure of a and β, and μ being defined by the congruence $\mu\equiv\gamma r+\delta s$, $\mathrm{mod}\,\frac{n}{\nu}$, in which r and s are any two integers satisfying the equation $ar+\beta s=\nu$.

Instead of η we may take any multiple of η by a number prime to n. Taking the logarithm and differentiating, we obtain from (4) the equation

$$\begin{aligned}\frac{d}{du}\log\vartheta_0\left[\frac{(a+\beta\Omega)\pi u}{2K},\Omega\right] &= \frac{2\pi\beta(a+\beta\Omega)u}{2K^2}+n\frac{d}{du}\log\vartheta_0\left[\frac{\pi u}{2K},\omega\right]\\ &+\Sigma_{j=1}^{j=\frac{1}{2}(n-1)}\frac{d}{du}\log[1-k^2\sin^2\mathrm{am}\,\eta\sin^2\mathrm{am}\,u], \quad (5)\end{aligned}$$

or finally the equation

$$\begin{aligned}&\frac{1}{M}Z\left(\frac{u}{M},\lambda^2\right)-nZ(u,k^2)+2Hu\\ &=2k^2\sin\mathrm{am}\,u\cos\mathrm{am}\,u\Delta\,\mathrm{am}\,u\times\Sigma_{j=1}^{j=\frac{1}{2}(n-1)}\frac{\sin^2\mathrm{am}\,4j\eta}{1-k^2\sin^2\mathrm{am}\,4j\eta\sin^2\mathrm{am}\,u};\end{aligned}$$

where

$$\begin{aligned}2H &= \frac{nJ(\omega)}{K(\omega)}-\frac{1}{M^2}\frac{J(\Omega)}{K(\Omega)}-\frac{i\pi\beta(a+\beta\Omega)}{2K^2(\omega)}\\ &= \frac{nJ(\omega)}{K(\omega)}-\frac{1}{M^2}\frac{J(\Omega)}{K(\Omega)}-\frac{i\pi\beta}{2MK(\omega)K(\Omega)}. \quad \ldots\ldots \quad (6)\end{aligned}$$

The formulæ (5) and (6) are given in fact by Jacobi for the transformation of the elliptic function of the second species.

For our present purpose we have to show however that the constant H is a rational function of k^2, λ^2, M, $\frac{dM}{d.k^2}$; and that the integrals $J(\omega)$, $J(\Omega)$, $K(\omega)$, $K(\Omega)$ enter into the expression (6) only apparently.

To verify the assertion we write $-Q(\omega)+k^2K(\omega)$ for $J(\omega)$, and consequently $-Q(\Omega)+\lambda^2K(\Omega)$ for $J(\Omega)$. Similarly for the function $J(\omega)$ defined by the equation

$$K(\omega)J'(\omega)-K'(\omega)J(\omega)=\tfrac{1}{2}\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

we write $-Q'(\omega)+k^2K'(\omega)$, so that

$$K'(\omega)\,Q(\omega)-K(\omega)\,Q'(\omega)=\tfrac{1}{2}\pi. \quad . \quad . \quad . \quad (8)$$

Then we have

$$\left.\begin{aligned}\frac{1}{nM}[\alpha Q(\Omega)+i\beta Q'(\Omega)]&=Q(\omega)-2k^2(1-k^2)\frac{K(\omega)}{M}\frac{dM}{d.k^2},\\ \frac{1}{nM}[\gamma Q(\Omega)+i\delta Q'(\Omega)]&=iQ'(\omega)-2k^2(1-k^2)\frac{K'(\omega)}{M}\frac{dM}{d.k^2},\end{aligned}\right\} \quad . \quad . \quad . \quad (9)$$

which are the general formulæ for the transformation of the complete integrals of the second kind represented by

$$Q(\omega)=\int_0^K k^2\cos^2\mathrm{am}\,x\,dx,$$

$$iQ'(\omega)=\int_0^{K+iK'} k^2\cos^2\mathrm{am}\,x\,dx.$$

Introducing the functions $Q(\omega)$, $Q'(\omega)$ into the equation (6), we have

$$2H=nk^2-\frac{\lambda^2}{M^2}-\frac{nQ(\omega)}{K(\omega)}+\frac{1}{M^2}\frac{Q(\Omega)}{K(\Omega)}-\frac{i\pi b}{2MK(\omega)K(\Omega)}. \quad . \quad . \quad (10)$$

But from (9)

$$\frac{Q(\Omega)}{M^2K(\Omega)}=\frac{\delta Q(\omega)-i\beta Q'(\omega)}{MK(\Omega)}-\frac{2nk^2(1-k^2)}{M^2}\frac{dM}{d.k^2}.$$

Substituting this value in (10), we obtain

$$2H=nk^2-\frac{\lambda^2}{M^2}-\frac{2nk^2(1-k^2)}{M}\frac{dM}{d.k^2}$$

$$-\frac{1}{MK(\omega)K(\Omega)}[nMK(\Omega)\,Q(\omega)-K(\omega)\{\delta Q(\omega)-i\beta Q'(\omega)\}-\tfrac{1}{2}i\pi\beta],$$

or finally,

$$2H=nk^2-\frac{\lambda^2}{M^2}-\frac{2nk^2(1-k^2)}{M}\frac{dM}{d.k^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the terms in the square brackets vanishing by virtue of the equations

$$nMK(\Omega) = \delta K(\omega) - i\beta K'(\omega), \quad K'(\omega)\,Q(\omega) - K(\omega)\,Q'(\omega) = \tfrac{1}{2}\pi.$$

If we now employ the developments

$$\sin^2 \mathrm{am}\, u = u^2 - \tfrac{1}{3}(1+k^2)\,u^4 + \tfrac{1}{45}(2+13k^2+2k^4)\,u^6 + \ldots$$

$$\frac{d.\sin^2 \mathrm{am}\, u}{du} = 2u - \tfrac{4}{3}(1+k^2)\,u^3 + \tfrac{2}{15}(2+13k^2+2k^4)\,u^5 + \ldots$$

$$Z(u) = \tfrac{1}{3}k^2u^3 - \tfrac{1}{15}k^2(1+k^2)\,u^5 + \tfrac{1}{315}k^2(2+13k^2+3k^4)\,u^7 + \ldots$$

(of which, however, the general terms are not known, and which are convergent only for values of u not surpassing a given certain limit), we can by equating the coefficients of like powers of u in the two members of equation (5) obtain successively the values of the sums

$$\Sigma_{j=1}^{j=\frac{1}{2}(n-1)} k^{2s} \sin^{2s} \mathrm{am}\, 4j\eta,$$

expressed rationally in terms of λ^2, k^2 and H. Thus, for example,

$$\Sigma k^2 \sin^2 \mathrm{am}\, 4j\eta = H,$$

$$\Sigma k^4 \sin^4 \mathrm{am}\, 4j\eta = \frac{\lambda^2}{6M^4} - \frac{\Delta k^2}{6} + \tfrac{1}{3}(1+k^2)\,H$$

$$= \tfrac{1}{6}\Big[\frac{\lambda^2}{M^2} - 2\lambda^2(1+k^2)\frac{1}{M^2} + \Delta k^2(1+2k^2) - 4nk^2(1-k^2)(1+k^2)\frac{1}{M}\frac{dM}{d.k^2}\Big].$$

But from the theory of transformation

$$1 + b_1x^2 + b_2x^4 \ldots + b_{\frac{1}{2}(n-1)}x^{n-1} = \Pi_j(1 - k^2x^2\sin^2 \mathrm{am}\, 4j\eta),$$

so that the coefficients $b_1, b_2, \ldots$ are rational because the quantities $\Sigma k^{2s}\sin^{2s}\mathrm{am}\, 4j\eta$ are so. In particular

$$b_1 = \frac{nk^2(1-k^2)}{M}\frac{dM}{d.k^2} + \tfrac{1}{2}\frac{\lambda^2}{M^2} - \tfrac{1}{2}nk^2,$$

$$b_3 = \tfrac{1}{24}\frac{\lambda^2(3\lambda^2-2)}{M^4} - \tfrac{1}{12}\frac{\lambda^2\{(3n-2)k^2-2\}}{M^2} + \frac{n\lambda^2k^2(1-k^2)}{2M^3}\frac{dM}{d.k^2}$$

$$- \frac{nk^2(1-k^2)\{(3n-2)k^2-2\}}{6M}\frac{dM}{d.k^2} + \tfrac{1}{2}\frac{n^2k^4(1-k^2)^2}{M^2}\Big(\frac{dM}{d.k^2}\Big)^2 + \frac{nk^2\{(3n-4)k^2-2\}}{24}.$$

The last coefficient is known, viz. we have

$$b_{\frac{1}{2}(n-1)} = (-)^{\frac{1}{2}(n-1)}k^{n-1}\Pi_{j=1}^{j=\frac{1}{2}(n-1)}\sin^2 \mathrm{am}\, 4j\eta = \frac{k^{\frac{1}{2}(n-2)}\sqrt{\lambda}}{M},$$

the expression being rational in k^2 and λ^2.

Lastly, the coefficients a are rational because

$$1 + a_1x^2 + a_2x^4 \ldots + a_{\frac{1}{2}(n-1)}x^{n-1} = \Pi_{j=1}^{j=\frac{1}{2}(n-1)}\Big[1 - \frac{x^2}{\sin^2 \mathrm{am}\, 4j\eta}\Big],$$

whence it follows that

$$\frac{a_s}{k^{2s}} = \frac{M}{k^{\frac{1}{2}(n-1)}\sqrt{\lambda}}\, b_{\frac{1}{2}(n-1)-s}.$$

V.—*Relations in the formulæ of Jacobi between the elements η and the transformation represented by the formula* $\omega = \dfrac{\gamma + \delta\Omega}{\alpha + \beta\Omega}$.

The general formulæ of Jacobi for any transformation of an uneven order n are

$$\left.\begin{aligned}
\operatorname{sn}\left(\frac{u}{M}, \lambda^2\right) &= \frac{\operatorname{sn} u}{MD} \Pi_{j=1}^{j=\frac{1}{2}(n-1)} \left[1 - \frac{\operatorname{sn}^2 u}{\operatorname{sn}^2 4jK}\right], \\
\operatorname{cn}\left(\frac{u}{M}, \lambda^2\right) &= \frac{\operatorname{cn} u}{D} \Pi_{j=1}^{j=\frac{1}{2}(n-1)} \left[1 - \frac{\operatorname{sn}^2 u}{\operatorname{sn}^2 \operatorname{co} 4jK}\right], \\
\Delta\left(\frac{u}{M}, \lambda^2\right) &= \frac{\Delta u}{D} \Pi_{j=1}^{j=\frac{1}{2}(n-1)} \left[1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 \operatorname{co} 4jK\right], \\
D &= \Pi_{j=1}^{j=\frac{1}{2}(n-1)} \left[1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 4jK\right];
\end{aligned}\right\} \quad \text{(i)}$$

$$\left.\begin{aligned}
\lambda^2 &= k^{2n} \times \Pi_{j=1}^{j=\frac{1}{2}(n-1)} \left[\frac{\operatorname{cn}^2 4jK}{\Delta^2 4jK}\right]^4, \\
\lambda'^2 &= k'^{2n} \times \Pi_{j=1}^{j=\frac{1}{2}(n-1)} \left[\frac{1}{\Delta^2 4jK}\right]^4, \\
\frac{1}{M^2} &= \frac{k^n}{\lambda} \times \Pi_{j=1}^{j=\frac{1}{2}(n-1)} \operatorname{sn}^4 4jK = \Pi_{j=1}^{j=\frac{1}{2}(n-1)} \frac{\operatorname{sn}^4 4jK}{\operatorname{sn}^4 \operatorname{co} 4jK};
\end{aligned}\right\} \quad \text{(ii)}$$

η representing an element of the form

$$\frac{pK + iqK'}{n}, \qquad \text{(iii)}$$

where p and q are integral numbers having no common divisor with n.

It is also known that the modules k^2 and λ^2 are connected by a relation of the form

$$\left.\begin{aligned}
k^2 &= \phi^8(\omega), \quad \lambda^2 = \phi^8(\Omega), \\
\omega &= \frac{\gamma + \delta\Omega}{\alpha + \beta\Omega} = \begin{vmatrix} \alpha, \beta \\ \gamma, \delta \end{vmatrix} \times \Omega,
\end{aligned}\right\} \quad \text{(iv)}$$

where $\left\| \begin{matrix} \alpha, \beta \\ \gamma, \delta \end{matrix} \right|$ is a primary and primitive matrix of integral numbers, having n for its determinant; and $\phi(\omega)$ is the function already defined in Note I.

The elliptic transformation is completely determined when either (1) the integral numbers (p, q), or (2) the integral numbers $\alpha, \beta, \gamma, \delta$ are given. Thus the two converse problems present themselves, (1) 'given any pair (p, q), to find all the corresponding matrices $\left\| \begin{matrix} \alpha, \beta \\ \gamma, \delta \end{matrix} \right|'$; and (2) 'given any matrix $\left\| \begin{matrix} \alpha, \beta \\ \gamma, \delta \end{matrix} \right|$, to find all the corresponding pairs (p, q).'

The solution of these problems is implicitly contained in the congruences

$$p\alpha + q\gamma \equiv 0, \quad p\beta + q\delta \equiv 0, \quad \text{mod } n, \quad \ldots\ldots \text{(v)}$$

which result from the theory of the transformation of the Theta functions; but for the sake of clearness it may be worth while to give the details of the solution, although the arithmetical considerations involved are quite elementary.

(*a*) If $p_1 \equiv p_2$, $q_1 \equiv q_2$, mod n, the pairs (p_1, q_1), (p_2, q_2), and the elements η_1, η_2, may be said to be *congruous*; if $hp_1 \equiv p_2$, $hq_1 \equiv q_2$, mod n, where h is an integral number prime to n, the pairs (p_1, q_1), (p_2, q_2), and the elements η_1, η_2, may be said to be *equivalent*. Congruous elements are always equivalent; and equivalent elements, when employed in the formulæ of Jacobi, give identical results.

(*b*) To find, for any given uneven number n, the number N of non-equivalent elements η, let $\chi(s)$ denote the number of numbers prime to any given number s, and not surpassing s, and let δ, δ' be two relatively prime divisors of n. The number of incongruous elements η, having δ and δ' for the greatest common divisors of n with p, and of n with q respectively, is $\chi\left(\frac{n}{\delta}\right) \times \chi\left(\frac{n}{\delta'}\right)$; hence the number of non-equivalent elements, having their greatest common divisors, is

$$\ldots \chi\left(\frac{n}{\delta}\right) \times \chi\left(\frac{n}{\delta'}\right) \div \chi(n) = \chi\left(\frac{n}{\delta\delta'}\right); \quad \ldots\ldots \text{(vi)}$$

and the whole number N of non-equivalent elements is given by the formula

$$N = \Sigma\chi\left(\frac{n}{\delta\delta'}\right),$$

the summation extending to every pair of relatively prime divisors of n, and the pairs, δ, δ' and δ', δ being regarded as different, except when $\delta' = \delta = 1$. It is easy to simplify the expression thus obtained for N; viz. if $n = A \times B$, A and B being two numbers relatively prime, we find

$$\Sigma\chi\left(\frac{n}{\delta\delta'}\right) = \Sigma\chi\left(\frac{A}{\delta\delta'}\right) \times \Sigma\chi\left(\frac{B}{\delta\delta'}\right);$$

also if $n = \theta^\mu$, θ being a prime,

$$\Sigma\chi\left(\frac{n}{\delta\delta'}\right) = \theta^\mu\left(1 + \frac{1}{\theta}\right).$$

Hence in general

$$N = \Sigma\chi\left(\frac{n}{\delta\delta'}\right) = n\Pi\left(1 + \frac{1}{\theta}\right);$$

the sign of multiplication extending to all the primes θ dividing n.

(*c*) Two matrices $\left\| \begin{matrix} \alpha_1, & \beta_1 \\ \gamma_1, & \delta_1 \end{matrix} \right\|$ and $\left\| \begin{matrix} \alpha_2, & \beta_2 \\ \gamma_2, & \delta_2 \end{matrix} \right\|$ are said to be equivalent (or, more properly, equivalent by primary post-multiplication), when

$$\left\| \begin{matrix} \alpha_1, & \beta_1 \\ \gamma_1, & \delta_1 \end{matrix} \right\| = \left\| \begin{matrix} \alpha_2, & \beta_2 \\ \gamma_2, & \delta_2 \end{matrix} \right\| \times \| \epsilon \| ;$$

$\| \epsilon \|$ being a primary unit matrix.

Equivalent matrices, employed in the formulæ (iv), give identical values for λ^2.

(*d*) The number of primitive and primary matrices $\left\| \begin{matrix} \alpha, & \beta \\ \gamma, & \delta \end{matrix} \right\|$ non-equivalent by primary post-multiplication, is also $n \Pi \left(1 + \frac{1}{\theta}\right)$; so that there are as many non-equivalent matrices as there are non-equivalent elements η; this we know, *à priori*, must be the case; because the number of different transformations must be equal to the number of non-equivalent matrices, and also to the number of non-equivalent elements.

(*e*) We may take as representatives of the N non-equivalent elements η the elements of the *reduced* system

$$\zeta = \frac{-2lK + igK'}{n},$$

where g is any divisor of n, and $2l$ is any term of a system of residues, even and prime to g, taken with respect to the modulus $\frac{n}{g}$. No two of the elements ζ can be equivalent; for if $2hl_1 \equiv 2hl_2$, mod n; $hg_1 \equiv hg_2$, mod n; we must have in the first place $g_1 = g_2$, because h is prime to n; and in the second place, writing g for g_1 or g_2, $h \equiv 1$, mod $\frac{n}{g}$; $l_1 \equiv l_2$, mod $\frac{n}{g}$; *i.e.* $l_1 = l_2$, and the two elements ζ, which were supposed equivalent, are identical. Again, any given element $\frac{pK + iqK'}{n}$ is equivalent to one of the elements ζ; for, if g be the greatest common divisor of q and n, let $-2l$ be the residue of $\frac{n}{g}$, which satisfies the congruence $-2l\frac{q}{g} \equiv p$, mod $\frac{n}{g}$, and which may be supposed prime to g, because p is prime to g; then the simultaneous congruences

$$\left. \begin{matrix} hp \equiv & -2l \\ hq \equiv & g \end{matrix} \right\}, \text{ mod } n,$$

are resoluble, because p, q, and n are respectively prime, and because the

determinant $pg+2lq$ is divisible by n; further, the value of h supplied by these congruences is prime to n, because $2l$ and g are relatively prime; thus the given element is equivalent to the reduced element $\frac{-2lK+igK'}{n}$.

Example.—Let $\eta=\frac{581K+4000iK'}{4725}$ be the given element. Here $n=4725$, $g=25$, $\frac{n}{g}=189$, $\frac{q}{g}=160$, $-2l\times 160\equiv 581$, mod 189; whence $-2l\equiv 182$, mod 189. The congruences $581h\equiv 182$, $4000h\equiv 25$, mod 4725, give $h\equiv 1147$, mod 4725; so that the reduced element $\frac{182K+25iK'}{4725}$ is congruous to $1147\times\eta$, and consequently equivalent to η.

(f) As representatives of the non-equivalent matrices of determinant n, we may take the matrices

$$\|\zeta\|=\begin{vmatrix} g, & 0 \\ 2l, & \frac{n}{g} \end{vmatrix},$$

g and $2l$ having the same significations as in (e).

(g) The reduced pair $(-2l, g)$, and the reduced matrix $\begin{Vmatrix} g, & 0 \\ 2l, & \frac{n}{g} \end{Vmatrix}$, satisfy the congruences (v); and the reduced pairs and matrices cannot be combined in any other way so as to satisfy those congruences.

(h) The congruences (v), which may also be written in the symbolic form

$$(p,\ q)\times\begin{Vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{Vmatrix}\equiv 0, \text{ mod } n,$$

admit, for any given primitive matrix $\begin{Vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{Vmatrix}$, $\psi(n)$ primitive and incongruous solutions (p, q); which, however, are all equivalent: and if we replace the matrix by any equivalent matrix, these solutions remain unchanged.

From these considerations it is evident that, to solve the two converse problems, we have only to determine the reduced pair equivalent to the given pair, or the reduced matrix equivalent to the given matrix, as the case may be; either of them being known, the other is known also; and the proposed problem is completely solved.

Example.—If $n=4725$, and (581, 4000) is the given pair, the equivalent reduced pair is, as we have already seen, (182, 25); the corresponding reduced matrix is therefore $\begin{Vmatrix} 25, & 0 \\ -182, & 189 \end{Vmatrix}$, and all the corresponding matrices are in-

cluded in the formula $\left\| \begin{matrix} 25, & 0 \\ -182, & 189 \end{matrix} \right\| \times \| \epsilon \|$,

where $\| \epsilon \|$ is a primary unit matrix.

Again, if $\left\| \begin{matrix} 75, & 100 \\ -168, & -161 \end{matrix} \right\|$ is a given matrix of the determinant 4725, the equivalent reduced matrix is $\left\| \begin{matrix} 25, & 0 \\ -182, & 189 \end{matrix} \right\|$; hence (182, 25) is the corresponding reduced pair, and all the corresponding pairs are included in the formula

$$(182h + 4725a, \quad 25h + 4725b),$$

where a, b, h are any integers, positive and negative, of which h is prime to 4725.

If we introduce the one-valued functions

$$\sqrt{k} = \phi^2(\omega), \quad \sqrt{\lambda} = \phi^2(\Omega), \quad \sqrt{k'} = \psi^2(\omega), \quad \sqrt{\lambda'} = \psi^2(\Omega),$$

we may write the first two of the equations (ii) in the form

$$\left.\begin{aligned} \phi^2(\Omega) &= i^{-\frac{1}{2}\gamma\delta}\phi^{2n}(\omega)\,\Pi\frac{\cos^2 \mathrm{am}\, 4j\eta}{\Delta^2 \mathrm{am}\, 4j\eta}, \\ \psi^2(\Omega) &= i^{\frac{1}{2}\alpha\beta}\psi^{2n}(\omega)\,\Pi\frac{1}{\Delta^2 \mathrm{am}\, 4j\eta}. \end{aligned}\right\} \quad \ldots\ldots \text{(vii)}$$

Similarly the last of the equations (ii) may be written

$$\left.\begin{aligned} \frac{1}{M} &= (-1)^{\frac{1}{2}(\delta-1)} i^{-\frac{1}{2}\gamma\delta}\frac{\phi^{2n}(\omega)}{\phi^2(\Omega)}\,\Pi \sin^2 \mathrm{am} 4j\eta, \\ &= (-1)^{\frac{1}{2}(\delta-1)}\,\Pi\frac{\sin^2 \mathrm{am}\, 4j\eta}{\sin^2 \mathrm{coam}\, 4j\eta}. \end{aligned}\right\} \quad \ldots\ldots \text{(viii)}$$

In the theory of the transformation of the Theta functions it is convenient to suppose that the matrix $\left\| \begin{matrix} \alpha, & \beta \\ \gamma, & \delta \end{matrix} \right\|$ satisfies the congruences $\beta \equiv 0$, $\gamma \equiv 0$, mod 8, $\alpha \equiv 1$, mod 4. In this case the equations (vii) and (viii) present themselves in the form

$$\sqrt{\lambda} = k^{\frac{1}{2}n}\Pi\frac{\cos^2 \mathrm{am}\, 4j\eta}{\Delta^2 \mathrm{am}\, 4j\eta},$$

$$\sqrt{\lambda'} = k'^{\frac{1}{2}n}\Pi\frac{1}{\Delta^2 \mathrm{am}\, 4j\eta},$$

$$\frac{1}{M} = (-1)^{\frac{1}{2}(n-1)}\frac{k^{\frac{1}{2}n}}{\sqrt{\lambda}}\,\Pi \sin^2 \mathrm{am}\, 4j\eta = (-1)^{\frac{1}{2}(n-1)}\Pi\frac{\sin^2 \mathrm{am}\, 4j\eta}{\sin^2 \mathrm{coam}\, 4j\eta},$$

in which they occur in the 'Fundamenta Nova.' From these special formulæ, in which $\sqrt{\lambda}$, $\sqrt{\lambda'}$, $\sqrt{k}$, $\sqrt{k'}$ represent the one-valued functions $\phi^2(\Omega)$, $\psi^2(\Omega)$, $\phi^2(\omega)$, $\psi^2(\omega)$, the general formula (vii) and (viii) may be immediately inferred by linear transformation.

If $\frac{n}{g} = \gamma\gamma'$, where γ is the greatest divisor of $\frac{n}{g}$ which is prime to g, the number of the residues of $\frac{n}{g}$ which are prime to g (*i. e.* the number of different numbers $2l$ in (*e*), p. 379) is $\gamma\chi(\gamma')$. Hence, the whole number of non-equivalent pairs is $\Sigma\gamma\chi(\gamma')$, the summation extending to every divisor g of n. We have, consequently,

$$\Sigma\gamma\chi(\gamma') = \Sigma\chi\left(\frac{n}{\delta\delta'}\right),$$

$$\Sigma\gamma\chi(\gamma') = n\Pi\left(1+\frac{1}{\theta}\right);$$

these results are easily verified independently (see a Memoir, on the 'Singularities of the Modular Equations and Curves,' 'Proceedings of the London Mathematical Society,' vol. ix., Art. 9) *.

It may deserve notice that the equation (vi) is the first of a series of elementary formulæ of the same general type.

The next in order is

$$\chi(n)\times\chi\left(\frac{n}{\delta_2\delta_3}\right)\chi\left(\frac{n}{\delta_3\delta_1}\right)\chi\left(\frac{n}{\delta_1\delta_2}\right) = \chi\left(\frac{n}{\delta_1}\right)\chi\left(\frac{n}{\delta_2}\right)\chi\left(\frac{n}{\delta_3}\right)\times\chi\left(\frac{n}{\delta_1\delta_2\delta_3}\right);$$

the general formula being

$$\chi(n)\times\Pi\chi\left(\frac{n}{\delta_1\delta_2}\right)\times\Pi\chi\left(\frac{n}{\delta_1\delta_2\delta_3\delta_4}\right)\times\ldots = \Pi\chi\left(\frac{n}{d}\right)\times\Pi\chi\left(\frac{n}{\delta_1\delta_2\delta_3}\right)\times\ldots.$$

The s divisors $\delta_1, \delta_2, \ldots, \delta_s$, are not necessarily relatively prime; but, if p be any prime dividing σ of them, and having a for its exponent in n, and b for the sum of its exponents in $\delta_1, \delta_2, \ldots$, we must have either $b < a$, or else $b = a$, $\sigma < s$. If $b = a$, $\sigma = s$, the factor $\frac{\chi(p)}{p} = \frac{p-1}{p}$ must be applied to the left or right-hand member, according as s is even or uneven.

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* [Vol. ii. p. 252.]

VI.—*Further Theory of the Functions* $Q(\omega)$, $Q'(\omega)$*.

The following is a Table of the linear transformations of $Q(\omega)$ and $Q'(\omega)$. For brevity the argument ω is omitted, and the functions $J = Q + k^2K$, $J' = Q' + k^2K'$ are introduced:

$$\omega = \frac{c + d\Omega}{a + b\Omega}.$$

$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$	$\frac{1}{M}[aQ(\Omega) + ibQ'(\Omega)]$	$\frac{1}{M}[cQ(\Omega) + idQ'(\Omega)]$	$\frac{1}{M}$
$\equiv$	$=$	$=$	$=$
I $\lvert 1 \rvert$	Q	iQ'	$(-1)^{\frac{1}{2}(a-1)}$
II $\lvert \psi \rvert$	Q	iQ'	$(-1)^{\frac{1}{2}(b-1)}i$
III $\lvert \sigma \rvert$	J	iJ'	$(-1)^{\frac{1}{2}(a-1)}k'$
IV $\lvert \tau \rvert$	$J - K$	$i(J' - K')$	$(-1)^{\frac{1}{2}(a+c-1)}k$
V $\lvert \rho \rvert$	$J - K$	$i(J' - K')$	$(-1)^{\frac{1}{2}(c+1)}ik$
VI $\lvert \rho^2 \rvert$	J	iJ'	$(-1)^{\frac{1}{2}(b-1)}ik'$

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* [In explanation of the notation, it is to be observed that every matrix of uneven determinant is, with regard to the modulus 2, of one of the six types

$$\begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}, \quad \begin{vmatrix} 0, & 1 \\ -1, & 0 \end{vmatrix}, \quad \begin{vmatrix} 1, & 0 \\ 1, & 1 \end{vmatrix}, \quad \begin{vmatrix} 1, & 1 \\ 0, & 1 \end{vmatrix}, \quad \begin{vmatrix} 1, & 1 \\ -1, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & 1 \\ -1, & -1 \end{vmatrix},$$

which are represented by the symbols

$$1, \ \psi, \ \sigma, \ \tau, \ \rho, \ \rho^2,$$

and considering these as unit matrices, *i.e.* supposing $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ to be equal, instead of only congruent, to the preceding values, they are represented by

$$\lvert 1 \rvert, \quad \lvert \psi \rvert, \quad \lvert \sigma \rvert, \quad \lvert \tau \rvert, \quad \lvert \rho \rvert, \quad \lvert \rho^2 \rvert.$$

Similarly the primitive matrices of any even determinant, considered with regard to the modulus 2, are of one or other of the following nine types:

$$\begin{vmatrix} 1, & 1 \\ 1, & 1 \end{vmatrix}, \quad \begin{vmatrix} 1, & 0 \\ 1, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & 1 \\ 0, & 1 \end{vmatrix},$$
$$\begin{vmatrix} 0, & 0 \\ 1, & 1 \end{vmatrix}, \quad \begin{vmatrix} 0, & 0 \\ 1, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & 0 \\ 0, & 1 \end{vmatrix},$$
$$\begin{vmatrix} 1, & 1 \\ 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} 1, & 0 \\ 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & 1 \\ 0, & 0 \end{vmatrix},$$

which are symbolized thus: $\begin{vmatrix} C_{1,1}, & C_{2,1}, & C_{3,1} \\ C_{1,2}, & C_{2,2}, & C_{3,2} \\ C_{1,3}, & C_{2,3}, & C_{3,3} \end{vmatrix}$.

See Arts. 21 and 23 of No. xliii.

The functions $Q(\omega)$ and $Q'(\omega)$ have opposite signs to the $Q(\omega)$ and $Q'(\omega)$ defined in Note III (p. 368). The quantity η denotes $e^{\frac{1}{2}i\pi}$.]

We add a list of useful particular cases; the formulæ are immediately deducible from the Table. The functions $P(\omega)$ and $P'(\omega)$ represent $J(\omega)-k'^2K(\omega)$ and $J'(\omega)-k'^2K'(\omega)$ respectively. For convenience, the transformations of K and K' are included in the list, and, as before, the argument ω is omitted.

II. $\omega=|\psi|\times\Omega=-\dfrac{1}{\Omega}$.

$$K(\Omega)=\quad K';\qquad iK'(\Omega)=\quad iK;$$
$$J(\Omega)=-J'+K';\qquad iJ'(\Omega)=-iJ+iK;$$
$$P(\Omega)=-P';\qquad iP'(\Omega)=-iP;$$
$$Q(\Omega)=-Q';\qquad iQ'(\Omega)=-iQ.$$

III. $\omega=\begin{vmatrix}1, & 0\\ \mp1, & 1\end{vmatrix}\times\Omega=\mp1+\Omega$.

$$K(\Omega)=k'K;\qquad iK'(\Omega)=k'(\pm K+iK');$$
$$J(\Omega)=\frac{1}{k'}Q;\qquad iJ'(\Omega)=\frac{1}{k'}(\pm Q+iQ');$$
$$P(\Omega)=\frac{1}{k'}(Q-K);\qquad iP'(\Omega)=\frac{1}{k'}[\pm(Q-K)+i(Q'-K')]$$
$$Q(\Omega)=\frac{J}{k'};\qquad iQ'(\Omega)=\frac{1}{k'}(\pm J+iJ').$$

IV. $\omega=\begin{vmatrix}1, & \mp1\\ 0, & 1\end{vmatrix}\times\Omega=\dfrac{\Omega}{1\mp\Omega}$.

$$K(\Omega)=k(K\pm iK');\qquad iK'(\Omega)=ikK';$$
$$J(\Omega)=\frac{1}{k}(J\pm iJ');\qquad iJ'(\Omega)=\frac{i}{k}J';$$
$$P(\Omega)=\frac{1}{k}(Q+K)\pm\frac{i}{k}(Q'+K');\qquad iP'(\Omega)=\frac{i}{k}(Q'+K');$$
$$Q(\Omega)=\frac{1}{k}(J-K)\pm\frac{i}{k}(J'-K');\qquad iQ'(\Omega)=\frac{i}{k}(J'-K').$$

V. $\omega=\begin{vmatrix}\mp1, & +1\\ -1, & 0\end{vmatrix}\times\Omega=-\dfrac{1}{\mp1+\Omega}$.

$$K(\Omega)=kK';\qquad iK'(\Omega)=ik[K\pm iK'];$$
$$J(\Omega)=-\frac{Q'}{k}\qquad iJ'(\Omega)=\frac{i}{k}[-Q\mp iQ'];$$
$$P(\Omega)=-\frac{Q'+K'}{k};\qquad iP'(\Omega)=\frac{i}{k}-[Q-K\mp i(Q'+K')];$$
$$Q(\Omega)=-\frac{J'-K'}{k};\qquad iQ'(\Omega)=\frac{i}{k}[-J+K\mp i(J'-K')].$$

VI. $\omega = \begin{vmatrix} 0, & 1 \\ -1, & \mp 1 \end{vmatrix} \times \Omega = \dfrac{-1 \mp \Omega}{\Omega}$.

$K(\Omega) = k'(K' \mp iK)$; $\qquad iK'(\Omega) = ik'K$;

$J(\Omega) = \dfrac{1}{k'}[K' - J' \mp i(K - J)]$; $\qquad iJ'(\Omega) = \dfrac{i}{k'}(K - J)$;

$P(\Omega) = \dfrac{1}{k'}[K' - Q' \mp i(K - Q)]$; $\qquad iP'(\Omega) = \dfrac{i}{k'}(K - Q)$;

$Q(\Omega) = -\dfrac{1}{k'}(J' \mp iJ)$; $\qquad iQ'(\Omega) = -\dfrac{i}{k'}J$.

The nine typical transformations of the second order (see Art. 34) give the following formulæ for Q and Q'. They are derived by differentiation from the corresponding formulæ for K and K' in accordance with the equations (1), Note III, p. 368.

I.

(*a*) $\omega = C_{1,1} \times \Omega = \begin{vmatrix} 1, & 1 \\ -1, & 1 \end{vmatrix} \times \Omega = \dfrac{\Omega - 1}{\Omega + 1}$.

$K(\Omega) = \eta^2 \sqrt{(kk')}\,(K - iK')$,

$K'(\Omega) = \eta^{-2} \sqrt{(kk')}\,(K + iK')$,

$Q(\Omega) = \dfrac{1}{i\sqrt{(2kk')}}[Q + (k^2 - \tfrac{1}{2})K]$,

$Q'(\Omega) = \dfrac{1}{i\sqrt{(2kk')}}[Q' + (k^2 - \tfrac{1}{2})K']$.

(*b*) $\omega = C'_{1,1} \times \Omega = \begin{vmatrix} -1, & 1 \\ -1, & -1 \end{vmatrix} \times \Omega = \dfrac{1 + \Omega}{1 - \Omega}$.

$K(\Omega) = \eta^{-2} \sqrt{(kk')}\,(K + iK')$,

$K'(\Omega) = \eta^2 \sqrt{(kk')}\,(K - iK')$,

$Q(\Omega) = \dfrac{i}{\sqrt{(2kk')}}[Q' + (k^2 - \tfrac{1}{2})K']$,

$Q'(\Omega) = \dfrac{i}{\sqrt{(2kk')}}[Q + (k^2 - \tfrac{1}{2})K]$.

II.

$\omega = C_{2,1} \times \Omega = \begin{vmatrix} 1, & 0 \\ 1, & 2 \end{vmatrix} \times \Omega = 1 + 2\Omega$.

$K(\Omega) = (k' - ik)\,K$,

$K'(\Omega) = \tfrac{1}{2}(k' - ik)\,(K' + iK)$,

$Q(\Omega) = 2(k' + ik)[Q + ikk'K]$,

$Q'(\Omega) = (k' + ik)[Q' + ikk'K' + i(Q + ikk'K)]$.

III.

$$\omega = C_{3,1} \times \Omega = \begin{vmatrix} 0, & 1 \\ -2, & 1 \end{vmatrix} \times \Omega = \frac{\Omega - 2}{\Omega}.$$

$$K(\Omega) = \tfrac{1}{2}(k' + ik)(K' + iK),$$

$$K'(\Omega) = (k' + ik) K,$$

$$Q(\Omega) = -(k' - ik)[Q' + ikk'K' + i(Q + ikk'K)],$$

$$Q'(\Omega) = -2(k' - ik)[Q + ikk'K].$$

IV.

$$\omega = C_{1,2} \times \Omega = \begin{vmatrix} 2, & 0 \\ -1, & 1 \end{vmatrix} \times \Omega = \frac{\Omega - 1}{2}.$$

$$K(\Omega) = \sqrt{k'}\, K,$$

$$K'(\Omega) = 2\sqrt{k'}(K' - \tfrac{1}{2}K),$$

$$Q(\Omega) = \frac{1}{2\sqrt{k'}}[Q + \tfrac{1}{2}k^2K],$$

$$Q'(\Omega) = \frac{1}{2\sqrt{k'}}[2(Q' + \tfrac{1}{2}k^2K') - i(Q + \tfrac{1}{2}k^2K)].$$

V.

$$\omega = C_{2,2} \times \Omega = \begin{vmatrix} 0, & 2 \\ -1, & 0 \end{vmatrix} \times \Omega = -\frac{1}{2\Omega}.$$

$$K(\Omega) = (1 + k')\, K',$$

$$K'(\Omega) = \tfrac{1}{2}(1 + k')K,$$

$$Q(\Omega) = -\frac{2}{1 + k'}[Q' + k'(1 - k')\, K'],$$

$$Q'(\Omega) = -\frac{1}{1 + k'}[Q + k'(1 - k')K].$$

VI.

$$\omega = C_{3,2} \times \Omega = \begin{vmatrix} 2, & 0 \\ 0, & 1 \end{vmatrix} \times \Omega = \tfrac{1}{2}\Omega.$$

$K($

$$K(\Omega) = \tfrac{1}{2}(1 + k')\, K,$$

$J(\Omega) =$ $\quad K'(\Omega) = (1 + k')\, K',$

$$P(\Omega) = -\frac{Q'}{k} = \frac{1}{1 + k'}[Q + k'(1 - k')\, K],$$

$$Q(\Omega) = -\frac{J' - K'}{k}; \frac{2}{1 + k'}[Q' + k'(1 - k')K'].$$

VII.

$$\omega = C_{1,3} \times \Omega = \begin{vmatrix} -1, & 1 \\ -2, & 0 \end{vmatrix} \times \Omega = \frac{2}{1-\Omega}.$$

$$K(\Omega) = \sqrt{k}K',$$

$$K'(\Omega) = 2\sqrt{k}\left[K - \tfrac{1}{2}iK'\right],$$

$$Q(\Omega) = -\frac{1}{2\sqrt{k}}(Q' - \tfrac{1}{2}k'^2K'),$$

$$Q'(\Omega) = -\frac{1}{2\sqrt{k}}\left[2(Q - \tfrac{1}{2}k'^2K) - i(Q' - \tfrac{1}{2}k'^2K')\right].$$

VIII.

$$\omega = C_{2,3} \times \Omega = \begin{vmatrix} 1, & 0 \\ 0, & 2 \end{vmatrix} \times \Omega = 2\Omega.$$

$$K(\Omega) = (1+k)K,$$

$$K'(\Omega) = \tfrac{1}{2}(1+k)K',$$

$$Q(\Omega) = \frac{2}{1+k}\left[Q - k(1-k)K\right],$$

$$Q'(\Omega) = \frac{1}{1+k}\left[Q' - k(1-k)K'\right].$$

IX.

$$\omega = C_{3,3} \times \Omega = \begin{vmatrix} 0, & 1 \\ -2, & 0 \end{vmatrix} \times \Omega = -\frac{2}{\Omega}.$$

$$K(\Omega) = \tfrac{1}{2}(1+k)K',$$

$$K'(\Omega) = (1+k)K,$$

$$Q(\Omega) = -\frac{1}{1+k}\left[Q' - k(1-k)K'\right],$$

$$Q'(\Omega) = -\frac{2}{1+k}\left[Q - k(1-k)K\right].$$

Let $h(\omega) = i\,\dfrac{Q'(\omega)}{Q(\omega)}$; the formulæ of the Table show that if $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ be a unit matrix of either of the types (1) or (ψ), the equation $\omega = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times \Omega$ implies the equation

$$h(\omega) = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times h(\Omega);$$

or, which is the same thing,

$$\omega = 2a_1 - \frac{1}{2a_2} - \frac{1}{2a_3} \ldots - \frac{1}{2a_s} - \frac{1}{\Omega}, \quad \ldots\ldots\ldots \quad (1)$$

the quotients being even and the continued fraction subtractive; then also

$$h(\omega) = 2a_1 - \frac{1}{2a_2} - \frac{1}{2a_3} \ldots - \frac{1}{2a_s} - \frac{1}{h(\Omega)}. \quad \ldots\ldots\ldots \quad (2)$$

The property of the function $h(\omega)$ expressed by the equations (1) and (2), and the characteristic property of the elliptic function of the second species, may be regarded as having a relation to one another comparable to that existing between the double periodicity of the elliptic functions of the first species and the quasi-periodicity of the modular functions. The plane on which the argument of the elliptic functions is represented is divided into elementary parallelograms, of which the sides are any two simultaneous elliptic periods; and the elliptic function of the first species has the same value at corresponding points in two different parallelograms, while the values of the function of the second species differ by a quantity of the same type as the difference of the vectors of the two points; *e. g.*

$$Z(x + 2mK + 2m'iK') = Z(x) + 2mQ + 2m'iQ'.$$

On the other hand, answering to the linear substitutions $\omega = \frac{c + d\Omega}{a + b\Omega}$, we have a division of the semi-plane on which ω is represented into lunular spaces equivalent to the reduced space (B) of fig. 2, referred to in Art. 38. At corresponding points in two such spaces the modular functions $\Phi(\omega)$ and $\Psi(\omega)$ have the same value, while the values of $h(\omega)$ are related to one another as the vectors of the two points.

VII. *Formulæ relating to the Elliptic Functions of the Second Species.**

The Formula of Addition.

Let x_1, x_2, x_3 be three arguments of which the sum is zero; we have, from the equation of Jacobi [Art. 10, (iii)],

$$Z(x_1) + Z(x_2) + Z(x_3) = -\Sigma_{s=1}^{s=3} \frac{d}{dx_s} \log \vartheta_0\left(\frac{\pi x_s}{2K}\right). \quad \ldots\ldots \quad (i)$$

In the equation (i) of Art. 6, let

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu, \qquad \nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu, \qquad x_1 + x_2 + x_3 = 0;$$

* [This Note is to some extent equivalent to IV, but the two Notes could not be united together conveniently.]

differentiate with regard to x_4, and write $x_4 = 0$ in the result. Representing for brevity $\vartheta_1'(0)\,\vartheta_1(x_1)\,\vartheta_1(x_2)\,\vartheta_1(x_3)$ by a_1, and

$$\vartheta_s(0)\,[\vartheta_s'(x_1)\,\vartheta_s(x_2)\,\vartheta_s(x_3)+\vartheta_s(x_1)\,\vartheta_s'(x_2)\,\vartheta_s(x_3)+\vartheta_s(x_1)\,\vartheta_s(x_2)\,\vartheta_s'(x_3)]$$

by a_s, $(s = 0, 2, 3)$, we attribute successively to μ, ν the values 0, 0 ; 0, 1 ; 1, 0 ; we thus obtain the equations

$$a_1+a_2+a_3+a_0=0,$$
$$a_1+a_2-a_3-a_0=0,$$
$$a_1-a_2-a_3+a_0=0,$$

or, which is the same thing,

$$a_1=-a_2=a_3=-a_0.$$

Transforming the right-hand member of (i) by means of the equality $a_0 = -a_1$, we arrive at the equation of addition in its usual form ; viz.

$$Z(x_1)+Z(x_2)+Z(x_3)=-k^2\operatorname{sn}x_1\operatorname{sn}x_2\operatorname{sn}x_3\,; \quad \ldots \quad \text{(ii)}$$
$$x_1+x_2+x_3=0.$$

The Formulæ of Transformation.

In the equation of Jacobi [Art. 10, (iii)] we write $\frac{x}{M}$ for x, and Ω for ω ; we thus obtain

$$\frac{1}{M}\,Z\Big[\frac{x}{M},\ \lambda\Big]=\frac{J(\Omega)}{K(\Omega)}\,\frac{x}{M^2}-\frac{d}{dx}\log\vartheta_0\Big(\frac{\pi x}{2MK(\Omega)},\ \Omega\Big)$$
$$=\frac{J(\Omega)}{K(\Omega)}\,\frac{x}{M^2}-\frac{d}{dx}\log\vartheta_0\Big(\frac{\pi x(a+b\Omega)}{2K},\ \Omega\Big), \quad \ldots \quad \text{(iii)}$$

where $\omega=\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}\times\Omega$, $\lambda=\phi^4(\Omega)$, and M is the multiplier corresponding to the transformation. For brevity, we suppose that the matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ is of the uneven determinant Δ, and satisfies the congruential conditions $b\equiv c\equiv 0$, $a\equiv 1$, mod 8, assumed to exist in Art. 33. Putting $h = 2K$ in equation (xii) of that article, and designating by C' a quantity not containing x, we find

$$\vartheta_0\Big[(a+b\Omega)\frac{\pi x}{2K},\ \Omega\Big]$$
$$=C'\times e^{-i\pi b(a+b\Omega)\frac{x^2}{4K^2}}\times\vartheta_0^{\Delta}\Big(\frac{\pi x}{2K},\ \omega\Big)\times\Pi_{j=1}^{j=\frac{1}{2}(\Delta-1)}[1-k^2\sin^2 4j\,\zeta\sin^2\operatorname{am}x],$$

or taking the logarithm, and differentiating,

$$\frac{d}{dx}\log\vartheta_0\Big[(a+b\Omega)\frac{\pi x}{2K},\ \Omega\Big]$$

$$=-\frac{i\pi b\,(a+b\Omega)x}{2K^2}+\Delta\frac{d}{dx}\log\vartheta_0\Big[\frac{\pi x}{2K},\ \omega\Big]+\Sigma_{j=1}^{j=\frac{1}{2}(\Delta-1)}\frac{d}{dx}\log\big[1-k^2\sin^2 4j\zeta\sin^2\mathrm{am}\,x\big],$$

whence finally

$$\frac{1}{M}Z\Big(\frac{x}{M},\lambda\Big)-\Delta Z(x,k^2)+2Cx$$

$$=2k^2\sin\mathrm{am}\,x\cos\mathrm{am}\,x\,\Delta\,\mathrm{am}\,x\,\Sigma_j\,\frac{\sin^2\mathrm{am}\,4j\zeta}{1-\sin^2\mathrm{am}\,4j\zeta\sin^2\mathrm{am}\,x},\quad\ldots\quad\text{(iv)}$$

the value of the constant C being

$$\left.\begin{aligned}2C&=\Delta\frac{J(\omega)}{K(\omega)}-\frac{1}{M^2}\frac{J(\Omega)}{K(\Omega)}-\frac{i\pi b\,(a+b\Omega)}{2K^2(\omega)}\\&=\Delta\frac{J(\omega)}{K(\omega)}-\frac{1}{M^2}\frac{J(\Omega)}{K(\Omega)}-\frac{i\pi b}{2MK(\omega)K(\Omega)}.\end{aligned}\right\}\quad\ldots\quad\text{(v)}$$

Dividing equation (iv) by x, putting $x=0$, and observing that $\lim\frac{Z(x)}{x}=0$, we also have

$$C=\Sigma_j\,k^2\sin^2\mathrm{am}\,4j\zeta\quad\ldots\quad\text{(vi)}$$

If we denote $\sin\mathrm{am}\,x$ by u, the value of $\sin\mathrm{am}\big(\frac{x}{M},\lambda\big)$, Art. 33, equation (xvii), is of the form

$$\frac{u}{M}\,\frac{1+\mathrm{A}_1u^2+A_2u^4+\ldots+A_{\frac{1}{2}(\Delta-1)}u^{\Delta-1}}{1+B_1u^2+B_2u^4+\ldots+B_{\frac{1}{2}(\Delta-1)}u^{\Delta-1}}=(-1)^{\frac{1}{2}(\Delta-1)}\frac{u^\Delta}{M\,.\,\Pi\sin^2\mathrm{am}4j\zeta}\times\frac{\chi\big(\frac{1}{k^2u^2}\big)}{\chi(u^2)}$$

$$=\frac{k^{\frac{1}{2}\Delta}u^\Delta}{\lambda^{\frac{1}{2}}}\times\frac{\chi\big(\frac{1}{k^2u^2}\big)}{\chi(u^2)},$$

if

$$\chi(u^2)=1+B_1u^2+B_2u^4+\ldots$$

$$=\Pi[1-k^2\sin^2\mathrm{am}\,4j\zeta\sin^2\mathrm{am}\,x].$$

Employing this notation, we have for the right-hand member of (i) the equivalent expression

$$-2\cos\mathrm{am}\,x\Delta\mathrm{am}\,x\frac{B_1u+2B_2u^3+3B_3u^5+\ldots}{1+B_1u^2+B_2u^4+\ldots},$$

so that

$$C=-B_1.$$

The constant C is a rational function of k^2 and λ^2; the integrals $K(\omega)$, $K(\Omega)$, $J(\omega)$, $J(\Omega)$ entering into the expressions (iii) only apparently: the value is in fact

$$2C = \Delta k^2 - \frac{\lambda^2}{M^2} - \frac{2\Delta k^2 k'^2}{M}\frac{dM}{d.k^2}*. \quad \ldots\ldots\ldots \text{(vii)}$$

It may be observed that the coefficients B_2, B_3, ... like B_1, are all rational functions of k^2 and λ^2; for these coefficients are linear functions of the quantities $\Sigma k^{2s}\sin^{2s}\mathrm{am}\,4j\zeta$, which are rational in k^2 and λ^2, as may be seen by expanding the elliptic functions in equation (ii) in series proceeding by powers of x, and equating coefficients of like powers of x. The same thing is true for the coefficients A, which are linear in the quantities $\Sigma \sin^{-2s}\mathrm{am}4j\zeta$, and which are also connected with the coefficients B by the relation

$$A_s = \frac{k^{2s+1}M}{k^{\frac{1}{2}\Delta}\lambda^{\frac{1}{2}}} B_{\frac{1}{2}(\Delta-1)-s},$$

where the coefficient of B is rational in k^2 and λ^2, because $\Pi \sin^2 \mathrm{am}\,4j\zeta$ is so.

The right-hand member of equation (i) may be transformed by means of the formula of addition; viz. this formula gives

$$Z(x+y)+Z(x-y)-2Z(x) = Z(x+y)+Z(x-y)-Z(2x)-[2Z(x)-Z(2x)],$$

whence

$$= \frac{2k^2 \sin\mathrm{am}\,x \cos\mathrm{am}\,x\,\Delta\mathrm{am}\,x \sin^2\mathrm{am}\,y}{1-k^2\sin^2\mathrm{am}\,x\sin^2\mathrm{am}\,y},$$

$$\frac{1}{M}Z\left(\frac{x}{M}, \lambda\right) - \Delta Z(x, k) + 2Cx = \Sigma Z(x+4j\zeta, k), \quad \ldots\ldots \text{(viii)}$$

the summation extending to every value of j' from $-\frac{1}{2}(\Delta-1)$ to $\frac{1}{2}(\Delta-1)$. The formula (viii) may also be obtained by integrating the sum of the squares of the roots of equation (xxv), Art. 33.

To complete the preceding theory we add the following list of the transformations of $Z(x)$ by unit matrices.

I.

For any unit matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix} \bmod 2,$

$$(-1)^{\frac{1}{2}(a-1)} Z[(-1)^{\frac{1}{2}(a-1)}x, k] = Z(x, k).$$

---

* [C is the same as the H of Note IV (pp. 373–376).]

II.

$$\omega = \begin{vmatrix} 0, & 1 \\ -1, & 0 \end{vmatrix} \times \Omega = -\frac{1}{\Omega},$$

$$iZ(ix, k') = Z(x, k) - \left[x - \frac{\sin\mathrm{am}\, x \Delta\mathrm{am}\, x}{\cos\mathrm{am}\, x}\right].$$

III.

$$\omega = \begin{vmatrix} 1, & 0 \\ \pm 1, & 1 \end{vmatrix} \times \Omega = \Omega \pm 1,$$

$$h'Z\left(k'x, \mp\frac{ik}{k'}\right) = Z(x, k) - k^2\left[x - \frac{\sin\mathrm{am}\, x \cos\mathrm{am}\, x}{\Delta\mathrm{am}\, x}\right].$$

IV.

$$\omega = \begin{vmatrix} \pm 1, & 1 \\ 0, & \pm 1 \end{vmatrix} \times \Omega = \frac{\Omega}{1 \pm \Omega},$$

$$kZ\left[kx, \frac{1}{k}\right] = Z[x, k].$$

V.

$$\omega = \begin{vmatrix} \pm 1, & 1 \\ -1, & 0 \end{vmatrix} \times \Omega = -\frac{1}{\Omega \pm 1},$$

$$ikZ\left(ikx, \mp\frac{ik'}{k}\right) = Z[x, k] - k^2\left(x - \frac{\sin\mathrm{am}\, x \cos\mathrm{am}\, x}{\Delta\mathrm{am}\, x}\right).$$

VI.

$$\omega = \begin{vmatrix} 0, & 1 \\ -1, & \pm 1 \end{vmatrix} \times \Omega = -\frac{1 \mp \Omega}{\Omega},$$

$$ik'Z\left(ik'x, \frac{1}{k'}\right) = Z[x, k] - \left(x - \frac{\sin\mathrm{am}\, x \Delta\mathrm{am}\, x}{\cos\ \mathrm{am}}\right).$$

These formulæ may be verified by transforming the function $\sin\mathrm{am} x$ in the equation $Z(x) = \int_0^x k^2 \sin^2\mathrm{am}\, x\, dx$. For example,

$$\begin{aligned} iZ(ix, k') - Z(x, k) + x &= i\int_0^{ix} k'^2 \sin^2\mathrm{am}(x, k')\, dx - \int_0^x k^2 \sin^2\mathrm{am}(x, k)\, dx + x \\ &= -\int_0^x [k'^2 \sin^2\mathrm{am}(ix, k') + k^2 \sin^2\mathrm{am}(x, k) - 1]\, dx \\ &= \int_0^x \left[k'^2 \frac{\sin^2\mathrm{am}(x, k)}{\cos^2\mathrm{am}(x, k)} + \Delta^2\mathrm{am}(x, k)\right] dx \\ &= \frac{\sin\mathrm{am}\, x \Delta\mathrm{am}\, x}{\cos\mathrm{am}\, x}. \end{aligned}$$

The formula IV (which is an immediate consequence of the corresponding formula for $\sin \operatorname{am} x$) changes II into VI, III into V, and *vice versâ*. The formulæ may also be deduced from the equation (iii) *suprà*, but the elimination of the complete integrals requires the use of some additional formulæ.

Taking again as an example the formula II, we have

$$\begin{aligned} iZ(ix, k') &= -\frac{J(\Omega)}{K(\Omega)}x - \frac{d}{dx}\log \vartheta_0\left(\frac{\pi x \Omega}{2K}, \Omega\right) \\ &= -\frac{J(\Omega)}{K(\Omega)}x + \frac{i\pi\Omega x}{2K^2} - \frac{d}{dx}\log \vartheta_2\left(\frac{\pi x}{2K}, \omega\right) \\ &= -\frac{J(\Omega)}{K(\Omega)}x + \frac{i\pi\Omega x}{2K^2} - \frac{d}{dx}\log \vartheta_0\left(\frac{\pi x}{2K}, \omega\right) - \frac{d}{dx}\log \cos \operatorname{am}(x, k) \\ &= Z(x, k) - x\left[\frac{J(\Omega)}{K(\Omega)} + \frac{J(\omega)}{K(\omega)} - \frac{i\pi\Omega}{2K^2}\right] - \frac{d}{dx}\log \cos \operatorname{am}(x, k). \end{aligned}$$

But
$$J(\Omega) = K'(\omega) - J'(\omega), \quad K(\Omega) = K'(\omega),$$
$$\Omega = \frac{iK(\omega)}{K'(\omega)};$$

and the coefficient of $-x$ becomes

$$1 + \frac{K'J - KJ' + \frac{1}{2}\pi}{KK'}, \text{ that is, } 1,$$

in accordance with the formula II.

Lastly, for the three typical quadratic transformations, writing for brevity s, c, d to denote $\sin \operatorname{am}(x, k)$, $\cos \operatorname{am}(x, k)$, $\Delta \operatorname{am}(x, k)$ respectively, we have the formulæ

$$\omega = \begin{vmatrix} 1, & 0 \\ 1, & 2 \end{vmatrix} \times \Omega, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$(k' - ik)Z\left[(k' - ik)x, \ \frac{2\sqrt{(ikk')}}{k + ik'}\right] - 2Z(x, k) = -2k(k + ik')\left[x - \frac{csd}{1 - k(k + ik')s^2}\right];$$

$$(1 + k')Z\left[(1 + k')x, \ \frac{1 - k'}{1 + k'}\right] - 2Z(x, k) = -k^2\left[x - \frac{sc}{d}\right]; \quad \ldots \quad (2)$$

$$(1 + k)Z\left[(1 + k)x, \ \frac{2\sqrt{k}}{1 + k}\right] - 2Z(x, k) = 2k\left[x - \frac{scd}{1 + ks^2}\right]; \quad \ldots \quad (3)$$

which may be verified by either of the methods exemplified in the case of the linear transformation $\omega = -\frac{1}{\Omega}$. Observing that

$$2Z(x, k) - Z(2x, k) = -k^2 \sin^2 \operatorname{am} x \sin \operatorname{am} 2x.$$

we may write the equation (3) in the form

$$(1+k)Z\Big[(1+k)\,x,\ \frac{2\sqrt{k}}{1+k}\Big]-Z(2x,k)=k(2x-\sin\operatorname{am}2x).$$

Employing the notation of Legendre,

$$F(\phi,k)=\int_0^{\phi}\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}},\quad E(\phi,k)=\int_0^{\phi}\sqrt{(1-k^2\sin^2\phi)}\,d\phi\,;$$

and writing

$$\phi=\operatorname{am}(2x,k),\quad \psi=\operatorname{am}\Big[(1+k)\,x,\frac{2\sqrt{k}}{1+k}\Big],$$

so that

$$Z(2x,k)=F(\phi,k)-E(\phi,k),$$

$$Z\Big[(1+k)x,\frac{2\sqrt{k}}{1+k}\Big]=F\Big(\psi,\frac{2\sqrt{k}}{1+k}\Big)-E\Big(\psi,\frac{2\sqrt{k}}{1+k}\Big)$$

$$=\tfrac{1}{2}(1+k)F(\phi,k)-E\Big(\psi,\frac{2\sqrt{k}}{1+k}\Big),$$

we find

$$\left.\begin{aligned}\tfrac{1}{2}k'^2F(\phi,k)&=E(\phi,k)-(1+k)\,E\Big(\psi,\frac{2\sqrt{k}}{1+k}\Big)+k\sin\phi,\\ \sin(2\psi-\phi)&=k\sin\phi,\end{aligned}\right\}\quad\ldots\quad(4)$$

a celebrated formula, due to Landen, which serves to express an elliptic integral of the first species having a real modulus less than unity by means of two elliptic integrals of the second species having moduli of the same character.

A similar, and indeed equivalent, formula is obtained from the equation (2); viz. writing

$$\phi=\operatorname{am}(x,k),\quad \psi=\operatorname{am}\Big[(1+k')\,x,\frac{1-k'}{1+k'}\Big],$$

we find

$$\left.\begin{aligned}k'F(\phi,k)&=\tfrac{1}{2}(1+k')\,E\Big(\psi,\frac{1-k'}{1+k'}\Big)-E(\phi,k)+\tfrac{1}{2}(1-k')\sin\psi,\\ \tan(\psi-\phi)&=\tan\phi.\end{aligned}\right\}\quad\ldots\quad(5)$$

VIII. *The Functions* Al (x) *of Weierstrass.*

The Abelian functions of M. Weierstrass are defined by the equations $[s=0,\ 1,\ 2,\ 3]$,

$$\operatorname{Al}_s(x)=e^{-\frac{J}{2K}x^2}\frac{\vartheta_s\left(\frac{\pi x}{2K}\right)}{a_s}\quad\ldots\quad(i)$$

where, if $s=1$,

$$a_s=\lim_{x=0}\Big[\frac{\vartheta_1\left(\frac{\pi x}{2K}\right)}{x}\Big]$$

and, if $s = 0, 2, 3,$ $$a_s = \vartheta_s(0),$$

and where $$J = \int_0^K k^2 \sin^2 \mathrm{am}\, x dx.$$

These functions which differ from the corresponding Theta functions of $\frac{\pi x}{2K}$ only by the presence of the exponential factor $e^{-\frac{J}{2K}x^2}$, and of a factor not containing x, possess, as is well known, the remarkable property that they can be developed in series proceeding by powers of x, of which the coefficients are integral functions of k^2 with integral coefficients, and which are convergent for all values of x. For our present purpose, there is a slight convenience in considering, instead of the functions $\mathrm{Al}(x)$, multiples of these functions by the exponential $e^{\frac{1}{2}k^2x^2}$. These multiples we shall call, in what follows, Abelian functions, and we shall denote them by the symbols $A_0(x)$, $A_1(x)$, $A_2(x)$, $A_3(x)$, so that *e. g.*

$$A_0(x) = e^{-\frac{Q}{2K}x} \times \frac{\vartheta_0\left(\frac{\pi x}{2K}\right)}{\vartheta_0(0)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

These functions respectively satisfy the partial differential equations

$$\frac{d^2A_s}{dx^2} + 4k^2(1-k^2)\frac{dA_s}{d.k^2} - k^2(1-k^2)x^2A_s + g_sA_s = 0, \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

where $$g_0 = -k^2, \quad g_1 = 1-2k^2, \quad g_2 = 1-k^2, \quad g_3 = 0;$$

or, which is the same thing, the four functions $A_0\sqrt{k'}$, $A_1\sqrt{(kk')}$, $A_2\sqrt{k}$, A_3 all satisfy one and the same equation

$$\frac{d^2A}{dx^2} + 4k^2(1-k^2)\frac{dA}{d.k^2} - k^2(1-k^2)x^2A = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

The equations (iii) are a little simpler than the corresponding equations satisfied by the functions of M. Weierstrass, viz.

$$\frac{d^2\,\mathrm{Al}}{dx^2} + 2k^2x\frac{d\,\mathrm{Al}}{dx} + 4k^2(1-k^2)\frac{d\,\mathrm{Al}}{d.k^2} + k^2x^2\mathrm{Al} + g_s\mathrm{Al} = 0, \quad . \quad . \quad . \quad \text{(v)}$$

where $$g_0 = 0, \quad g_1 = 1-k^2, \quad g_2 = 1, \quad g_3 = k^2;$$

but the gain in simplicity is apparent rather than real, as the determination of the coefficients is not more easily effected in the functions A_s than in the

functions Al$_s$. The equations (iii) or (iv), as well as (v), may be derived directly from the partial differential equation of the Theta functions

$$\frac{d^2\vartheta(x,\omega)}{dx^2} = \frac{4i}{\pi}\frac{d\vartheta(x,\omega)}{d\omega},$$

which gives

$$\frac{d\vartheta\left(\frac{\pi x}{2K}\right)}{d\omega} = -\frac{x}{K}\frac{d\vartheta\left(\frac{\pi x}{2K}\right)}{dx}\frac{dK}{d\omega} + \frac{K^2}{i\pi}\frac{d^2\vartheta\left(\frac{\pi x}{2K}\right)}{dx^2},$$

and in which we are to put

$$\vartheta_s\left(\frac{\pi x}{2K}\right) = e^{\frac{Q}{2K}x^2} \times A_s(x) \times \vartheta_s(0),$$

and to eliminate $d\omega$ by means of the equation

$$\frac{d.k^2}{d\omega} = -\frac{4}{i\pi}k^2(1-k^2)K^2,$$

the differentiations with regard to k^2 being effected by means of the formulæ

$$\frac{dK}{d.k^2} = -\frac{Q}{2k^2k'^2}, \qquad \frac{dQ}{d.k^2} = -\tfrac{1}{2}K.$$

The Abelian functions are connected by a relation which may be inferred from the corresponding relation between the Theta functions; viz., using for a moment a notation with double suffixes,

$$A_{0,1} = A_0, \quad A_{1,1} = A_1, \quad A_{1,0} = A_2, \quad A_{0,0} = A_3,$$

and supposing $A_{m,n} = A_{\mu,\nu}$, if $m \equiv \mu$, $n \equiv \nu$, mod 2, then from the formula

$$\vartheta_{m+\mu,n+\nu}(x) = e^{\mu i\pi\left[\frac{x}{\pi}+\frac{1}{2}\mu\omega+\frac{1}{2}(n+\nu)\right]} \times \vartheta_{m,n}(x+\tfrac{1}{2}\mu\omega\pi+\tfrac{1}{2}\nu\pi)$$

we obtain the following

$$A_{m,n}(x+\nu K+i\mu K') = e^{(\nu Q+i\mu Q')(x+\frac{1}{2}\nu K+\frac{1}{2}i\mu K)} \times e^{-\frac{1}{4}i\pi\mu(2n+\nu)} \times \xi \times A_{m+\mu,n+\nu}(x),$$

where

$$\xi = \frac{\alpha_{m+\mu,n+\nu}}{\alpha_{m,n}}.$$

General formulæ for the transformation of the Abelian functions are immediately deducible from the formulæ for the transformation of the Theta functions.

Thus, since in general, whatever be the transformation $\omega = \dfrac{c+d\Omega}{a+b\Omega}$, we have

$$\vartheta_{\mu,\nu}\left[(a+\beta\Omega)\frac{\pi x}{2K}, \Omega\right] = e^{-i\pi\beta(a+\beta\Omega)\frac{x^2}{4K^2}} \times T,$$

where T is a homogeneous function of order n of two of the Theta functions $\Im\left(\frac{\pi x}{h}, \omega\right)$; then, attending to the equation

$$\frac{1}{M^2}\frac{Q(\Omega)}{2K(\Omega)} - n\frac{Q(\omega)}{2K(\omega)} - \frac{i\pi\beta(\alpha+\beta\Omega)}{4K^2(\omega)} = -nk^2(1-k^2)\frac{1}{M}\frac{dM}{d.k^2},$$

we find

$$A_s\left(\frac{x}{M}, \lambda^2\right) = e^{-nk^2(1-k^2)\frac{1}{M}\frac{dM}{d.k^2}x^2} \times T_s,$$

where T_s is a homogeneous function of the order n of two of the functions $A(x, k^2)$.

In particular, if n is uneven and the transformation is primary, we have

$$A_s\left(\frac{x}{M}, \lambda^2\right) = e^{-nk^2(1-k^2)\frac{1}{M}\frac{dM}{d.k^2}x^2} \times A_0^n(x, k^2) \times U_s,$$

where U_s may be expressed in terms of the Abelian functions; *e. g.*

$$U_3 = \Delta\,\mathrm{am}x\,\Pi\left[1 - k^2\sin^2\mathrm{coam}\,4j\eta\,\sin^2\mathrm{am}x\right]$$
$$= \frac{A_3(x)}{A_0(x)}\,\Pi\left[1 - k^2\frac{A_2^2(4j\eta)}{A_3^2(4j\eta)}\,\frac{A_1^2(x)}{A_0^2(x)}\right].$$

For the linear transformations of the Abelian functions we find

I. If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}$, mod 2,

$$\lambda^2 = k^2\,\frac{1}{M} = (-1)^{\frac{1}{2}(a-1)},$$

$$A_s = A_0.$$

II. If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 0, & 1 \\ =1, & 0 \end{vmatrix}$, mod 2,

$$\lambda^2 = 1 - k^2,\ \frac{1}{M} = (-1)^{\frac{1}{2}(b-1)}\,i,$$

$$A_1(ix, 1-k^2) = iA_1(x, k^2),$$
$$A_2(ix, 1-k^2) = A_0(x, k^2),$$
$$A_3(ix, 1-k^2) = A_3(x, k^2),$$
$$A_0(ix, 1-k^2) = A_2(x, k^2).$$

These equations signify that the coefficients of x^{4n+1} and x^{4n} in A_1 and A_3 respectively are functions of $k^2(1-k^2)$, and the coefficients of x^{4n+3} and x^{4n+2} are functions of $k^2(1-k^2)$ multiplied by $1-2k^2$; while corresponding coefficients in

A_0 and A_2 are of the form $p \pm (1-2k^2)q$ or $\pm p + (1-2k^2)q$, according as the exponent of x is evenly or unevenly even.

III. If $\begin{vmatrix} a, b \\ c, d \end{vmatrix} \equiv \begin{vmatrix} 1, 0 \\ 1, 1 \end{vmatrix}$, mod 2, $\lambda^2 = \dfrac{k^2}{k^2-1}$,

$$\frac{1}{M} = k', \quad k^2(1-k^2)\frac{d\log M}{d.k^2} = \tfrac{1}{2}k^2.$$

$$A_1\left(k'x, \frac{k^2}{k^2-1}\right) = k' e^{-\frac{1}{2}k^2x^2} A_1(x, k^2),$$

$$A_2\left(k'x, \frac{k^2}{k^2-1}\right) = e^{-\frac{1}{2}k^2x^2} A_2(x, k^2),$$

$$A_3\left(k'x, \frac{k^2}{k^2-1}\right) = e^{-\frac{1}{2}k^2x^2} A_0(x, k^2),$$

$$A_0\left(k'x, \frac{k^2}{k^2-1}\right) = e^{-\frac{1}{2}k^2x^2} A_3(x, k^2).$$

IV. If $\begin{vmatrix} a, b \\ c, d \end{vmatrix} \equiv \begin{vmatrix} 1, 1 \\ 0, 1 \end{vmatrix}$, mod 2, $\lambda^2 = \dfrac{1}{k^2}$,

$$\frac{1}{M} = k, \quad k^2(1-k^2)\frac{d\log M}{d.k^2} = \tfrac{1}{2}(1-k^2),$$

$$A_1\left(kx, \frac{1}{k^2}\right) = k e^{-\frac{1}{2}k'^2x^2} A_1(x, k^2),$$

$$A_2\left(kx, \frac{1}{k^2}\right) = e^{-\frac{1}{2}k'^2x^2} A_3(x, k^2),$$

$$A_3\left(kx, \frac{1}{k^2}\right) = e^{-\frac{1}{2}k'^2x^2} A_2(x, k^2)$$

$$A_0\left(kx, \frac{1}{k^2}\right) = e^{-\frac{1}{2}k'^2x^2} A_0(x, k^2).$$

V. If $\begin{vmatrix} a, b \\ c, d \end{vmatrix} \equiv \begin{vmatrix} 1, 1 \\ -1, 0 \end{vmatrix}$, mod 2, $\lambda^2 = \dfrac{1}{1-k^2}$,

$$\frac{1}{M} = ik, \quad k^2(1-k^2)\frac{d\log M}{d.k^2} = \tfrac{1}{2}(1-k^2),$$

$$A_1\left(ikx, \frac{1}{1-k^2}\right) = ik e^{-\frac{1}{2}k'^2x^2} A_1(x, k^2),$$

$$A_2\left(ikx, \frac{1}{1-k^2}\right) = e^{-\frac{1}{2}k'^2x^2} A_0(x, k^2),$$

$$A_3\left(ikx, \frac{1}{1-k^2}\right) = e^{-\frac{1}{2}k'^2x^2} A_2(x, k^2),$$

$$A_0\left(ikx, \frac{1}{1-k^2}\right) = e^{-\frac{1}{2}k'^2x^2} A_3(x, k^2).$$

VI. If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 0, & 1 \\ -1, & -1 \end{vmatrix}$, mod 2, $\lambda^2 = 1 - \frac{1}{k^2}$,

$$\frac{1}{M} = ik', \quad k^2(1-k^2)\frac{d\log M}{d.k^2} = \tfrac{1}{2}k^2,$$

$$A_1\left(ik'x, 1-\frac{1}{k^2}\right) = ik'e^{-\frac{1}{2}k^2x^2}A_1(x, k^2),$$

$$A_2\left(ik'x, 1-\frac{1}{k^2}\right) = \quad e^{-\frac{1}{2}k^2x^2}A_3(x, k^2),$$

$$A_3\left(ik'x, 1-\frac{1}{k^2}\right) = \quad e^{-\frac{1}{2}k^2x^2}A_0(x, k^2),$$

$$A_0\left(ik'x, 1-\frac{1}{k^2}\right) = \quad e^{-\frac{1}{2}k^2x^2}A_2(x, k^2).$$

The functions $\Sigma = \frac{1}{\sqrt{M}}e^{hx^2}\bar{A}\left(\frac{x}{M}, \lambda^2\right)$ satisfy a partial differential equation with respect to x and k^2. We have from that equation, if $A = \bar{A}\left(\frac{x}{M}, \lambda^2\right)$,

$$M^2\frac{d^2A}{dx^2} + 4\lambda^2(1-\lambda^2)\left[\frac{dA}{d.\lambda^2}\right] - \frac{\lambda^2(1-\lambda^2)}{M^2}x^2A = 0,$$

where

$$\left[\frac{dA}{d.\lambda^2}\right] = \frac{dA}{d.\lambda^2} - Mx\frac{dA}{dx}\cdot\frac{d\left(\frac{1}{M}\right)}{d.\lambda^2}.$$

Observing that

$$\frac{\lambda^2(1-\lambda^2)}{M^2}\frac{d}{d.\lambda^2} = nk^2(1-k^2)\frac{d}{d.k^2},$$

we find

$$\frac{d^2A}{dx^2} + 4nk^2(1-k^2)\frac{1}{M}\frac{dM}{d.k^2}.x\frac{dA}{dx} + 4nk^2(1-k^2)\frac{dA}{d.k^2} - \frac{\lambda^2(1-\lambda^2)}{M^4}x^2A = 0;$$

or, substituting for A its value $\sqrt{M}.e^{-hx^2}\Sigma$, and dividing by $e^{-hx^2}\sqrt{M}$,

$$\frac{d^2\Sigma}{dx^2} + 4nk^2(1-k^2)\frac{d\Sigma}{d.k^2} - x^2\left[\frac{\lambda^2(1-\lambda^2)}{M^4} + 4h^2 + 4nk^2(1-k^2)\frac{dh}{d.k^2}\right]\Sigma = 0.$$

But the coefficient of $-x^2\Sigma$ in this equation is equal to $n^2k^2(1-k^2)$. Hence, we have finally

$$\frac{d^2\Sigma}{dx^2} + 4nk^2(1-k^2)\frac{d\Sigma}{d.k^2} - n^2k^2(1-k^2)x^2\Sigma = 0,$$

which is satisfied by the four functions

$$\sqrt{\frac{\lambda'}{M}}e^{hx^2}A_0\left(\frac{x}{M},\lambda^2\right),\quad \sqrt{\frac{\lambda\lambda'}{M}}e^{hx^2}A_1\left(\frac{x}{M},\lambda^2\right),$$

$$\sqrt{\frac{\lambda}{M}}e^{hx^2}A_2\left(\frac{x}{M},\lambda^2\right),\quad \sqrt{\frac{1}{M}}e^{hx^2}A_3\left(\frac{x}{M},\lambda^2\right).$$

The determinant of the transformation n may have any positive integral value, and the constituents of the matrix of transformation may have any integral values.

If we replace x by $\frac{x}{\sqrt{n}}$ and $\frac{1}{M}$ by $\frac{\sqrt{n}}{\mu}$, we see that the functions

$$\sqrt{\mu}\times\overline{A}_s\left(\frac{x}{\mu},\lambda^2\right)\times e^{k^2(1-k^2)\frac{d\log M}{d.k^2}x^2}$$

satisfy an equation of the same form as that satisfied by $\overline{A}_s(x, k^2)$.

We next proceed to form the equation satisfied by

$$\sigma=\Sigma\div\overline{A}^n(x, k^2).$$

Substituting for Σ and dividing by A^n, and observing that

$$\frac{d\Sigma}{dx}=A^n\frac{d\sigma}{dx}+nA^{n-1}\frac{dA}{dx}\sigma,$$

$$\frac{d^2\Sigma}{dx^2}=A^n\frac{d^2\sigma}{dx^2}+2nA^{n-1}\frac{dA}{dx}\frac{d\sigma}{dx}+n(n-1)A^{n-2}\left(\frac{dA}{dx}\right)^2\sigma+nA^{n-1}\frac{d^2A}{dx^2}\sigma,$$

$$\frac{d\Sigma}{d.k^2}=A^n\frac{d\sigma}{d.k^2}+nA^{n-1}\frac{dA}{d.k^2}\sigma,$$

we find

$$\frac{d^2\sigma}{dx^2}+2n\frac{1}{A}\frac{dA}{dx}\frac{d\sigma}{dx}+4nk^2(1-k^2)\frac{d\sigma}{d.k^2}$$

$$+n\sigma\left[(n-1)\left(\frac{1}{A}\frac{dA}{dx}\right)^2+\frac{1}{A}\frac{d^2A}{dx^2}+4nk^2(1-k^2)\frac{1}{A}\frac{dA}{d.k^2}-nk^2(1-k^2)x^2\right]=0;$$

or, after all reductions,

$$\frac{d^2\sigma}{dx^2}+2nZ_0\frac{d\sigma}{dx}+4nk^2(1-k^2)\frac{d\sigma}{d.k^2}-n(n-1)k^2\mathrm{cn}^2x.\sigma=0.$$

Lastly, if in this equation we introduce $s = \sin \operatorname{am} x$ and k^2 as the independent variables instead of x and k^2; we have

$$\frac{d\sigma}{dx} = \frac{d\sigma}{ds} \cdot \frac{ds}{dx} = cd \frac{d\sigma}{ds},$$

$$\frac{d^2\sigma}{dx^2} = c^2 d^2 \frac{d^2\sigma}{ds^2} - s \frac{d\sigma}{ds}(d^2 + k^2 c^2),$$

$$\left[\frac{d\sigma}{d.k^2}\right] = \frac{d\sigma}{d.k^2} + \frac{d\sigma}{ds}\frac{ds}{d.k^2} = \frac{d\sigma}{d.k^2} + \frac{d\sigma}{ds}\frac{1}{2k^2k'^2}[k^2sc^2 - cdZ_0];$$

where $\left[\frac{d\sigma}{d.k^2}\right]$ is the differential coefficient of σ taken on the supposition that σ is a function of the independent variables k^2 and x, and $\frac{d\sigma}{d.k^2}$ is the differential coefficient taken on the supposition that σ is a function of k^2 and s, and that k^2 does not vary in s, and we obtain finally

$$(1-s^2)(1-k^2s^2)\frac{d^2\sigma}{ds^2} + [(2n-1)k^2 - 1 - 2(n-1)k^2s^2]\, s\frac{d\sigma}{ds} - n(n-1)\, k^2(1-s^2)\sigma + 4nk^2(1-k^2)\frac{d\sigma}{d.k^2} = 0, \quad . \quad (\alpha)$$

in which equation the coefficient of $\frac{d\sigma}{ds}$ may be written in either of the two forms $(2n-1)k^2c^2 - d^2$ and $2(n-1)k^2c^2 - k'^2$.

The equation is satisfied by

$$\sqrt{\frac{\lambda'}{Mk'^n}}\, U_0, \quad \sqrt{\frac{\lambda\lambda'}{Mk'^n}}\, U_1, \quad \sqrt{\frac{\lambda}{Mk'^n}}\, U_2, \quad \sqrt{\frac{1}{Mk'^n}}\, U_3.$$

For our immediate purpose we add the term $n^2k^2\sigma$ to this equation in order to obtain the equation satisfied by

$$\sqrt{\frac{\lambda'}{M}}\, U_0, \quad \sqrt{\frac{\lambda\lambda'}{M}}\, U_1, \quad \sqrt{\frac{\lambda}{M}}\, U_2, \quad \sqrt{\frac{1}{M}}\, U_3.$$

There is some difficulty in applying the equation

$$\frac{d^2\Sigma}{dx^2} + 4nk^2(1-k^2)\frac{d\Sigma}{d.k^2} - n^2k^2(1-k^2)x^2\Sigma = 0$$

to the actual determination of the functions U_0, U_1, U_2, U_3. The following method serves to exhibit these functions in what may be termed their 'canonical form.'

Let the operating symbol

$$\frac{d^2}{dx^2} + 4nk^2(1-k^2)\frac{d}{d.k^2} - n^2k^2(1-k^2)x^2$$

of this equation be denoted by ∇; write also for brevity

$$4k^2(1-k^2)\frac{d}{d.k^2} = \delta,$$

and, considering first the three even functions $e^{hx^2}A_s\left(\frac{x}{M}, \lambda^2\right)$, let $\nu = \sqrt{\frac{\lambda'}{M}}$, $\sqrt{\frac{\lambda}{M}}$, $\sqrt{\frac{1}{M}}$ in the three cases $s=0$, 2, 3 respectively. We find, by expansion,

$$\sqrt{\frac{\lambda'}{M}}\, e^{hx^2}A_0\left(\frac{x}{M}, \lambda^2\right) = \sqrt{\frac{\lambda'}{M}}\left[1 + \left(2h + \frac{\lambda^2}{M^2}\right)\frac{x^2}{1.2} + \ldots\right],$$

$$\sqrt{\frac{\lambda}{M}}\, e^{hx^2}A_2\left(\frac{x}{M}, \lambda^2\right) = \sqrt{\frac{\lambda}{M}}\left[1 + \left(2h - \frac{1-\lambda^2}{M^2}\right)\frac{x^2}{1.2} + \ldots\right],$$

$$\sqrt{\frac{1}{M}}\, e^{hx^2}A_3\left(\frac{x}{M}, \lambda^2\right) = \sqrt{\frac{1}{M}}\left[1 + 2h\,\frac{x^2}{1.2} + \ldots\right];$$

so that we have in the three cases alike

$$\nu e^{hx^2}A\left(\frac{x}{M}, \lambda^2\right) = \nu - n\delta\nu\,\frac{x^2}{1.2} + \ldots.$$

Let
$$\phi_0 = 1 + \alpha_2\frac{x^4}{4!} - \alpha_3\frac{x^6}{6!} + \ldots$$

be a series satisfying the equation $\nabla\phi_0 = 0$, so that the coefficients α_2, α_3, ... are rational and integral functions of k^2 with integral coefficients; similarly, let

$$\phi_1 = \frac{x^2}{2!} + b_3\frac{x^6}{6!} - b_4\frac{x^8}{8!} + \ldots$$

be a function determined by the equation

$$\nabla\phi_1 = \phi_0;$$

let
$$\phi_2 = \frac{x^4}{4!} + c_4\frac{x^8}{8!} - c_5\frac{x^{10}}{10!} + \ldots$$

be determined by the equation
$$\nabla\phi_2 = \phi_1;$$

and so on continually. It will be found that these determinations are always possible; and that, as indicated in the case of the functions ϕ_0, ϕ_1, ϕ_2, the first s coefficients in the function ϕ_s are zero, and the values of the two following coefficients are unity and zero. All the coefficients are rational and integral in k^2, and the highest power of k^2 in the coefficient of $\frac{x^{2s+2s'}}{(2s+2s')!}$ in ϕ_s is s'. We then have the theorem

$$\nu e^{hx^2}A\left(\frac{x}{M}, \lambda^2\right) = \nu\phi_0 - n\delta\nu.\phi_1 + n^2\delta^2\nu.\phi_2 - n^3\delta^3\nu.\phi_3 + \ldots.$$

For if Φ represent the series on the right-hand side, we have, evidently,

$$\nabla\Phi = \nu\nabla\phi_0 - (\nabla\phi_1 - \phi_0)\, n\delta\nu + (\nabla\phi_2 - \phi_1)\, n^2\delta^2\nu - \dots ;$$

i.e.
$$\nabla\Phi = 0 \text{ ; also } \Phi = \nu - n\delta\nu \,.\, \frac{x^2}{1.2} + \dots ;$$

so that the series Φ satisfies the same partial differential equation as the function $\nu e^{hx^2} A\left(\frac{x}{M}, \lambda^2\right)$, while the first two terms of the series coincide with the first two terms of the expansion of the function; and this establishes the theorem, because, given the first two terms of the expansion, it can be continued by means of the partial differential equation in one way only.

We next denote the operating factor

$$\frac{d^2}{dx^2} + 2nZ_0(x)\frac{d}{dx} + 4nk^2(1-k^2)\frac{d}{d.k^2} - n(n-1)k^2\,\mathrm{cn}^2 x + n^2k^2$$

by D; and we observe that we have not only

$$D\left[\frac{\Phi}{A_0^n(x, k^2)}\right] = 0,$$

but also, separately,

$$D\left[\frac{\phi_0}{A_0^n}\right] = 0, \quad D\left[\frac{\phi_1}{A_0^n}\right] = \frac{\phi_0}{A_0^n}, \quad D\left[\frac{\phi_2}{A_0^n}\right] = \frac{\phi_1}{A_0^n}, \dots ;$$

viz. each of these equations is derived from the corresponding equation $\nabla\phi_s = \phi_{s-1}$ precisely in the same way in which the equation $D\sigma = 0$ is derived from the equation $\nabla\Sigma = 0$.

We now expand the functions $\frac{\phi_r}{A_0^n}$ in series proceeding by powers of s; we observe that if $\psi_r(s)$ be this expansion, it follows from the properties of the functions ϕ_r that the first term in $\psi_r(s)$ is $\frac{s^{2r}}{2r!}$, and that

$$\psi_r(s) = \frac{s^{2r}}{(2r)!} + \rho_1\frac{s^{2r+2}}{(2r+2)!} + \rho_2\frac{s^{2r+4}}{(2r+4)!} + \dots,$$

where ρ_m is a rational and integral function of k^2 with integral coefficients of an order not exceeding m in k^2; the coefficient ρ is however not zero, but

$$-nk^2\frac{(2r+1)(2r+2)}{1.2} - \frac{2r(2r+1)(2r+2)}{1.2.3}(1+k^2)$$
$$= -\frac{(2r+1)(2r+2)}{1.2}\left[nk^2 + \tfrac{2}{3}r(1+k^2)\right].$$

We thus obtain for U_0 the expression

$$\nu U_0 = \nu\psi_0(s) + n\delta\nu \,.\, \psi_1(s) + n^2\delta^2\nu \,.\, \psi_2(s) + \ldots.$$

But U_0 is a rational and integral function of order $n-1$. Hence we may omit in $\psi_2(s)$ all powers of s above s^{n-1}, and consequently all the functions $\delta^2\nu \,.\, \psi_2(s)$ after $\delta^{\frac{1}{2}(n-1)}\nu \,.\, \psi_{\frac{1}{2}(n-1)}(s)$; the higher powers of s disappearing of themselves. And if we denote by I the operating factor in the left-hand member of the equation (a), p. 401, increased by n^2k^2, viz.

$$I = (1-s^2)(1-k^2s^2)\frac{d^2}{ds^2} + [(2n-1)k^2 - 1 - 2(n-1)k^2s^2]s\frac{d}{ds} + nk^2[1+(n-1)s^2] + 4nk^2(1-k^2)\frac{d}{dk^2},$$

we have the equations

$$I\psi_0(s) = 0, \quad I\psi_1(s) = \psi_0(s), \quad I\psi_2(s) = \psi_1(s), \ldots,$$

by which the functions $\psi_0(s)$, $\psi_1(s)$, ... may be successively calculated; viz. if

$$\psi_r(s) = C_r\frac{s^{2r}}{(2r)} + C_{r+1}\frac{x^{2r+2}}{(2r+2)!} + \ldots,$$

we have

$$C_{\sigma+1} - [4\sigma^2 + (4\sigma^2 - 4n\sigma - n)k^2]C_\sigma + 2\sigma(2\sigma-1)(2\sigma-n-1)(2\sigma-n-2)k^2C_{\sigma-1} + n\delta C_\sigma = B_\sigma,$$

if B_σ be the coefficient of $\frac{x^{2\sigma}}{2\sigma!}$ in $\psi_{r-1}(s)$.

IX. *On Modular Curves**.

Let $[\alpha, \beta, \gamma]$ or ω be a given quadratic point, and let it be required to assign all the modular curves of any species passing through the point

$$X + iY = -\tfrac{1}{2} + \Phi(\omega).$$

Let $\left|\begin{matrix} A, B, C \\ A', B', C' \end{matrix}\right|$ be a pair of relatively prime solutions of the equation

$$\gamma A - 2\beta B + \alpha C = 0,$$

so that

$$|\gamma, -2\beta, \alpha| = \left|\begin{matrix} A, B, C \\ A', B', C' \end{matrix}\right|,$$

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* [On the back of the last page of the manuscript of this Note Professor Smith has written :— 'These papers relate to the problem, &c., "Given a quadratic point, to find all the modular curves passing through it," &c., &c. It ought to be worked into a memoir "On the ordinary multiple points of a modular curve; and on the intersections of two modular curves."']

and so that all the solutions of this equation are comprised in the formula

$$\lambda' A - \lambda A', \quad \lambda' B - \lambda B', \quad \lambda' C - \lambda C'.$$

The determinants $B^2 - AC = D$, $B'^2 - A'C' = D'$ are positive, because $\beta^2 - \alpha\gamma = -\Delta$ is negative, and two pairs of conjugate imaginary points cannot be harmonic. Hence the symbols $[A, B, C]$, $[A', B', C']$ represent real semicircles passing through the point ω. The determinants of all semicircles passing through ω are included in the formula

$$D'\lambda^2 - 2J\lambda\lambda' + D\lambda^2,$$

where $2J$ is the invariant $2BB' - AC' - A'C$, and where $J^2 - DD' = -\Delta$. The form $(D', -J, D)$ is the duplicate of the form (α, β, γ), and is transformed into the product of that form by itself by means of the bipartite linear substitution

$$\begin{vmatrix} A, B, B, C \\ A', B', B', C' \end{vmatrix}.$$

We thus obtain the theorem:

'The modular curves of order D pass through the point $[\alpha, \beta, \gamma]$ as often as there are primitive representations of D by the duplicate of the form (α, β, γ).'

To determine (by means of the corresponding semicircles) the ovals which pass through a given point $[\alpha, \beta, \gamma]$ we have the following rule:

'Let (P, Q, R) be the duplicate of (α, β, γ), and let

$$\begin{vmatrix} p_1, q_1, q_1, r_1 \\ p_2, q_2, q_2, r_2 \end{vmatrix}$$

be the substitution transforming (P, Q, R) into $(\alpha, \beta, \gamma)^2$; let also

$$(P, Q, R) \times (\mu_1, \mu_2)^2 = D$$

be the given representation of D; the semicircle

$$[\mu_1 p_2 - \mu_2 p_1, \ \mu_1 q_2 - \mu_2 q_1, \ \mu_1 r_2 - \mu_2 r_1]$$

of determinant D passes through $[\alpha, \beta, \gamma]$.'

To determine which of the modular curves of order D passes through $[\alpha, \beta, \gamma]$ we should have to distinguish the cases in which $\alpha, \beta, \gamma, \mu_1, \mu_2$ have different congruential values for the modulus 2. This discussion, for brevity, we omit.

If we apply to the form $(D', -J, D)$ the reducing substitution of Lagrange, we obtain the orders of the two lowest modular curves, which can pass through the point $-\frac{1}{2} + \Phi(\omega)$, or P. If on the tangents to these two curves at P we measure lengths PT, PT' equal to the square roots of the corresponding determinants, and if, completing the parallelogram $PTP'T'$, we construct the

parallelism of which $PTP'T'$ is an elementary parallelogram, the tangents to the modular curves passing through P are the lines joining P to the nodes of this parallelism; and, if N be one of these nodes such that PN contains no other node besides P and N, PN is the square root of the determinant of the modular curve touching PN at P.

Gauss has shown that every class of the principal genus of any determinant is the duplicate of as many classes as there are ambiguous classes; *i.e.* classes producing the principal class by duplication. Thus no point of determinant $-\Delta$ can lie on a modular curve of order D unless D is represented by some class of the principal genus of determinant $-\Delta$; and, if there be such representations, there are, corresponding to each of them, as many points of determinant $-\Delta$ lying on the modular curves of order D, as there are sub-classes of determinant $-\Delta$ producing the principal class by duplication.

The necessary and sufficient condition that a given point $[\alpha, \beta, \gamma]$ should lie simultaneously on a modular curve of order D, and also on a modular curve of order D', is that D and D' should both be capable of primitive representation by the duplicate of $[\alpha, \beta, \gamma]$. Let

$$D = (P, Q, R) \times (\mu_1, \mu_2)^2, \quad D' = (P, Q, R) \times (\nu_1, \nu_2)^2$$

be the two representations; let $\begin{vmatrix} \mu_1, & \nu_1 \\ \mu_2, & \nu_2 \end{vmatrix} = v$, and let (P, Q, R) be transformed by $\begin{vmatrix} \mu_1, & \nu_1 \\ \mu_2, & \nu_2 \end{vmatrix}$ into $(D, -J, D')$ of determinant Δv^2. The form $(D, -J, D')$ is then transformed by $\begin{vmatrix} \nu_2, & -\nu_1 \\ -\mu_2, & \mu_1 \end{vmatrix}$ into $v^2 \times (P, Q, R)$; that is, it is transformed by the bipartite transformation

$$\begin{vmatrix} \nu_2, & -\nu_1 \\ -\mu_2, & \mu_1 \end{vmatrix} \times \begin{vmatrix} p_1, & q_1, & q_1, & r_1 \\ p_2, & q_2, & q_2, & r_2 \end{vmatrix}$$

into $v^2 \times (\alpha, \beta, \gamma)^2$. The semicircles

$$[\nu_2 p_1 - \nu_1 p_2, \ \nu_2 q_1 - \nu_1 q_2, \ \nu_2 r_1 - \nu_1 r_2],$$
$$[\mu_2 p_1 - \mu_1 p_2, \ \mu_2 q_1 - \mu_1 q_2, \ \mu_2 r_1 - \mu_1 r_2],$$

which are of determinants D', D, and which have J for their harmonic invariant and $v \times [\alpha, \beta, \gamma]$ for their covariant, are the two semicircles corresponding to the given representation and intersecting at the point $[\alpha, \beta, \gamma]$.

The equation $DD' = J^2 + v^2\Delta$ always subsists (as the preceding analysis implies) whenever any point of determinant Δ lies on two modular curves of the orders D and D'. But this condition, though necessary, is not sufficient; viz., confining ourselves to the case in which D and D' are relatively prime, J and

v must be relatively prime, and the numbers D, D' must have each the characters of the principal genus of determinant $-\Delta$; when these additional conditions are satisfied, D and D' are necessarily represented by the same form of the principal genus; and the two sets of curves have as many points of determinant $-\Delta$ in common as there are ambiguous classes of determinant Δ.

Let $[\alpha_1, \beta_1, \gamma_1]$ and $[\alpha_2, \beta_2, \gamma_2]$ be two quadratic points ω_1, ω_2, of which the determinants are to one another as two squares; let $\Delta_1 = \Delta'\theta_1^2$, $\Delta_2 = \Delta'\theta_2^2$, Δ' being the greatest common divisor of Δ_1 and Δ_2; and let it be required to assign all the modular curves with regard to which $-\frac{1}{2}+\Phi(\omega_1)$ and $-\frac{1}{2}+\Phi(\omega_2)$ are inverse. Let (P, Q, R) be a form of determinant Δ' compounded of $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$. If $-\frac{1}{2}+\Phi(\omega_1)$ and $-\frac{1}{2}+\Phi(\omega_2)$ are inverse with regard to any modular curve of determinant D, D is divisible by $\theta_1\theta_2$, and the quotient is capable of primitive representation by (P, Q, R). Conversely, if these conditions are satisfied, a modular curve of determinant D exists with regard to which the two points are inverse. Let $(P, Q, R) \times (\mu_1, \mu_2)^2 = \dfrac{D}{\theta_1\theta_2}$; and let

$$\begin{vmatrix} p_1, & q_1, & q_1', & r_1 \\ p_2, & q_2, & q_2', & r_2 \end{vmatrix}$$

be the substitution transforming (P, Q, R) into

$$(\alpha_1, \beta_1, \gamma_1) \times (\alpha_2, \beta_2, \gamma_2);$$

the points $\omega_1 = x_1 + iy_1$, $\omega_2 = x_2 + iy_2$ will satisfy the equation

$$x_2 + iy_2 = \begin{vmatrix} \mu_1 q_2' & -\mu_2 q_1', & \mu_1 r_2 - \mu_2 r_1 \\ \mu_1 p_2 & -\mu_2 p_1, & \mu_1 q_2 - \mu_2 q_1 \end{vmatrix} \times (x_1 + iy_1),$$

the determinant of the matrix being D; *i.e.* $-\frac{1}{2}+\Phi(\omega_1)$ and $-\frac{1}{2}+\Phi(\omega_2)$ are inverse with regard to a modular curve of determinant D. If D and D' are two uneven numbers relatively prime, the necessary and sufficient condition that two given points should be inverse with regard to modular curves appertaining to each of those determinants is that the two points should have the same determinant Δ, and that D and D' should be primitively represented by the same class of determinant Δ; or, which is the same thing, that $D \times D'$ should be represented by the principal class of determinant Δ. If this condition be satisfied, every point of determinant Δ is inverse to another point of that determinant with regard to a modular curve (D) and also with regard to a modular curve (D'). If the class by which D and D' are represented is not a class of the principal genus, all the points of intersection thus obtained are

imaginary, no two inverse points coinciding. But if D and D' are represented by a class Γ of the principal genus, the classes which by their duplication produce the class Γ give real points of intersection.

X. *On the Quarter Periods* $K, \frac{1}{2}iK'$.

The fundamental pair of quarter periods $(K, \frac{1}{2}iK')$ is not in general the absolutely least pair of quarter periods appertaining to the doubly periodic function; for it does not follow that the parallelism $(K, \frac{1}{2}iK')$ is absolutely reduced because the parallelism (K, iK') is primarily reduced.

Problem. To determine, for any given value of k^2, the absolutely reduced parallelism equivalent to (K, iK').

From the preceding discussion it appears that the reduced triangle is formed by the vectors

$$K,\ iK',\quad -K-iK',$$

or

$$K-iK',\quad iK',\ -K,$$

according as the angle from K to iK' is obtuse or acute. It may be proved by the considerations already employed in the demonstration of the theorem (...)*, or by a method presently to be explained, that this angle is obtuse or acute according as the coefficient of i in k^2 is negative or positive.

We have now to show how, for any given value of k^2, the vectors of the reduced triangle can be arranged in order of magnitude. For this purpose we employ the principles contained in a Memoir 'Sur les équations modulaires,' which will be found in the Transactions of the 'Accademia dei Lincei,' vol. i. Ser. iii. (1877)†; it will suffice to consider only one of the propositions to be demonstrated. We understand by $[z]$ the absolute value, *i.e.* the analytical modulus, of any complex quantity z.

'The inequalities

$$[K]<[K\pm iK']$$

subsist simultaneously, if $[k^2]<1$: if $[k^2]>1$, we have

$$[K-iK']<[K]<[K+iK'],$$

or

$$[K+iK']<[K]<[K-iK'],$$

according as the coefficient of i in the imaginary part of k^2 is positive or negative.'

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* [Blanks in the manuscript are denoted by dots enclosed in parentheses, thus (...).]

† [Vol. ii. p. 224].

Let $\frac{iK'}{K} = X + iY$; the inequalities $[K] \leqq [K+iK']$ are equivalent to the inequalities $0 \leqq 2X + X^2 + Y^2$. But the quadrant of the circle $X^2 + Y^2 + 2X = 0$, which lies within the reduced space defined by the inequalities in question, and which runs from the cusp at 0 to the point $-1+i$, is represented in the plane $k^2 = \frac{1}{2} + x + iy$ by a semicircle of radius 1, running below the axis of x from the point $x = +\frac{1}{2}$, to the point $x = -\frac{3}{2}$; and the two regions containing the points -1 and $+1$ respectively, into which the reduced space is divided by the quadrant are represented in the plane of xy by the two regions, containing the points infinitely far off on the axis of y in the negative and positive directions respectively, into which that plane is divided by the axis of x from $+\infty$ to $\frac{1}{2}$, by the semicircle, and by the axis of x from $-\frac{3}{2}$ to $-\infty$. Hence according as $k^2 - \frac{1}{2}$ lies in the first or second of these regions or on the boundary between them, we have

$$[K] > [K+iK'], \text{ or } [K] < [K+iK'], \text{ or } [K] = [K+iK'].$$

Similarly, if we divide the plane xy into two regions, one lying above the axis of x and external to the circle

$$(x+\tfrac{1}{2})^2 + y^2 = 1,$$

the other comprising the rest of the plane, it will be found that according as $k^2 - \frac{1}{2}$ lies in the first or second of these regions, or on the boundary between them, we have

$$[K] > [K-iK'], \text{ or } [K] < [K-iK'], \text{ or } [K] = [K-iK'].$$

These two results taken together are equivalent to the proposition which we have enunciated.

To complete the solution of the problem we have to discuss the inequalities $[iK'] \gtrless [K \pm iK']$; their theory depends on the representation of the lines $2X \pm 1 = 0$, or, rather, of those portions of them which lie within the reduced space, by the semicircles $(x - \frac{1}{2})^2 + y^2 = 1$.

The final result is perhaps most simply expressed in the following form: The plane xy is divided by the axes and by the circles $(x \pm \frac{1}{2})^2 + y^2 = 1$ into twelve regions. The regions within both circles are designated by A, those within one circle only by B, those outside both by C; the four regions A are numbered 1, 2, 3, 4 according to the quadrants in which they lie; the regions B and C are similarly distinguished (see fig. 1). The Table indicates the arrangement of the four periods K, iK', $K+iK'$, $K-iK'$, in ascending order of

absolute magnitude, according to the region in which the vector point $x+iy=k^2-\frac{1}{2}$ is situated.

A.

(1) iK', K, $K-iK'$, $K+iK'$;
(2) K, iK', $K-iK'$, $K+iK'$;
(3) K, iK', $K+iK'$, $K-iK'$;
(4) iK', K, $K+iK'$, $K-iK'$.

B.

(1) iK', $K-iK'$, K, $K+iK'$;
(2) K, $K-iK'$, iK', $K+iK'$;
(3) K, $K+iK'$, iK', $K-iK'$;
(4) iK', $K+iK'$, K, $K-iK'$.

C.

(1) $K-iK'$, iK', K, $K+iK'$;
(2) $K-iK'$, K, iK', $K+iK'$;
(3) $K+iK'$, K, iK', $K-iK'$;
(4) $K+iK'$, iK', K, $K-iK'$.

It will be seen that along the axis of x,

$$[K-iK']=[K+iK'];$$

along the axis of y, $$[iK']=[K];$$

along the upper and lower semicircles of $(y+\frac{1}{2})^2+y^2=1$,

$$[K]=[K-iK'],\quad [K]=[K+iK']\quad \text{respectively};$$

along the upper and lower semicircles of $(x-\frac{1}{2})^2+y^2=1$,

$$[iK']=[K-iK'],\quad [iK']=[K+iK']\quad \text{respectively};$$

and that when we traverse any one of these six boundaries the quantities, which on it are equal, change places with one another.

Cor. 1. The zeros of the Theta functions

$$\vartheta_1\left(\frac{\pi x}{2K}\right),\quad \vartheta_2\left(\frac{\pi x}{2K}\right),\quad \vartheta_3\left(\frac{\pi x}{2K}\right),\quad \vartheta_0\left(\frac{\pi x}{2K}\right),$$

being respectively $2rK+2siK'$, $(2r+1)K+2siK'$,

$$(2r+1)K+(2s+1)iK', \quad 2rK+(2s+1)iK',$$

it follows that the least zeros of $\vartheta_2\left(\frac{\pi x}{2K}\right)$ are always $\pm K$, and the least zeros of $\vartheta_0\left(\frac{\pi x}{2K}\right)$ are always $\pm iK'$. But the least zeros of $\vartheta_3\left(\frac{\pi x}{2K}\right)$ are $\pm(K+iK')$ or $\pm(K-iK')$, according as the coefficient of i in the imaginary part of k^2 is negative or positive; and the least zeros of $\vartheta_1\left(\frac{\pi x}{2K}\right)$ are $\pm 2K$, $\pm 2iK'$, $\pm 2(K-iK')$, $\pm 2(K+iK')$, according as $k^2-\frac{1}{2}$ lies (1) inside the circle $(x+\frac{1}{2})^2+y^2=1$ to the left of the axis of y, (2) inside the circle $(y-\frac{1}{2})^2+y^2=1$ to the right of the axis of y, (3) outside the two circles and above the axis of x, (4) outside the two circles and below the axis of x.

These determinations assign in all cases the circles of convergence of the developments of $\frac{\sin \operatorname{am} u}{u}$, $\cos \operatorname{am} u$, $\Delta \operatorname{am} u$,

and their reciprocals, in series proceeding by powers of u.

Cor. 2. The Table also assigns in every case the absolutely least pair of conjugate periods of the functions $\sin^2 \operatorname{am} u$, $\cos^2 \operatorname{am} u$, $\Delta^2 \operatorname{am} u$; viz. this pair consists of the least and least but one of the four quantities $2K$, $2iK'$, $2(K \pm iK')$. But to obtain the absolutely least pairs of periods appertaining to the functions $\sin \operatorname{am} u$, $\cos \operatorname{am} u$, and $\Delta \operatorname{am} u$ themselves, we have to consider the modular curves of the square determinant 4, instead of the lines and circles of determinant $+1$.

Problem. To determine, for any given value of k^2, the absolutely reduced parallelisms equivalent to $(K, \frac{1}{2}iK')$, $(\frac{1}{2}[K-iK'], \frac{1}{2}[K+iK'])$, and $(\frac{1}{2}K, iK')$.

Since the reduced triangle of the parallelism (K, iK') is acute-angled, one of the two triangles into which it is divided by the line joining any of its vertices to the middle point of the opposite side is certainly acute-angled, and the half side is always less than the bisecting line. Hence the reduced triangle of the parallelism $(K, \frac{1}{2}K')$ is one of the four

$$K, \quad \tfrac{1}{2}iK', \quad -K-\tfrac{1}{2}iK', \quad \ldots\ldots\ldots \quad (1)$$

$$K+\tfrac{1}{2}iK', \quad \tfrac{1}{2}iK', \quad -K-iK', \quad \ldots\ldots\ldots \quad (2)$$

$$K-\tfrac{1}{2}iK', \quad \tfrac{1}{2}iK', \quad -K, \quad \ldots\ldots\ldots \quad (3)$$

$$K-iK', \quad \tfrac{1}{2}iK', \quad -K+\tfrac{1}{2}iK', \quad \ldots\ldots\ldots \quad (4)$$

in each of which the side $\frac{1}{2}iK'$ is less than the side $K \pm \frac{1}{2}iK'$. In fact, we have the case (1), (2), (3), or (4), according as

$$(1) \quad \text{amplitude of } \frac{iK'}{K} > \tfrac{1}{2}\pi; \quad [K] < [K+iK'];$$

$$(2) \quad \text{amplitude of } \frac{iK'}{K} > \tfrac{1}{2}\pi; \quad [K] > [K+iK'];$$

$$(3) \quad \text{amplitude of } \frac{iK'}{K} < \tfrac{1}{2}\pi; \quad [K] < [K-iK'];$$

$$(4) \quad \text{amplitude of } \frac{iK'}{K} < \tfrac{1}{2}\ ; \quad [K] > [K-iK'].$$

These inequalities have been already examined; in addition to these we have to consider the inequalities

$$[K] \lesseqgtr [\tfrac{1}{2}iK']; \qquad [K] \lesseqgtr [K \pm iK'];$$
$$[K \pm iK] \lesseqgtr [K \pm iK']; \quad [K \pm iK] \lesseqgtr [\tfrac{1}{2}iK'];$$

which determine the order of magnitude of the sides in the four triangles; and which are equivalent to the following:

$$X^2 + Y^2 \lesseqgtr 4; \qquad X^2 + Y^2 \pm 4X \gtreqless 0;$$
$$3(X^2 + Y^2) \pm 4X \lesseqgtr 0; \quad 3(X^2 + Y^2) \pm X + 4 \lesseqgtr 0.$$

In the plane $k^2 = \frac{1}{2} + x + iy$, the circle $X^2 + Y^2 = 4$, and the three pairs of circles

$$X^2 + Y^2 \pm 4X = 0, \quad 3(X^2 + Y^2) \pm 4X = 0, \quad 3(X^2 + Y^2) \pm 8X + 4 = 0,$$

are represented by the four loops of the modular curve (...), beginning with the innermost and proceeding in order to the outermost. If we designate the regions (taken in the same order from within outwards) into which the plane is divided by the curve by A, B, C, D, E, the Table gives the three least quarter periods corresponding to a value of $k^2 - \frac{1}{2}$ lying within any given region:

$$A.$$
$$K, \tfrac{1}{2}iK'; \quad K \pm \tfrac{1}{2}iK'.$$

$$B.$$
$$-\tfrac{1}{2}iK', K; \quad K \pm \tfrac{1}{2}iK'$$

$$C_1.$$
$$-\tfrac{1}{2}iK', K \pm \tfrac{1}{2}iK'; \quad K.$$

C_2.

$$-\tfrac{1}{2}iK',\ K\pm\tfrac{1}{2}iK';\quad K\pm iK'.$$

D.

$$-\tfrac{1}{2}iK',\ K\pm iK';\quad K\pm\tfrac{1}{2}iK'.$$

E.

$$K\pm iK',\ \tfrac{1}{2}iK';\quad K\pm\tfrac{1}{2}iK'.$$

The upper signs are to be taken in the region below the axis of x, and *vice versâ*. The region C is supposed to be divided into two, C_1 and C_2, the first within, the second without the circle $(x+\frac{1}{2})^2+y^2=1$.

To obtain the reduced parallelism equivalent to $(\frac{1}{2}K,\ iK')$ we have only to take the modular curve (...) which is symmetrical to (...) with respect to the axis of y, and to interchange K and K', dividing at the same time by i. We thus obtain the following Table:

A.

$$-iK',\ \tfrac{1}{2}K;\quad iK'\pm\tfrac{1}{2}K.$$

B.

$$\tfrac{1}{2}K,\ iK';\quad iK'\pm\tfrac{1}{2}K.$$

C_1.

$$\tfrac{1}{2}K,\ iK'\pm\tfrac{1}{2}K;\quad iK.$$

C_2.

$$\tfrac{1}{2}K,\ iK'\pm\tfrac{1}{2}K;\quad iK'\pm K.$$

D.

$$\tfrac{1}{2}K,\ iK'\pm K;\quad iK'\pm\tfrac{1}{2}K.$$

E.

$$iK'\pm K,\ \tfrac{1}{2}K;\quad iK'\pm\tfrac{1}{2}K.$$

The reduction of the parallelism $(\frac{1}{2}[K-iK'],\ \frac{1}{2}[K+iK'])$ depends on the remaining modular curve of determinant 4, ... which is a symmetrical with regard to both axes. The reduced triangle of this parallelism always has the vectors $\frac{1}{2}(K\pm iK')$ for two of its sides, and either K or iK' for its third side: in this enunciation, which is obtained by bisecting the side $K\pm iK'$ of the reduced triangle of the parallelism $(K,\ iK')$, the signs of the vectors are neglected.

Therefore, besides the inequalities $[K] \gtrless [iK']$, $[K+iK'] \gtrless [K-iK']$, which we have already considered, we have to examine the inequalities

$$[K] \gtrless \tfrac{1}{2}[K \pm iK']; \quad [iK'] \gtrless \tfrac{1}{2}[K \pm iK'];$$

or, which is the same thing, the inequalities

$$(X \pm 1)^2 + Y^2 \lesseqgtr 4; \quad (X \pm 1)^2 + Y^2 \lesseqgtr 4(X^2 + Y^2).$$

The four circles

$$X^2 + Y^2 \pm 2X - 3 = 0; \quad 3(X^2 + Y^2) \pm 2X - 1 = 0;$$

or rather the arcs of those circles which lie within the reduced space are represented in the plane $k^2 - \frac{1}{2} = x + iy$ by the curve (...), of which the form may be roughly compared to that of a hyperbola having an internal loop at each vertex. We designate by A the central infinite region between the two branches of the curve; by B and B' the infinite regions internal to the two branches of the curve on the right and left of the axis of y respectively; by C and C' the regions internal to the loops: we then have the following scheme, the upper signs being taken in the upper part of the plane, and *vice versâ*:

A.

$\frac{1}{2}(K \pm iK'), \frac{1}{2}(\mp K + iK'); \quad K, iK.$

A'.

$\frac{1}{2}(K \pm iK'), \frac{1}{2}(\mp K + iK'); \quad iK', K.$

B.

$\frac{1}{2}(K \pm iK'), \mp K; \quad \frac{1}{2}(K \mp iK'), iK.$

B'.

$\frac{1}{2}(K \pm iK'), iK'; \quad \frac{1}{2}(K \mp iK'), K.$

C.

$K, \frac{1}{2}(\pm K + iK'); \quad \frac{1}{2}(K \mp iK'), iK'.$

C'.

$-iK', \frac{1}{2}(K \pm iK'); \quad \frac{1}{2}(K \mp iK'), K.$

XLIII.

MEMOIR ON THE THETA AND OMEGA FUNCTIONS.

Arts. 1–14. Definitions and Elementary Properties of the Theta, Omega, and Elliptic Functions.

Art. 1.] *THE Theta Functions.*

An exponential series of the type

$$\sum_{m=-\infty}^{m=+\infty} e^{am^2+2bm+c} \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

is termed a Theta series. The necessary and sufficient condition for the convergence of the Theta series is that the real part of the coefficient a shall be different from zero, and negative; subject to this restriction, the coefficients a, b, c, may be any quantities whatever, real or complex.

For the purposes of this Memoir it is convenient to employ a notation for the Theta functions somewhat different from that which has been adopted in the Tables.* Writing $q = e^{i\pi\omega}$, we define the function $\vartheta_{\mu,\nu}(x, q)$ by the equation

$$\left.\begin{aligned} \vartheta_{\mu,\nu}(x,q) &= i^{\mu\nu}\sum_{m=-\infty}^{m=+\infty}(-)^{m\nu}q^{\frac{1}{4}(2m+\mu)^2}e^{(2m+\mu)ix} \\ &= \sum_{m=-\infty}^{m=+\infty} e^{i\pi\left[\frac{\omega}{4}(2m+\mu)^2+\frac{1}{2}(2m+\mu)\left(2\frac{x}{\pi}+\nu\right)\right]} \end{aligned}\right\} \quad \ldots\ldots \quad \text{(ii)}$$

μ and ν denoting any positive or negative integral numbers. We thus have

$$\left.\begin{aligned} \vartheta_{0,1}(x,q) &= \Theta\left(\frac{2Kx}{\pi}\right), \\ \vartheta_{1,-1}(x,q) &= \sqrt{k}\,\Theta_1\left(\frac{2Kx}{\pi}\right), \\ \vartheta_{1,0}(x,q) &= \sqrt{\frac{k}{k'}}\,\Theta_2\left(\frac{2Kx}{\pi}\right), \\ \vartheta_{0,0}(x,q) &= \frac{1}{\sqrt{k'}}\,\Theta_3\left(\frac{2Kx}{\pi}\right). \end{aligned}\right\} \quad \ldots\ldots\ldots \quad \text{(iii)}$$

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* [This Memoir was written to accompany the Tables of the Theta Functions calculated by Mr. J. W. L. Glaisher.]

For brevity, we shall often write

$$\vartheta_{0,1}(x,q) = \vartheta_0(x,q), \qquad \vartheta_{1,-1}(x,q) = \vartheta_1(x,q),$$
$$\vartheta_{1,0}(x,q) = \vartheta_2(x,q), \qquad \vartheta_{0,0}(x,q) = \vartheta_3(x,q);$$

we shall also omit the second argument q, when no ambiguity is likely to arise from this abbreviation.

From the equation of definition (ii.) we infer immediately

$$\vartheta_{\mu+2,\nu}(x) = \vartheta_{\mu,\nu}(x), \quad \text{(iv)}$$
$$\vartheta_{\mu,\nu+2}(x) = (-)^{\mu}\vartheta_{\mu,\nu}(x), \quad \text{(v)}$$
$$\vartheta_{\mu,\nu}(-x) = (-)^{\mu\nu}\vartheta_{\mu,\nu}(x), \quad \text{(vi)}$$
$$\vartheta_{\mu,\nu}(x+\pi) = (-)^{\mu}\vartheta_{\mu,\nu}(x), \quad \text{(vii)}$$
$$\vartheta_{\mu,\nu}(x+\omega\pi) = (-)^{\nu}e^{-i\pi\left(\frac{x}{\pi}+\omega\right)}\vartheta_{\mu,\nu}(x), \quad \text{(viii)}$$
$$\vartheta_{\mu+\mu',\nu+\nu'}(x) = e^{\mu' i\pi\left(\frac{x}{\pi}+\frac{1}{2}\mu'\omega+\frac{\nu+\nu'}{2}\right)}\vartheta_{\mu,\nu}(x+\tfrac{1}{2}\pi[\mu'\omega+\nu']). \quad \text{(ix)}$$

Thus there are only four distinct Theta functions:

$$\vartheta_{0,1}(x) = \vartheta_0(x), \qquad \vartheta_{1,-1}(x) = \vartheta_1(x) = -\vartheta_{1,1}(x),$$
$$\vartheta_{1,0}(x) = \vartheta_2(x), \qquad \vartheta_{0,0}(x) = \vartheta_3(x)$$

(equations iv and v). Of these, $\vartheta_1(x)$ is an uneven function, while the other three are even (equation vi). The Theta functions are singly periodic, having π or 2π for their period according as μ is even or uneven (equation vii); the quotient of any one of them divided by any other is doubly periodic, the periods of $\vartheta_{\mu,\nu}(x) \div \vartheta_{\mu',\nu'}(x)$ being (1) π, $2\omega\pi$, (2) $\pi+\omega\pi$, $\pi-\omega\pi$, (3) 2π, $\omega\pi$, in the three cases (1) $\mu-\mu'$ even, $\nu-\nu'$ uneven, (2) $\mu-\mu$ uneven, $\nu-\nu'$ uneven, (3) $\mu-\mu'$ uneven, $\nu-\nu'$ even (equations vii and viii); lastly, any one of the four can be expressed as a product of any other by an exponential factor (equation ix); so that, in particular,

$$\left.\begin{aligned}
\vartheta_1(x) &= e^{i\pi\left(\frac{x}{\pi}+\frac{1}{2}\omega-\frac{1}{2}\right)}\vartheta_3(x+\tfrac{1}{2}\pi(\omega-1)),\\
\vartheta_0(x) &= \vartheta_3(x+\tfrac{1}{2}\pi),\\
\vartheta_2(x) &= e^{i\pi\left(\frac{x}{\pi}+\frac{1}{4}\omega\right)}\vartheta_3(x+\tfrac{1}{2}\pi\omega) = \vartheta_1(x+\tfrac{1}{2}\pi).
\end{aligned}\right\} \quad \text{(x)}$$

To obtain the Theta function $\vartheta_{\mu,\nu}(x)$ from the series (i), we have only to write

$$a = i\pi\omega,$$
$$b = \tfrac{1}{2}i\pi\left(\mu\omega+\nu+2\frac{x}{\pi}\right),$$
$$c = \mu i\pi\left[\tfrac{1}{4}\mu\omega+\tfrac{1}{2}\left(2\frac{x}{\pi}+\nu\right)\right].$$

Thus, in some sort, the function $\vartheta_3(x)$ is the simplest of the Theta functions, the values of the coefficients a, b, c being, for this function, $a = i\pi\omega$, $b = ix$, $c = 0$.

2. *The Omega and Modular Functions.*

The Theta functions are themselves functions of two arguments x and q; but if we give to x the value zero, or any numerical value, or, again, any value depending on the value of q, we obtain a series of functions containing the single argument q or ω. In this Memoir we propose to direct our attention chiefly but not exclusively to these functions of a single argument, which we propose to term the *Omega functions*. They are important not only in the theory of elliptic functions, but also in other parts of analysis; and they are intimately connected with several interesting questions of arithmetic, algebra, and geometry. An exhaustive account of the researches which have been undertaken by geometers on this subject would exceed our present limits; we propose therefore to give a brief outline of the most essential parts of the theory, and to select for a fuller treatment certain recent investigations which appear to have some interest.

We shall usually find it convenient to regard ω, and not q, as the independent variable, although for brevity we shall often employ q as an abbreviation for $e^{i\pi\omega}$. It will be observed that ω may have any complex value of the form $x+iy$, in which y is positive. If y is negative, the analytical modulus of q is greater than unity, and the Theta series are divergent; the value $y=0$ is also excluded, though y may be as small a positive quantity as we please *.

The following are the expressions, in terms of ω, of the Omega functions which appear at the top of each page of the Tables. We write $\vartheta[\omega]$ for $\vartheta(0, q)$.

$$\sqrt{k} = \frac{\vartheta_2[\omega]}{\vartheta_3[\omega]}, \qquad \sqrt{k'} = \frac{\vartheta_0[\omega]}{\vartheta_3[\omega]}; \quad \ldots\ldots\ldots \text{(i)}$$

$$\left.\begin{aligned} \sqrt{\frac{2K}{\pi}} &= \vartheta_3[\omega], \\ K &= K(\omega) = \tfrac{1}{2}\pi\vartheta_3^2[\omega], \\ K' &= K'(\omega) = \frac{\omega}{i}K = \frac{\pi\omega}{2i}\vartheta_3^2[\omega]; \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(ii)}$$

* The collected edition of Riemann's works (Leipzig, 1876) contains some fragmentary notes on the limiting values of the Omega functions, when the analytical modulus of q converges to unity. To these the editor, Professor Dedekind, has added some interesting researches of his own on the same subject.

$$\left.\begin{aligned} J = J(\omega) &= \frac{i\pi}{K}\frac{1}{\vartheta_0[\omega]}\frac{d\vartheta_0[\omega]}{d\omega} = \frac{i\pi}{K(\omega)}\frac{\vartheta_0'[\omega]}{\vartheta_0[\omega]}, \\ E = E(\omega) &= K(\omega) - J(\omega), \\ J' = J'(\omega) &= \frac{\omega}{i}J(\omega) + \frac{\pi}{2K(\omega)}, \\ E' = E'(\omega) &= J'(\omega). \end{aligned}\right\} \quad \ldots\ldots \text{(iii)}$$

When ω is a pure imaginary (*i.e.* a quantity of the form $i\sigma$, σ being real and positive), q is real, and k^2 is real, positive, and less than unity; the functions K, K', J, J', E, E' are also real and positive. This, in all practical applications, is the only important case, and it is only for such values of ω that the values of the Omega functions are given in the Tables; but the theory of these functions requires us to attend to all complex values of ω for which the analytical modulus of q is less than unity.

The equations (i), (ii), (iii) are to be regarded as defining the functions $\sqrt{k}$, $\sqrt{k'}$, $\sqrt{K}$, K', J, J', and E. To these we add four other equations of definition,

$$\left.\begin{aligned} \phi(\omega) &= \sqrt{2}\,q^{\frac{1}{8}}\prod_{m=1}^{m=\infty}\frac{1+q^{2m}}{1+q^{2m-1}}, & \psi(\omega) &= \prod_{m=1}^{m=\infty}\frac{1-q^{2m-1}}{1+q^{2m-1}}, \\ \chi(\omega) &= \sqrt[6]{2}\,q^{\frac{1}{24}}\prod_{m=1}^{m=\infty}\frac{1}{1+q^{2m-1}}, & T(\omega) &= \frac{[1-\chi^{24}(\omega)]^3}{\chi^{48}(\omega)}; \end{aligned}\right\} \quad \ldots \text{(iv)}$$

and we observe that these four functions, as well as the seven preceding, are one-valued functions of ω. (We understand by $\sqrt{2}$ and $\sqrt[6]{2}$ real and positive roots of 2; and by $q^{\frac{1}{8}}$ and $q^{\frac{1}{24}}$ we understand $e^{\frac{1}{8}i\pi\omega}$ and $e^{\frac{1}{24}i\pi\omega}$.) Any rational and integral function of $\phi(\omega)$, $\psi(\omega)$, $\chi(\omega)$ is termed a *Modular* function.

We also note the equations

$$\left.\begin{aligned} \vartheta_0(\tfrac{1}{2}\pi) &= \vartheta_3(0), & \vartheta_1(\tfrac{1}{2}\pi) &= \vartheta_2(0), \\ \vartheta_2(\tfrac{1}{2}\pi) &= \vartheta_1(0) = 0, & \vartheta_3(\tfrac{1}{2}\pi) &= \vartheta_0(0); \end{aligned}\right\} \quad \ldots\ldots \text{(v)}$$

$$\left.\begin{aligned} e^{\frac{1}{4}i\pi\omega}\vartheta_0(\tfrac{1}{2}\pi\omega) &= i\,\vartheta_1(0) = 0, & e^{\frac{1}{4}i\pi\omega}\vartheta_1(\tfrac{1}{2}\pi\omega) &= i\,\vartheta_0(0), \\ e^{\frac{1}{4}i\pi\omega}\vartheta_2(\tfrac{1}{2}\pi\omega) &= \vartheta_3(0), & e^{\frac{1}{4}i\pi\omega}\vartheta_3(\tfrac{1}{2}\pi\omega) &= \vartheta_2(0); \end{aligned}\right\} \quad \ldots \text{(vi)}$$

$$\left.\begin{aligned} e^{\frac{1}{4}i\pi\omega}\vartheta_0(\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\omega) &= \vartheta_2(0), & e^{\frac{1}{4}i\pi\omega}\vartheta_1(\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\omega) &= \vartheta_3(0), \\ e^{\frac{1}{4}i\pi\omega}\vartheta_2(\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\omega) &= -i\,\vartheta_0(0), & e^{\frac{1}{4}i\pi\omega}\vartheta_3(\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\omega) &= i\,\vartheta_1(0) = 0, \end{aligned}\right\} \quad \text{(vii)}$$

which are particular cases of equation (ix), Art. 1, and which serve to express the Theta functions of the half periods in terms of $\vartheta_0[\omega]$, $\vartheta_2[\omega]$, $\vartheta_3[\omega]$.

3. *The Theta and Omega Functions as Infinite Products.*

The theory of the Omega functions is so closely dependent on the theory of the Theta functions, of which indeed it forms a subordinate part, that we shall find it convenient to give a brief outline of the principal properties of the Theta functions, and to deduce from these, as we proceed, the characteristic properties of the Omega functions. We shall, in the first instance, confine ourselves to those properties which are independent of the theory of the transformation of the Theta and Omega functions.

The identity

$$\left.\begin{aligned} &1+q(v^2+v^{-2})+q^4(v^4+v^{-4})+q^9(v^6+v^{-6})+\dots \\ &=\prod_{m=1}^{m=\infty}(1-q^{2m})\times\prod_{m=1}^{m=\infty}(1+q^{2m-1}v^2)\times\prod_{m=1}^{m=\infty}(1+q^{2m-1}v^{-2}) \end{aligned}\right\},\quad\dots\quad\text{(i)}$$

which has been demonstrated by Jacobi and by Cauchy, expresses a fundamental property of the Theta functions. If we replace v by $q^{\frac{1}{2}}v$, and multiply by $q^{\frac{1}{4}}v$, this identity assumes the form

$$\left.\begin{aligned} &q^{\frac{1}{4}}(v+v^{-1})+q^{\frac{9}{4}}(v^3+v^{-3})+q^{\frac{25}{4}}(v^5+v^{-5})+\dots \\ &=\prod_{m=1}^{m=\infty}(1-q^{2m})\times q^{\frac{1}{4}}(v+v^{-1})\times\prod_{m=1}^{m=\infty}(1+q^{2m}v^2)\times\prod_{m=1}^{m=\infty}(1+q^{2m}v^{-2}). \end{aligned}\right\}\quad\dots\quad\text{(ii)}$$

Writing successively $v=e^{ix}$, and $v=ie^{ix}$, in the equations (i) and (ii), we obtain

$$\vartheta_3(x)=1+2q\cos 2x+2q^4\cos 4x+2q^9\cos 6x+\dots$$
$$=\prod_1^\infty(1-q^{2m})\prod_1^\infty[1+2q^{2m-1}\cos 2x+q^{4m-2}];\qquad\text{(iii)}$$

$$\vartheta_0(x)=1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+\dots$$
$$=\prod_1^\infty(1-q^{2m})\prod_1^\infty[1-2q^{2m-1}\cos 2x+q^{4m-2}];\qquad\text{(iv)}$$

$$\vartheta_2(x)=2q^{\frac{1}{4}}\cos x+2q^{\frac{9}{4}}\cos 3x+2q^{\frac{25}{4}}\cos 5x+\dots$$
$$=2q^{\frac{1}{4}}\prod_1^\infty(1-q^{2m})\cos x\prod_1^\infty[1+2q^{2m}\cos 2x+q^{4m}];\qquad\text{(v)}$$

$$\vartheta_1(x)=2q^{\frac{1}{4}}\sin x-2q^{\frac{9}{4}}\sin 3x+2q^{\frac{25}{4}}\sin 5x-\dots$$
$$=2q^{\frac{1}{4}}\prod_1^\infty(1-q^{2m})\sin x\prod_1^\infty[1-2q^{2m}\cos 2x+q^{4m}].\qquad\text{(vi)}$$

The *zeros* of the Theta functions (*i.e.* the values of x for which these functions vanish) are consequently as follows:

$$\left.\begin{aligned} \vartheta_0(x) &= 0, \quad \text{if } x = r\pi + \tfrac{1}{2}(2s+1)\omega\pi ; \\ \vartheta_1(x) &= 0, \quad \text{if } x = r\pi + s\,\omega\pi ; \\ \vartheta_2(x) &= 0, \quad \text{if } x = \tfrac{1}{2}(2r+1)\pi + s\,\omega\pi ; \\ \vartheta_3(x) &= 0, \quad \text{if } x = \tfrac{1}{2}(2r+1)\pi + \tfrac{1}{2}(2s+1)\omega\pi ; \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(vii)}$$

i. e. the zeros of $\vartheta_{\mu,\nu}(x)$ are

$$\tfrac{1}{2}(2r+\nu-1)\pi + \tfrac{1}{2}(2s+\mu-1)\omega\pi.$$

Putting $x = 0$, we obtain from the equations (iii), (iv), (v):

$$\vartheta_3(0) = \sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + \ldots = \prod_1^\infty (1-q^{2m})(1+q^{2m-1})^2 ; \quad \text{(viii)}$$

$$\vartheta_0(0) = \sqrt{\frac{2k'K}{\pi}} = 1 - 2q + 2q^4 - 2q^9 + \ldots = \prod_1^\infty (1-q^{2m})(1-q^{2m-1})^2 ; \quad \text{(ix)}$$

$$\vartheta_2(0) = \sqrt{\frac{2kK}{\pi}} = 2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + \ldots = 2q^{\frac{1}{4}} \prod_1^\infty (1-q^{2m})(1+q^{2m})^2. \quad . \quad . \quad \text{(x)}$$

Since $\vartheta_1(x)$ is an uneven function, we have

$$\vartheta_1(0) = 0 ;$$

but we find

$$\vartheta_1'(0) = 2q^{\frac{1}{4}} \prod_1^\infty (1-q^{2m})^3 = 2q^{\frac{1}{4}} - 6q^{\frac{9}{4}} + 10q^{\frac{25}{4}} - 14q^{\frac{49}{4}} + \ldots \quad . \quad . \quad \text{(xi)}$$

If ν denote any uneven number, we have, by an identity due to Euler,

$$\prod_{\nu=1}^{\nu=\infty}(1-q^\nu) \prod_{m=1}^{m=\infty}(1+q^m) = \prod_{\nu=1}^{\nu=\infty}(1-q^\nu)(1+q^\nu)(1+q^{2\nu})(1+q^{4\nu})\ldots = 1. \quad . \quad \text{(xii)}$$

Multiplying together the values of $\vartheta_0(0)$, $\vartheta_3(0)$, $\vartheta_2(0)$, and attending to this identity, we obtain

$$\vartheta_0(0)\,\vartheta_2(0)\,\vartheta_3(0) = \frac{2K}{\pi}\sqrt{\frac{2kk'K}{\pi}} = 2q^{\frac{1}{4}} \prod_1^\infty (1-q^{2m})^3 = \vartheta_1'(0). \quad . \quad . \quad \text{(xiii)}$$

Dividing (x) and (ix) respectively by (viii), we find

$$\phi^2(\omega) = \sqrt{k}, \quad \psi^2(\omega) = \sqrt{k'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiv)}$$

We also have from (xii), and from Art. 2, (iv),

$$\chi(\omega+1) = e^{\frac{1}{4}i\pi}\frac{\chi(\omega)}{\psi(\omega)} = 2^{\frac{1}{6}} q^{\frac{1}{24}} e^{\frac{1}{24}i\pi} \Pi(1+q^m), \quad \phi(\omega) \times \psi(\omega) = \chi^3(\omega). \quad . \quad \text{(xv)}$$

For brevity, we shall often write

$$\Phi(\omega) = k^2 = \phi^8(\omega), \qquad \Psi(\omega) = k'^2 = \psi^8(\omega).$$

4. *Expressions for the Modular Functions* $\phi(\omega)$, $\psi(\omega)$, $\chi(\omega)$ *as Infinite Products.*

The following different expressions for $\phi(\omega)$, for $\psi(\omega)$, and for the Eulerian

product $\Pi(1+q^m)$, or, which comes to the same thing, for $\chi(\omega+1)$ have been given by Jacobi (Crelle's Journal, vol. xxxvii. pp. 67–77):

I.

$$(a)\ \phi(\omega)=\sqrt{2}\,q^{\frac{1}{8}}\prod_{m=1}^{m=\infty}\frac{1-q^{4m}}{(1+q^{2m-1})(1-q^{2m})}=\sqrt{2}\,q^{\frac{1}{8}}\frac{\sum_{m=-\infty}^{m=+\infty}(-)^m q^{6m^2+2m}}{\sum(-)^{\frac{1}{2}m(m+1)}q^{\frac{1}{2}(3m^2+m)}};$$

$$(b)\ \phi(\omega)=\sqrt{2}\,q^{\frac{1}{8}}\Pi\frac{(1+q^{4m-2})(1-q^{8m})}{(1+q^{2m-1})(1-q^{4m})}=\sqrt{2}\,q^{\frac{1}{8}}\frac{\sum q^{4m^2+2m}}{\sum q^{2m^2+m}}=\sqrt{2}\,\frac{\vartheta_2[\omega]}{\vartheta_2[\frac{1}{2}\omega]};$$

$$(c)\ \phi(\omega)=\sqrt{2}\,q^{\frac{1}{8}}\Pi\frac{(1-q^{2m-1})(1-q^{4m})}{(1-q^{4m-2})^2(1-q^{4m})}=\sqrt{2}\,q^{\frac{1}{8}}\frac{\sum(-)^m q^{2m^2+m}}{\sum(-)^m q^{2m^2}}=\frac{e^{\frac{1}{8}i\pi}}{\sqrt{2}}\,\frac{\vartheta_2[\frac{1}{2}(\omega+1)]}{\vartheta_0[2\omega]};$$

$$(d)\ \phi(\omega)=\sqrt{2}\,q^{\frac{1}{8}}\Pi\frac{(1+q^{2m-1})(1-q^{4m})}{(1+q^{2m-1})^2(1-q^{2m})}=\sqrt{2}\,q^{\frac{1}{8}}\frac{\sum q^{2m^2+m}}{\sum q^{m^2}}=\frac{1}{\sqrt{2}}\,\frac{\vartheta_2[\frac{1}{2}\omega]}{\vartheta_3[\omega]}.$$

II.

$$(a)\ \psi(\omega)=\Pi\frac{1-q^m}{(1+q^{2m-1})(1-q^{2m})}=\frac{\sum(-)^m q^{\frac{1}{2}(3m^2+m)}}{\sum(-)^{\frac{1}{2}m(m+1)}q^{\frac{1}{2}(3m^2+m)}};$$

$$(b)\ \psi(\omega)=\Pi\frac{(1-q^{2m-1})(1-q^{4m})}{(1+q^{2m-1})(1-q^{4m})}=\frac{\sum(-)^m q^{2m^2+m}}{\sum q^{2m^2+m}}=e^{-\frac{1}{8}i\pi}\,\frac{\vartheta_2[\frac{1}{2}(\omega+1)]}{\vartheta_2[\frac{1}{2}\omega]};$$

$$(c)\ \psi(\omega)=\Pi\frac{(1-q^{2m-1})^2(1-q^{2m})}{(1-q^{4m-2})^2(1-q^{4m})}=\frac{\sum(-)^m q^{m^2}}{\sum(-)^m q^{2m^2}}=\frac{\vartheta_0[\omega]}{\vartheta_0[2\omega]};$$

$$(d)\ \psi(\omega)=\Pi\frac{(1-q^{4m-2})^2(1-q^{4m})}{(1+q^{2m-1})^2(1-q^{2m})}=\frac{\sum(-)^m q^{2m^2}}{\sum q^{m^2}}=\frac{\vartheta_0[2\omega]}{\vartheta_3[\omega]}.$$

III.

$$(a)\ \frac{\chi(\omega+1)}{2^{\frac{1}{6}}e^{\frac{1}{24}i\pi}}=q^{\frac{1}{24}}\Pi\frac{1-q^{2m}}{1-q^m}=\frac{\sum(-)^m q^{\frac{1}{12}(6m+1)^2}}{\sum(-)^m q^{\frac{1}{24}(6m+1)^2}};$$

$$(b)\ \frac{\chi(\omega+1)}{2^{\frac{1}{6}}e^{\frac{1}{24}i\pi}}=q^{\frac{1}{24}}\Pi\frac{(1+q^{2m-1})(1-q^{4m})}{1-q^{2m}}=\frac{\sum q^{\frac{1}{6}(4m+1)^2}}{\sum(-)^m q^{\frac{1}{12}(6m+1)^2}};$$

$$(c)\ \frac{\chi(\omega+1)}{2^{\frac{1}{6}}e^{\frac{1}{24}i\pi}}=q^{\frac{1}{24}}\Pi\frac{1-q^m}{(1-q^{2m-1})^2(1-q^{2m})}=\frac{\sum(-)^m q^{\frac{1}{24}(6m+1)^2}}{\sum(-)^m q^{m^2}};$$

$$(d)\ \frac{\chi(\omega+1)}{2^{\frac{1}{6}}e^{\frac{1}{24}i\pi}}=q^{\frac{1}{24}}\Pi\frac{(1+q^{2m-1})(1-q^{2m})}{(1-q^{4m-2})^2(1-q^{4m})}=\frac{\sum(-)^{\frac{1}{2}m(m+1)}q^{\frac{1}{24}(6m+1)^2}}{\sum(-)^m q^{2m^2}};$$

$$(e)\quad \frac{\chi(\omega+1)}{2^{\frac{1}{6}}e^{\frac{1}{24}i\pi}} = q^{\frac{1}{24}}\Pi\frac{1-q^{4m}}{(1-q^{2m-1})(1-q^{4m})} = \frac{\Sigma(-)^m q^{\frac{1}{6}(6m+1)^2}}{\Sigma(-)^m q^{\frac{1}{6}(4m+1)^2}};$$

$$(f)\quad \frac{\chi(\omega+1)}{2^{\frac{1}{6}}e^{\frac{1}{24}i\pi}} = q^{\frac{1}{24}}\Pi\frac{(1+q^{3m-2})(1+q^{3m-1})(1-q^{3m})}{(1-q^{6m-3})^2(1-q^{6m})} = \frac{\Sigma q^{\frac{1}{24}(6m+1)^2}}{\Sigma(-)^m q^{3m^2}};$$

$$(g)\quad \frac{\chi(\omega+1)}{2^{\frac{1}{6}}e^{\frac{1}{24}i\pi}} = q^{\frac{1}{24}}\Pi\frac{(1+q^{6m-3})(1-q^{12m})}{(1-q^{6m-5})(1-q^{6m-1})(1-q^{6m})} = \frac{\Sigma q^{\frac{3}{8}(4m+1)^2}}{\Sigma(-)^m q^{\frac{1}{3}(3m+1)^2}}.$$

We have also for the square and cube of $\chi(\omega+1)$ the analogous formulæ :

IV.

$$(a)\quad \frac{\chi^2(\omega+1)}{2^{\frac{1}{3}}e^{\frac{1}{12}i\pi}} = q^{\frac{1}{12}}\Pi\frac{(1+q^{2m-1})(1-q^{4m})}{1-q^m} = q^{\frac{1}{12}}\frac{\Sigma q^{2m^2+m}}{\Sigma(-)^m q^{\frac{1}{2}(3m^2+m)}} = \frac{\Sigma q^{\frac{1}{8}(4m+1)^2}}{\Sigma(-)^m q^{\frac{1}{24}(6m+1)^2}};$$

$$(b)\quad \frac{\chi^2(\omega+1)}{2^{\frac{1}{3}}e^{\frac{1}{12}i\pi}} = q^{\frac{1}{12}}\Pi\frac{1-q^{2m}}{(1-q^{2m-1})^2(1-q^{2m})} = q^{\frac{1}{12}}\frac{\Sigma(-)^m q^{\frac{3}{2}(m^2+m)}}{\Sigma(-)^m q^{m^2}} = \frac{\Sigma(-)^m q^{\frac{1}{12}(6m+1)^2}}{\Sigma(-)^m q^{m^2}}.$$

V.

$$(a)\quad \frac{\chi^3(\omega+1)}{2^{\frac{1}{2}}e^{\frac{1}{8}i\pi}} = q^{\frac{1}{8}}\Pi\frac{(1+q^{2m-1})(1-q^{4m})}{(1-q^{2m-1})^2(1-q^{2m})} = q^{\frac{1}{8}}\frac{\Sigma q^{2m^2+m}}{\Sigma(-)^m q^{m^2}} = \frac{\Sigma q^{\frac{1}{8}(4m+1)^2}}{\Sigma(-)^m q^{m^2}} = \frac{1}{2}\frac{\vartheta_2[\frac{1}{2}\omega]}{\vartheta_0[\omega]};$$

$$(b)\quad \frac{\chi^3(\omega+1)}{2^{\frac{1}{2}}e^{\frac{1}{8}i\pi}} = q^{\frac{1}{8}}\Pi\Big(\frac{1-q^{2m}}{1-q^m}\Big)^3 = q^{\frac{1}{8}}\frac{\Sigma(4m+1)q^{4m^2+2m}}{\Sigma(4m+1)q^{2m^2+m}} = \frac{\vartheta_1'(0,q)}{\vartheta_1'(0,q^{\frac{1}{2}})}.$$

These expressions may be verified by comparing their type factors with the type factors in the equations defining $\phi(\omega)$, $\psi(\omega)$, and $\chi(\omega+1)$; the transformation of the products into sums is effected by employing appropriate particularisations of the formula (i) of Art. 3 : among these, besides the formulæ (vii)–(x), of that article, we may mention the following :

VI.

$$\begin{cases} (a) & \Pi(1-q^{2m-1})\Pi(1-q^{4m}) = \Sigma(-)^m q^{2m^2+m}, \\ (a') & \Pi(1+q^{2m-1})\Pi(1-q^{4m}) = \Sigma q^{2m^2+m}, \\ (b) & \Pi(1-q^m) = \Sigma(-)^m q^{\frac{1}{2}(3m^2+m)}, \\ (b') & \Pi(1-(-)^m q^m) = \Sigma(-)^{\frac{1}{2}m(m+1)} q^{\frac{1}{2}(3m^2+m)}. \end{cases}$$

We give three examples of these verifications. In the formula (I. c) we have

$$\Pi(1-q^{2m-1})(1-q^{4m}) = \Sigma(-)^m q^{2m^2+m} \text{ by (VI. } a\text{)},$$

and

$$\Pi(1-q^{4m-2})^2(1-q^{4m}) = \Sigma(-)^m q^{2m^2} \text{ by Art. 3 (ix)};$$

also

$$\Pi\frac{(1-q^{2m-1})(1-q^{4m})}{(1-q^{4m-2})^2(1-q^{4m})} = \Pi\frac{1-q^{2m-1}}{(1-q^{4m-2})^2} = \Pi\frac{1}{(1-q^{2m-1})(1-q^{4m-2})} = \Pi\frac{1+q^{2m}}{1+q^{2m-1}},$$

since by Art. 3, equation (xii),

$$\Pi\frac{1}{1-q^{4m-2}} = \Pi(1+q^{2m}).$$

Again, to verify the equation (III. g) we have

$$\Pi(1+q^{6m-3})(1-q^{12m}) = \Sigma q^{6m^2+3m} \text{ by (VI. } a'\text{)};$$

and, writing q^3 for q, and $-q^2$ for v^2, in the formula (i) of Art. 3, we obtain

$$\Pi(1-q^{6m})(1-q^{6m-1})(1-q^{6m-5}) = \Sigma(-)^m q^{3m^2+2m};$$

also

$$\Pi\frac{(1+q^{6m-3})(1-q^{12m})}{(1-q^{6m-5})(1-q^{6m-1})(1-q^{6m})} = \Pi\frac{(1+q^{6m-3})(1+q^{6m})}{(1-q^{6m-5})(1-q^{6m-1})}$$

$$= \Pi\frac{1+q^{3m}}{(1-q^{6m-5})(1-q^{6m-1})} = \Pi\frac{1}{(1-q^{6m-3})(1-q^{6m-5})(1-q^{6m-1})} = \Pi\frac{1}{1-q^{2m-1}}.$$

Lastly, to establish (VI. b'), we write $e^{\frac{1}{3}i\pi}q^{\frac{3}{2}}$ for q, and $e^{\frac{1}{3}i\pi}q^{\frac{1}{2}}$ for v^2; we thus find

$$\Sigma(-)^{\frac{1}{2}m(m+1)}q^{\frac{1}{2}(3m^2+m)} = \Pi[1-(-)^m q^{3m}]\times\Pi[1+(-)^m q^{3m-1}]\times[1-(-)^m q^{3m-2}]$$

$$= \Pi[1-(-)^m q^m].$$

5. *Expansions of the Omega Functions in series proceeding by powers of q.*

Let

$$\frac{\phi(\omega)}{\sqrt{2}\,.\,q^{\frac{1}{8}}} = f(q);$$

Jacobi has given the formula (*Fundamenta Nova*, Art. 40, equation 27)

$$\log f(q) = \sum_{h=1}^{h=\infty}\frac{\Delta(h)}{h}q^h,$$

where $\Delta(h)$ is $[1+(-)^h 2]$ times the sum of the uneven divisors of h. Let

$$f^s(q) = \sum_{h=1}^{h=\infty}[s, h]q^h,$$

s being any quantity whatever; since

$$f^s(q) = e^{s\Sigma\frac{\Delta(h)}{h}q^h},$$

we have, if $h=0$, $[s, 0]=1$, and, if $h>0$,

$$[s, h]=s\frac{\Delta(h)}{h}+\frac{s^2}{1.2}\Sigma_2\frac{\Delta(a)\Delta(b)}{ab}+\frac{s^3}{1.2.3}\Sigma_3\frac{\Delta(a)\Delta(b)\Delta(c)}{abc}+\dots$$

where Σ_i extends to every set of i numbers (equal or unequal) which satisfy the equation $a+b+c+\dots=h$.

But, when s is a positive or negative integer, the coefficients $[s, h]$ are more easily calculated by a recurring formula, similar to that given by Euler for the development of the product $\Pi(1+q^m)$, and deduced by him from the equation III. (a) of the last article. Several such formulæ are supplied by the equations of Jacobi. For example, from equation I. (c), we have

$$f(q)=\Sigma(-1)^m q^{2m^2+m}\div\Sigma(-1)^m q^{2m^2},$$

whence $\Sigma_m(-1)^m\times[1, h-2m^2]=(-1)^h$ or 0, according as h is or is not of the form $2m^2+m$. Similarly, since $f^{s+1}(q)=f^s(q)\times f(q)$, we find, from equation I. (b),

$$\Sigma_m[s+1, h-2m^2-m]=\Sigma_m[s, h-4m^2-2m],$$

by which the coefficients in the successive powers of $f(q)$ may be calculated.

Formulæ, similar to these in their general character, exist for the other modular functions.

6. *The Formula for the Multiplication of Four Theta Functions.*

We next give a formula which enables us to obtain the algebraical relations connecting the four Theta functions, and the differential equations satisfied by them.

Let $2s=x_1+x_2+x_3+x_4$, $2\sigma=\mu_1+\mu_2+\mu_3+\mu_4$, $2\sigma'=\nu_1+\nu_2+\nu_3+\nu_4$,

the integral numbers $\mu_1, \mu_2, \mu_3, \mu_4$ and $\nu_1, \nu_2, \nu_3, \nu_4$ being subject to the restriction that σ and σ' are to be integral. Multiplying together the four Theta functions

$$\vartheta_{\mu_r, \nu_r}(x_r), \quad r=1, 2, 3, 4,$$

and transforming, in the general term of the product, the indices of $e^{i\pi\omega}$, of $e^{i\pi x}$, and of $e^{\pi i}$ by the identities

$$a^2+b^2+c^2+d^2=(S-a)^2+(S-b)^2+(S-c)^2+(S-d)^2$$

$$a\alpha+b\beta+c\gamma+d\delta=(S-a)(\Sigma-\alpha)+(S-b)(\Sigma-\beta)+(S-c)(\Sigma-\gamma)+(S-d)(\Sigma-\delta),$$

where

$$2S=a+b+c+d,$$

$$2\Sigma=\alpha+\beta+\gamma+\delta,$$

we find

$$\left.\begin{aligned}2\Pi_r\vartheta_{\mu_r, \nu_r}(x_r)=\Pi_r\vartheta_{\sigma-\mu_r, -\nu_r}(s-x_r)+(-)^\sigma\Pi_r\vartheta_{\sigma-\mu_r, \sigma'-\nu_r+1}(s-x_r)\\+\Pi_r\vartheta_{\sigma-\mu_r+1, \sigma'-\nu_r}(s-x_r)+(-)^{1+\sigma}\Pi_r\vartheta_{\sigma-\mu_r+1, \sigma'-\nu_r+1}(s-x_r).\end{aligned}\right\}\quad\text{(i)}$$

Giving in this formula to the symbols

$$\left.\begin{matrix} \mu_1, & \mu_2, & \mu_3, & \mu_4 \\ \nu_1, & \nu_2, & \nu_3, & \nu_4 \\ x_1, & x_2, & x_3, & x_4 \end{matrix}\right\}, \quad \ldots\ldots\ldots \quad \text{(ii)}$$

first, the values

$$\begin{matrix} 0, & 0, & 0, & 0, \\ 0, & 0, & 0, & 0, \\ x_1, & x_2, & x_3, & x_4\,; \end{matrix}$$

secondly, the values

$$\begin{matrix} 1, & 1, & 1, & 1, \\ 0, & 0, & 0, & 0, \\ x_1, & x_2, & x_3, & x_4\,; \end{matrix}$$

and adding the results, we obtain the equation

$$\Pi_r \vartheta_3(x_r) + \Pi_r \vartheta_2(x_r) = \Pi_r \vartheta_3(s - x_r) + \Pi_r \vartheta_2(s - x_r). \quad \ldots \quad \text{(iii)}$$

This is the fundamental formula of Jacobi's Lectures (see Rosenhain, 'Mémoires des Savants Étrangers,' vol. xi. p. 361). Owing to the relations (Art. 1, ix) which connect the Theta functions of different indices, it is no less general than the equation (i) from which it is derived; it is, however, less easily manipulated.

Of the conclusions which may be derived from the formula (i), we shall in this place mention only those which serve to establish the elementary properties of the elliptic functions.

7. *The Elliptic Functions of the First Species.*

Attributing successively to the symbols of the scheme (ii) Art. 6, the values

$$\left|\begin{matrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{matrix}\right|,$$

$$\left|\begin{matrix} 1, & 1, & 0, & 0 \\ 1, & 1, & 0, & 0 \\ x, & x, & 0, & 0 \end{matrix}\right|,$$

$$\left|\begin{matrix} 0, & 0, & 0, & 0 \\ 1, & 1, & 0, & 0 \\ x, & x, & 0, & 0 \end{matrix}\right|,$$

we find

$$\vartheta_2^4(0) + \vartheta_0^4(0) = \vartheta_3^4(0), \quad \ldots\ldots\ldots \quad \text{(i)}$$

$$\vartheta_0^2(0)\,\vartheta_2^2(x) = -\vartheta_3^2(0)\,\vartheta_1^2(x) + \vartheta_2^2(0)\,\vartheta_0^2(x), \quad \ldots\ldots \quad \text{(ii)}$$

$$\vartheta_0^2(0)\,\vartheta_3^2(x) = \vartheta_3^2(0)\,\vartheta_0^2(x) - \vartheta_2^2(0)\,\vartheta_1^2(x)\,; \quad \ldots\ldots \quad \text{(iii)}$$

or, using the notations of Art. 2,

$$k^2+k'^2 = \phi^8(\omega)+\psi^8(\omega)=1, \quad \text{(iv)}$$

$$k'\vartheta_2^2(x) = -\vartheta_1^2(x)+k\vartheta_0^2(x), \quad \text{(v)}$$

$$k'\vartheta_3^2(x) = \vartheta_0^2(x)-k\vartheta_1^2(x). \quad \text{(vi)}$$

Again, attributing to the same symbols the values

$$\begin{vmatrix} 0, & 1, & 0, & 1 \\ 1, & 1, & 0, & 0 \\ x-y, & x+y, & 0, & 0 \end{vmatrix},$$

$$\begin{vmatrix} 0, & 1, & 0, & 1 \\ 1, & 0, & 1, & 0 \\ x-y, & x+y, & 0, & 0 \end{vmatrix},$$

$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 1, & 0, & 0, & 1 \\ x-y, & x+y, & 0, & 0 \end{vmatrix},$$

we obtain the three equations,

$$\vartheta_0(x-y)\vartheta_1(x+y)\vartheta_3(0)\vartheta_2(0) = \vartheta_2(y)\vartheta_3(y)\vartheta_1(x)\vartheta_0(x)+\vartheta_1(y)\vartheta_0(y)\vartheta_2(x)\vartheta_3(x), \quad \text{(vii)}$$

$$\vartheta_0(x-y)\vartheta_2(x+y)\vartheta_0(0)\vartheta_2(0) = \vartheta_2(y)\vartheta_0(y)\vartheta_2(x)\vartheta_0(x)-\vartheta_1(y)\vartheta_3(y)\vartheta_1(x)\vartheta_3(x), \quad \text{(viii)}$$

$$\vartheta_0(x-y)\vartheta_3(x+y)\vartheta_3(0)\vartheta_0(0) = \vartheta_0(y)\vartheta_3(y)\vartheta_0(x)\vartheta_3(x)-\vartheta_1(y)\vartheta_2(y)\vartheta_1(x)\vartheta_3(x). \quad \text{(ix)}$$

We divide each of these equations by y, and then cause y to decrease without limit; attending to equation (xiii) Art. 3, and observing that

$$\vartheta_0'(0)=\vartheta_2'(0)=\vartheta_3'(0)=0,$$

we deduce the three differential equations

$$\frac{d}{dx}\left(\frac{\vartheta_1(x)}{\vartheta_0(x)}\right) = \vartheta_0^2(0)\frac{\vartheta_2(x)\vartheta_3(x)}{\vartheta_0^2(x)}, \quad \text{(x)}$$

$$\frac{d}{dx}\left(\frac{\vartheta_2(x)}{\vartheta_0(x)}\right) = -\vartheta_3^2(0)\frac{\vartheta_1(x)\vartheta_3(x)}{\vartheta_0^2(x)}, \quad \text{(xi)}$$

$$\frac{d}{dx}\left(\frac{\vartheta_3(x)}{\vartheta_0(x)}\right) = -\vartheta_2^2(0)\frac{\vartheta_1(x)\vartheta_2(x)}{\vartheta_0^2(x)}. \quad \text{(xii)}$$

If we write $\frac{\pi x}{2K}-\frac{x}{\vartheta_3^2(0)}$ for x, and introduce the elliptic functions $\mathrm{sinam}x$, $\mathrm{cosam}x$, $\Delta\mathrm{am}x$ (*i.e.* the sine, the cosine, and the Delta of the amplitude of x),

which are respectively defined by the equations

$$\sin\operatorname{am} x = \frac{1}{\sqrt{k}} \frac{\vartheta_1\left(\frac{\pi x}{2K}\right)}{\vartheta_0\left(\frac{\pi x}{2K}\right)} = \frac{\Theta_1(x)}{\Theta(x)}, \quad \ldots\ldots\ldots \text{(xiii)}$$

$$\cos\operatorname{am} x = \sqrt{\frac{k'}{k}} \frac{\vartheta_2\left(\frac{\pi x}{2K}\right)}{\vartheta_0\left(\frac{\pi x}{2K}\right)} = \frac{\Theta_2(x)}{\Theta(x)}, \quad \ldots\ldots\ldots \text{(xiv)}$$

$$\Delta\operatorname{am} x = \sqrt{k'} \frac{\vartheta_3\left(\frac{\pi x}{2K}\right)}{\vartheta_0\left(\frac{\pi x}{2K}\right)} = \frac{\Theta_3(x)}{\Theta(x)}, \quad \ldots\ldots\ldots \text{(xv)}$$

the differential equations (x), (xi), (xii) become

$$\frac{d.\sin\operatorname{am} x}{dx} = \cos\operatorname{am} x \Delta\operatorname{am} x, \quad \ldots\ldots\ldots \text{(xvi)}$$

$$\frac{d.\cos\operatorname{am} x}{dx} = -\sin\operatorname{am} x \Delta\operatorname{am} x, \quad \ldots\ldots\ldots \text{(xvii)}$$

$$\frac{d.\Delta\operatorname{am} x}{dx} = -k^2 \cos\operatorname{am} x \sin\operatorname{am} x. \quad \ldots\ldots\ldots \text{(xviii)}$$

The equations (v) and (vi) give at the same time

$$\left.\begin{aligned} \sin^2\operatorname{am} x + \cos^2\operatorname{am} x &= 1, \\ k^2 \sin^2\operatorname{am} x + \Delta^2\operatorname{am} x &= 1. \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(xix)}$$

The functions $\sin\operatorname{am} x$, $\cos\operatorname{am} x$, and $\Delta\operatorname{am} x$ are all doubly periodic (Art. 1), their periods being $4K, 2iK'$; $2(K+iK'), 2(K-iK')$; $2K, 4iK'$. The zeros of the three functions are respectively (see Art. 3, vii)

$$\left.\begin{aligned} x &= 2rK + 2siK', \\ x &= (2r+1)K + 2siK', \\ x &= (2r+1)K + (2s+1)iK'. \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(xx)}$$

The infinite points are the same for all three, viz.

$$x = 2rK + (2s+1)iK'.$$

Lastly, in the formula (i) of Art. 6, we successively attribute to the symbols (ii) the two sets of value

$$\left|\begin{array}{cccc} 0, & 0, & 0, & 0 \\ 1, & 1, & 1, & 1 \\ x-y, & x+y, & 0, & 0 \end{array}\right|,$$

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ 1, & 1, & 1, & 1 \\ x-y, & x+y, & 0, & 0 \end{vmatrix};$$

adding the results, we find

$$\vartheta_0(x-y)\,\vartheta_0(x+y)\,\vartheta_0^2(0) = \vartheta_0^2(x)\,\vartheta_0^2(y) - \vartheta_1^2(x)\,\vartheta_1^2(y). \quad . \quad . \quad . \quad . \quad . \quad \text{(xxi)}$$

Dividing by the terms of this equation the terms of the equations (vii), (viii), (ix), and introducing the elliptic functions, we obtain the formulæ of addition

$$\sin\operatorname{am}(x+y) = \frac{\sin\operatorname{am} x \cos\operatorname{am} y\,\Delta\operatorname{am} y + \sin\operatorname{am} y \cos\operatorname{am} x\,\Delta\operatorname{am} x}{1-k^2\sin^2\operatorname{am} x \sin^2\operatorname{am} y}, \quad . \quad \text{(xxii)}$$

$$\cos\operatorname{am}(x+y) = \frac{\cos\operatorname{am} x \cos\operatorname{am} y - \sin\operatorname{am} x \sin\operatorname{am} y\,\Delta\operatorname{am} x\,\Delta\operatorname{am} y}{1-k^2\sin^2\operatorname{am} x \sin^2\operatorname{am} y}, \quad . \quad \text{(xxiii)}$$

$$\Delta\operatorname{am}(x+y) = \frac{\Delta\operatorname{am} x\,\Delta\operatorname{am} y - k^2\sin\operatorname{am} x \sin\operatorname{am} y \cos\operatorname{am} x \cos\operatorname{am} y}{1-k^2\sin^2\operatorname{am} x \sin^2\operatorname{am} y}. \quad \text{(xxiv)}$$

The quantity $K-x$ is termed the complement of x; the amplitude of the complement of x is the coamplitude of x, and is written coam x. From Art. 2, equations (v) we have

$$\sin\operatorname{am} K = 1,\ \cos\operatorname{am} K = 0,\ \Delta\operatorname{am} K = k';$$

and hence

$$\sin\operatorname{coam} x = \sin\operatorname{am}(K-x) = \frac{\cos\operatorname{am} x}{\Delta\operatorname{am} x},$$

$$\cos\operatorname{coam} x = \cos\operatorname{am}(K-x) = \frac{k'\sin\operatorname{am} x}{\Delta\operatorname{am} x},$$

$$\Delta\operatorname{coam} x = \Delta\operatorname{am}(K-x) = \frac{k'}{\Delta\operatorname{am} x}.$$

The theory of the elliptic functions may be treated in two different ways:

(1) We may begin, as we have done in this memoir, with the definition of the Theta functions. We then define the elliptic functions by the equations (xiii), (xiv), (xv), and we show, as has been shown here, that these three one-valued functions are doubly periodic, and that they satisfy the differential equations (xvi), (xvii), (xviii), the algebraical relations (xix), and the formulæ of addition (xxii), (xxiii), (xxiv).

(2) Or we may begin with the definition of the elliptic integral

$$\int_0^u \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxv)}$$

We then define the function u as the synectic integral which satisfies the differential equation

$$\frac{du^2}{dx^2} = (1-u^2)(1-k^2u^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxvi)}$$

and the initial conditions

$$x=0, \quad u=0, \quad \frac{du}{dx}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxvii)}$$

This definition implies the theorem that the equation (xxvi) always admits of one, and only of one, synectic integral satisfying the initial conditions (xxvii). For a demonstration of this theorem, we may refer to the work of MM. Briot and Bouquet*. Assuming the theorem, we evidently have the equation

$$u = \operatorname{sinam} x;$$

for $\operatorname{sinam} x$ is a synectic integral of the equation (xxvi), and satisfies the initial conditions (xxvii). Hence also

$$v = \cos \operatorname{am} x, \qquad w = \Delta \operatorname{am} x,$$

if v and w are two functions defined by the equations

$$v^2 = 1-u^2, \qquad w^2 = 1-k^2u^2,$$

coupled with the initial conditions $x=0$, $v=1$, $w=1$.

In this way the identity of the functions obtained by considering the differential equation (xxvi) with the functions $\operatorname{sinam} x$, $\cos \operatorname{am} x$, $\Delta \operatorname{am} x$, as defined by the equations (xiii), (xiv), (xv), may be completely established.

That the functions u, v, w are doubly periodic, as well as synectic, can be inferred directly from the differential equation (xxvi), without employing the expressions of u, v, w by means of the Theta functions; the formulæ of addition can also be established in the same manner.

8. *The Complete Elliptic Integrals.*

We have already obtained the equations

$$\sin \operatorname{am} K = 1, \quad \cos \operatorname{am} K = 0, \quad \Delta \operatorname{am} K = k';$$

we have also (Art. 2, equations vii),

$$\sin \operatorname{am} (K+iK') = \frac{1}{k}, \quad \cos \operatorname{am} (K+iK') = \frac{k'}{ik}, \quad \Delta \operatorname{am} (K+iK') = 0.$$

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* 'Théorie des Fonctions Elliptiques,' par MM. Briot et Bouquet, ed. 2, Paris, 1875. See livre v. chap. iii.

Hence, if in the equation

$$x = \int_0^u \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}},$$

we put successively $x = K$, $x = K + iK'$, we find that K is *one* of the values of the definite integral

$$\int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}; \quad \ldots\ldots\ldots \text{(i)}$$

and that iK' is *one* of the values of the integral

$$\int_1^{\frac{1}{k}} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \quad \ldots\ldots\ldots \text{(ii)}$$

which, by either of the substitutions

$$u = (1-k'^2y^2)^{-\frac{1}{2}}, \quad u = \frac{1}{k}(1-k'^2y^2)^{\frac{1}{2}}, \quad \ldots\ldots\ldots \text{(iii)}$$

is changed into

$$i\int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k'^2u^2)}}.$$

We thus have the two equations

$$K = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \quad K' = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k'^2u^2)}}, \quad \ldots \text{(iv)}$$

in which, however, the track of the integration has not been determined. With this determination we shall occupy ourselves hereafter; for the present we observe that, when ω is a pure imaginary, k^2 and k'^2 are real, positive, and less than unity; the quantities K and K' are also real and positive. In this case, therefore, the integrals in the equations (iv) are the *rectilineal* integrals obtained by causing u to pass from 0 to 1 through a series of real values, the initial value of the radical being in each case $+1$.

9. *The Partial Differential Equation of the Theta Functions.*

The Theta functions satisfy the partial differential equation

$$\frac{d\vartheta}{d\omega} = -\frac{i\pi}{4}\frac{d^2\vartheta}{dx^2}, \quad \ldots\ldots\ldots \text{(i)}$$

which enables us to express their differential coefficients, taken with respect to ω, by means of their differential coefficients of an even order taken with respect to x. Thus, from equation (iii) Art. 2, we find

$$J = \frac{\pi^2}{4K}\frac{\vartheta_0''(0)}{\vartheta_0(0)}, \qquad \frac{J}{K} = \frac{\vartheta_0''(0)}{\vartheta_0(0)\,\vartheta_3^4(0)} \quad \ldots\ldots \text{(ii)}$$

10. *The Elliptic Function of the Second Species.*

If we differentiate the equation (xxi) of Art. 7 twice with regard to y, and put $y=0$ in the result, we find

$$\frac{d}{dx}\left[\frac{\vartheta_0'(x)}{\vartheta_0(x)}\right]=\frac{\vartheta_0''(0)}{\vartheta_0(0)}-\frac{\vartheta_1'^2(0)}{\vartheta_0^2(0)}\frac{\vartheta_1^2(x)}{\vartheta_0^2(x)}; \quad \ldots\ldots\ldots \quad \text{(i)}$$

or, writing $\frac{\pi x}{2K}$ for x, and attending to the equations (ii) of Art. 9, (xiii) of Art. 3, and (xiii) of Art. 7,

$$k^2\sin^2\mathrm{am}\,x=\frac{J}{K}-\frac{\pi}{2K}\frac{d}{dx}\left[\frac{\vartheta_0'\left(\frac{\pi x}{2K}\right)}{\vartheta_0\left(\frac{\pi x}{2K}\right)}\right]. \quad \ldots\ldots\ldots \quad \text{(ii)}$$

Integrating between the limits 0 and x, we obtain

$$\int_0^x k^2\sin^2\mathrm{am}\,x\,dx=\frac{J}{K}x-\frac{\pi}{2K}\frac{\vartheta_0'\left(\frac{\pi x}{2K}\right)}{\vartheta_0\left(\frac{\pi x}{2K}\right)}, \quad \ldots\ldots\ldots \quad \text{(iii)}$$

where the right-hand member is Jacobi's expression* for the elliptic function of the second species defined by the equation

$$Z(x)=\int_0^x k^2\sin^2\mathrm{am}\,x\,dx. \quad \ldots\ldots\ldots \quad \text{(iv)}$$

The function $Z(x)$ is a one-valued function of x, whatever be the value, real or complex, which we assign to x, and whatever be the course of the integration from the lower to the upper limit in the equation (iv). This may be inferred from the theory of definite integration, since the residue of $\sin^2\mathrm{am}\,x$, corresponding to any one of its infinite points, is zero; or it may be proved by considering the simultaneous equations

$$Z(x)=\int_0^u\frac{k^2u^2\,du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \qquad x=\int_0^u\frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \quad \text{(v)}$$

in each of which the initial value of the radical is + , and the integrations are to follow any one and the same track.

---

* [Jacobi's function $Z(x)$ is equal to $\frac{J}{K}x-\int_0^x d^2\sin^2\mathrm{am}\,x\,dx$, and therefore differs by the term $\frac{J}{K}x$, as well as by a change of sign, from $Z(x)$ as defined in the text. The $Z(x)$ in the text, which is the same as the $Z(x)$ used by M. Hermite, differs only in sign from the $Z(x)$ of Weierstrass.]

From equation (ix) Art. 1, we find

$$\vartheta_0'(\tfrac{1}{2}\pi)=0\,; \qquad \vartheta_0'(\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\omega)=-i\vartheta_0(\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\omega)\,;$$

hence, attributing to x in equation (iii) the values K and $K+iK'$ in succession, we obtain

$$\left.\begin{aligned} Z(K)&=\int_0^K k^2\sin^2\mathrm{am}\,x\,dx=J,\\ Z(K+iK')&=\int_0^{K+iK'}k^2\sin^2\mathrm{am}\,x\,dx=J+\omega J+\frac{i\pi}{2K}=J+iJ'.\end{aligned}\right\} \quad . \quad . \quad \text{(vi)}$$

The equations

$$J=Z(K), \qquad iJ'=Z(K+iK')-Z(K), \quad . \quad . \quad . \quad . \quad . \quad \text{(vii)}$$

are often taken as the equations defining J and J'; we have preferred the definitions which are given in Art. 2, because they exhibit $J(\omega)$ and $J'(\omega)$ more directly as one-valued functions of ω.

A characteristic property of the elliptic function of the second species is expressed by the equation

$$Z(x+2mK+2niK')=Z(x)+2mJ+2niJ', \quad . \quad . \quad . \quad . \quad \text{(viii)}$$

in which m and n are any positive or negative integral numbers. This equation may be verified immediately by means of the equation (iii); or, again, its truth may be inferred from the simultaneous equations (v).

Lastly, we may write

$$J(\omega)=\int_0^1\frac{k^2u^2du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \qquad iJ'(\omega)=\int_1^{\frac{1}{k}}\frac{k^2u^2du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \quad . \quad \text{(ix)}$$

the course of the integration being the same as in the integrals (i) and (ii) of Art. 8.

The second of these equations, by the substitutions (iii) of Art. 8, is changed into

$$J'=\int_0^1\frac{k^2du}{\sqrt{(1-u^2)(1-k'^2u^2)^3}}=\int_0^1\frac{\sqrt{1-k'^2u^2}}{\sqrt{1-u^2}}du, \quad . \quad . \quad . \quad . \quad . \quad \text{(x)}$$

where the course of the integration is the same as in the second integral (iv) of Art. 8. It will be noticed that we have also

$$E=K-J=\int_0^1\frac{k'^2u^2du}{\sqrt{(1-u^2)(1-k^2u^2)^3}}du=\int_0^1\frac{\sqrt{1-k^2u^2}}{\sqrt{1-u^2}}du, \quad . \quad . \quad . \quad . \quad \text{(xi)}$$

which are of the same form as the integrals (x).

The symmetrical substitution

$$u^2 = \frac{1-v^2}{1-k'^2v^2}, \quad \ldots\ldots\ldots\ldots \quad \text{(xii)}$$

which results from combining the inverse of either of the two substitutions (iii) of Art. 8 with the other, changes the integrals (x) into one another, and the second integral (iv) of Art. 8 into itself.

11. *The Differential Coefficients of the Omega Functions.*

Differentiating twice the equations (ii) and (iii) of Art 7, putting $x = 0$ in the result, and substituting for $\vartheta_1'(0)$ from equation (xiii), Art. 3, we find

$$\frac{\vartheta_2''(0)}{\vartheta_2(0)} - \frac{\vartheta_0''(0)}{\vartheta_0(0)} = -\vartheta_3^4(0), \quad \ldots\ldots\ldots \quad \text{(i)}$$

$$\frac{\vartheta_3''(0)}{\vartheta_3(0)} - \frac{\vartheta_0''(0)}{\vartheta_0(0)} = -\vartheta_2^4(0); \quad \ldots\ldots\ldots \quad \text{(ii)}$$

whence also, subtracting, and employing equation (i), Art. 7,

$$\frac{\vartheta_2''(0)}{\vartheta_2(0)} - \frac{\vartheta_3''(0)}{\vartheta_3(0)} = -\vartheta_0^4(0). \quad \ldots\ldots\ldots \quad \text{(iii)}$$

Attending to the partial differential equation of Art. 9, we obtain from these equations the following:

$$\frac{d.\log\sqrt{\frac{k'}{k}}}{d\omega} = \frac{1}{i\pi}K^2, \qquad \frac{d.\log\sqrt{k'}}{d\omega} = \frac{1}{i\pi}k^2K^2, \qquad \frac{d.\log\sqrt{k}}{d\omega} = -\frac{1}{i\pi}k'^2K^2; \quad \text{(iv)}$$

which may be otherwise written

$$\frac{d.k^2}{d\omega} = -\frac{d.k'^2}{d\omega} = -\frac{4}{i\pi}k^2k'^2K^2; \quad \ldots\ldots\ldots \quad \text{(v)}$$

or again, employing the notation of Arts. 2 and 3,

$$\left.\begin{aligned} &\Phi'(\omega) = -\Psi'(\omega) = -\frac{4}{i\pi}\Phi(\omega)\Psi(\omega)K^2, \qquad \frac{\phi'(\omega)}{\phi(\omega)} = -\frac{1}{2i\pi}\Psi(\omega)K^2; \\ &\frac{\psi'(\omega)}{\psi(\omega)} = \frac{1}{2i\pi}\Phi(\omega)K^2, \qquad \frac{\chi'(\omega)}{\chi(\omega)} = \frac{1}{6i\pi}\times[\Phi(\omega)-\Psi(\omega)]K^2. \end{aligned}\right\} \quad \ldots \quad \text{(vi)}$$

Since, by equation (ix) of Art. 3,

$$\vartheta_0[\omega] = \vartheta_0(0) = \sqrt{\frac{2k'K}{\pi}},$$

the equation defining J (Art. 2, iii) may be written in the form

$$\frac{d.k'K}{d\omega} = \frac{2}{i\pi}k'K^2J.$$

Substituting in this equation for $\frac{dk'}{d\omega}$ its value from (iv) or (v), we find

$$\frac{dK}{d\omega} = \frac{2}{i\pi} K^2(J - k^2 K). \quad \ldots \quad \text{(vii)}$$

Again, differentiating twice the equation (i) of Art. 10, and putting $x = 0$ in the result, we have

$$\tfrac{1}{2}\frac{\vartheta_0^{iv}(0)}{\vartheta_0(0)} - \tfrac{3}{2}\left[\frac{\vartheta_0''(0)}{\vartheta_0(0)}\right]^2 = -\vartheta_2^4(0)\,\vartheta_3^4(0); \quad \ldots \quad \text{(viii)}$$

but, by the partial differential equation of Art. 9,

$$\vartheta_0^{iv}(0) = -\frac{16}{\pi^2}\vartheta''[\omega], \qquad \vartheta_0''(0) = \frac{4i}{\pi}\vartheta_0'[\omega];$$

hence the equation (viii) becomes

$$\frac{\vartheta_0''[\omega]}{\vartheta_0[\omega]} - 3\left[\frac{\vartheta_0'[\omega]}{\vartheta_0[\omega]}\right]^2 = \frac{2k^2K^4}{\pi^2},$$

or, finally,

$$\frac{d.KJ}{d\omega} = \frac{2}{i\pi} K^2(J^2 - k^2K^2). \quad \ldots \quad \text{(ix)}$$

In (v), (vii), (ix), we write for simplicity $\omega = i\pi\tau$; we find

$$\frac{d.k^2}{d\tau} = -4k^2k'^2K^2, \quad \frac{d.K^2}{d\tau} = 4K^2(KJ - k^2K^2), \quad \frac{d.KJ}{d\tau} = 2K^2(J^2 - k^2K^2). \quad \text{(x)}$$

These equations enable us to form without difficulty the differential coefficients of the various Omega functions defined in Art. 3; the following are useful formulæ:

$$\left.\begin{aligned}
&\frac{d}{d\tau}\left(\frac{1}{K}\right) = -2(J - k^2K), \qquad \frac{dJ}{d\tau} = 2k^2K^2(J - K),\\
&\frac{d}{d\tau}\left(\frac{J}{K}\right) = -2(J^2 - 2k^2KJ + k^2K^2),\\
&\frac{d.\log k}{d\tau} = -2k'^2K^2, \qquad \frac{d.\log k'}{d\tau} = 2k^2K^2,\\
&\frac{d.\log kk'}{d\tau} = 2(2k^2 - 1)K^2, \qquad \frac{d.\log\left(\frac{k'}{k}\right)}{d\tau} = 2K^2,\\
&\tfrac{1}{2}\frac{d.\log K}{d\tau} = K(J - k^2K), \qquad \tfrac{1}{2}\frac{d.\log kK}{d\tau} = K(J - K) = -KE,\\
&\tfrac{1}{2}\frac{d.\log k'K}{d\tau} = KJ, \qquad \tfrac{1}{2}\frac{d.\log kk'K}{d\tau} = K(J - k'^2K).
\end{aligned}\right\} \quad \text{(xi)}$$

When ω is a pure imaginary (in which case τ is real and positive), the Omega functions of Art. 3, as has been already said, are all real and positive. Hence $\frac{d\psi(\omega)}{d\tau}$ is always positive, $\frac{d\phi(\omega)}{d\tau}$ is always negative; so that, as τ increases from zero to infinity, $\psi(\omega)$ continually increases from 0 to 1, and $\phi(\omega)$ continually decreases from 1 to 0. On the same hypothesis, K continually decreases from ∞ to $\frac{1}{2}\pi$, J from ∞ to 0, $E'=J'$ from $\frac{1}{2}\pi$ to 1; K' continually increases from $\frac{1}{2}\pi$ to ∞, and $E=K-J$ from 1 to $\frac{1}{2}\pi$. The demonstration of these assertions depends partly on the expressions for K, K', J, J', E, E' as definite integrals (equations (iv), Art. 8; (x), (xi), (xii), Art. 10), partly on the differential formulæ (x) and (xi) of this article.

If $n > 1$, it follows from the formulæ (x), that $\frac{d^n.k^2}{d\tau^n}$, $\frac{d^n.K^2}{d\tau^n}$, and $\frac{d^n.KJ}{d\tau^n}$ are respectively of the forms

$$2^{n+2}k^2k'^2K^{n+1}V_{n-1},\ 2^{n+1}K^{n+2}V_n,\ \text{and}\ 2^{n+1}K^{n+1}V_{n+1};$$

where V_{n-1}, V_n, and V_{n+1} denote integral and homogeneous functions of K and J, of the orders $n-1$, n, and $n+1$, in which the coefficients are rational and integral functions of k^2, with integral numerical coefficients.

More generally, if $P=k^{\frac{2}{a}}k'^{\frac{2}{b}}K^{\frac{2}{c}}$, $\frac{1}{P}\times\frac{d^nP}{d\tau^n}$ is of the form $K^n\times V_n$, the numerical coefficients of the powers of k^2 in V_n being, in general, fractional; these coefficients, however, are integral if $\frac{4}{a}$, $\frac{4}{b}$, $\frac{4}{c}$ are all integral.

It will be observed that no new transcendent is obtained by differentiating the Omega functions, since all their differential coefficients of any order can be expressed in terms of k^2, K, and J.

12. *The Differential Equations satisfied by the Omega Functions.*

Eliminating $d\tau$ from the last two of the equations (x), Art. 11, by means of the first, we find

$$\frac{dK}{dk}=-\frac{1}{kk'^2}(J-k^2K),\qquad \frac{dJ}{dk}=\frac{k}{k'^2}(K-J)=\frac{k}{k'^2}E;\quad\ldots\quad\text{(i)}$$

whence by subtraction

$$\frac{J}{k}=-\frac{dE}{dk}=\frac{d(J-K)}{dk}.\quad\ldots\quad\text{(ii)}$$

Eliminating in turn J and K from the equations (i), we have

$$(1-k^2)\frac{d^2K}{dk^2}+\frac{(1-3k^2)}{k}\frac{dK}{dk}-K=0, \quad \ldots\ldots\ldots \text{(iii)}$$

$$(1-k^2)\frac{d^2J}{dk^2}-\frac{1+k^2}{k}\frac{dJ}{dk}+J=0, \quad \ldots\ldots\ldots \text{(iv)}$$

which are the differential equations satisfied by K and J regarded as functions of k.

Observing that by the definitions of Art. 2,

$$K'=\frac{\omega}{i}K, \quad J'=\frac{\omega}{i}J+\frac{\pi}{2K},$$

and that by equation (v) of Art. 11,

$$\frac{d\omega}{dk}=-\frac{i\pi}{2kk'^2K^2},$$

we find from the equations (i)

$$\frac{dK'}{dk}=-\frac{1}{kk'^2}(J'-k^2K'), \quad \frac{dJ'}{dk}=\frac{k}{k'^2}(K'-J'). \quad \ldots\ldots \text{(v)}$$

These equations are of the same form as the equations (i), but contain K', J', instead of K, J. The differential equations (iii) and (iv) are therefore respectively satisfied by K' and J'; and the expressions

$$CK+C'K', \quad CJ+C'J'$$

are the complete solutions of those differential equations.

We have deduced this result from the definitions of K, K', J, J' as one-valued functions of ω without making any use of the expressions of these functions as definite integrals. A different demonstration may be obtained by employing the definite integrals; which, however, it must be remembered, cannot be used to define the one-valued functions. Writing, for brevity, Δu for $\sqrt{(1-u^2)(1-k^2u^2)}$, let us designate by P and Q any two corresponding values of the integrals

$$P=\int_0^1\frac{du}{\Delta u}, \quad Q=\int_0^1\frac{k^2u^2du}{\Delta u} \quad \ldots\ldots\ldots \text{(vi)}$$

(*i.e.* any two values in which the track of the integration is the same). It is not difficult to show that the equations (i), and hence also the equations (iii) and (iv), are verified if we write P for K and Q for J; for this purpose

we have only to differentiate the equations (vi) with regard to k, and to make use of the identities

$$\frac{k'^2}{k}\frac{d}{dk}\left(\frac{1}{\Delta u}\right)=\frac{1-u^2}{\Delta u}-\frac{d}{du}\left(\frac{u-u^3}{\Delta u}\right), \quad \ldots\ldots \text{(vii)}$$

$$\frac{k'^2}{k}\frac{d}{dk}\left(\frac{k^2u^2}{\Delta u}\right)=\frac{1-k^2u^2}{\Delta u}-\frac{d}{du}\left(\frac{u-u^3}{\Delta u}\right). \quad \ldots\ldots \text{(viii)}$$

The equations (iii) and (iv) are therefore satisfied by K and J, because these are values of P and Q respectively; but so also are $(2n+1)K+2miK'$, $(2n+1)J+2miJ'$; *i.e.* the equations (iii) and (iv) are respectively satisfied by K' and J' as well as by K and J.

Since the integral expressions for J and J' become $K'-E'$ and E respectively, when k^2 is changed into k'^2, we find that the functions E and $K'-E'$ satisfy the differential equation

$$(1-k'^2)\frac{d^2E}{dk'^2}-\frac{1+k'^2}{k'}\frac{dE}{dk'}+E=0,$$

or

$$(1-k^2)\frac{d^2E}{dk^2}+\frac{1-k^2}{k}\frac{dE}{dk}+E=0, \quad \ldots\ldots \text{(ix)}$$

and that the complete solution of this equation is $CE+C'(K'-E')$.

If V is any algebraical function of k, K, and J (*i.e.* a function defined by an algebraical equation of which the coefficients are rational functions of k, K, and J), V satisfies a differential equation of the second order containing only V, k, and the first two differential coefficients of V with regard to k, and a differential equation of the third order containing only V and its first three differential coefficients with regard to ω or τ. The differential equations satisfied by K and J are instances of the former assertion; as instances of the latter we may take the differential equations satisfied by K and k regarded as functions of τ*. These differential equations, which have been given by Jacobi, are as follows (we write c for $\frac{1}{K}$, and y for k, and we represent by accents the differential coefficients of c and y with regard to τ):

$$c^2(cc'''+3c'c'')^2=16c''^2(1+c^3c''); \quad \ldots\ldots \text{(x)}$$

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*Jacobi, 'Ueber die Differentialgleichung, welcher die Reihen $1\pm2q+2q^4\pm2q^9+\ldots$, $2q^{\frac{1}{4}}+2q^{\frac{9}{4}}+2q^{\frac{25}{4}}+\ldots$, Genüge leisten:' Crelle's Journal, vol. xxxvi. p. 97.

$$2y'y''' - 3y''^2 + \left(\frac{1+y^2}{y-y^3}\right)^2 y'^4 = 0. \quad \ldots\ldots\ldots \text{(xi)}$$

To obtain the equation (x) we deduce from (x) or (xi), Art. 11, the equation

$$c^3c'' = -4k^2k'^2,$$

and we eliminate k^2 between this equation and its first derived equation with regard to τ, viz.

$$c^4(cc''' + 3c'c'') = 16k^2k'^2(1-2k^2).$$

If we put $z = \sqrt[4]{\pi}\sqrt{\frac{2K}{\pi}}$, and introduce ω instead of τ as the independent variable, the equation (x) becomes

$$(z^2z''' - 15zz'z'' + 30z'^3)^2 + 32(zz'' - 3z'^2)^3 = z^{10}(zz'' - 3z'^2). \quad \ldots \text{(xii)}$$

To verify the equation (xi) we write it in the form

$$2\frac{d^2}{d\tau^2}\left[\log\frac{dy}{d\tau}\right] - \left[\frac{d}{d\tau}\log\frac{dy}{d\tau}\right]^2 + \left(\frac{1+y^2}{y-y^3}\right)^2\frac{dy^2}{d\tau^2} = 0, \quad \ldots\ldots \text{(xiii)}$$

and we substitute in it the values of the first two differential coefficients of $\log\frac{dy}{d\tau}$, deduced from the equations (x) and (xi) of Art. 11, viz.

$$\frac{d}{d\tau}\left[\log\frac{dy}{d\tau}\right] = 2K(2J - k'^2K),$$

$$\frac{d^2}{d\tau^2}\left[\log\frac{dy}{d\tau}\right] = 8K^2(J-K)(J+k^2K).$$

The integrals $z = \sqrt[4]{\pi} \times \sqrt{\frac{2K}{\pi}}$, $y = k$, are, of course, only particular integrals of the equations (xii) and (xiii). We proceed to give the general integrals of these equations, which have been assigned by Jacobi; the form of the solution (as we shall see later) is suggested by the theory of the transformation of the Theta and Omega functions.

Let $\omega = \frac{c+d\Omega}{a+b\Omega}$, the quantities a, b, c, d being supposed real, and $ad - bc = n$ being positive, so that the real part of $i\Omega$ is negative when the real part of $i\omega$ is negative, and *vice versâ*.

Replacing ω by Ω, we find from the equations (v), (vii), (ix) of Art. 11:

$$\left.\begin{aligned}
\frac{d\Phi(\Omega)}{d\Omega} &= \frac{4i}{\pi}\Phi(\Omega)\times\Psi(\Omega)\times K^2(\Omega);\\
\frac{dK(\Omega)}{d\Omega} &= -\frac{2i}{\pi}[J(\Omega)-\Phi(\Omega)K(\Omega)]\times K^2(\Omega);\\
\frac{dJ(\Omega)}{d\Omega} &= -\frac{2i}{\pi}\Phi(\Omega)[J(\Omega)-K(\Omega)]\times K^2(\Omega).
\end{aligned}\right\} \quad \ldots\ldots \quad \text{(xiv)}$$

Let

$$K_1(\Omega)=\frac{a+b\Omega}{\sqrt{n}}K(\Omega), \qquad J_1(\Omega)=\frac{a+b\Omega}{\sqrt{n}}J(\Omega)+\frac{bi\pi}{2\sqrt{n}}\frac{1}{K(\Omega)};$$

observing that $\frac{d\Omega}{d\omega}=\left(\frac{a+b\Omega}{\sqrt{n}}\right)^2$, we find that the equations (xiv) become

$$\left.\begin{aligned}
\frac{d\Phi(\Omega)}{d\omega} &= \frac{4i}{\pi}\Phi(\Omega)\times\Psi(\Omega)\times K_1^2(\Omega),\\
\frac{dK_1(\Omega)}{d\omega} &= -\frac{2i}{\pi}[J_1(\Omega)-\Phi(\Omega)K_1(\Omega)]\times K_1^2(\Omega),\\
\frac{dJ_1(\Omega)}{d\omega} &= -\frac{2i}{\pi}\Phi(\Omega)[J_1(\Omega)-K_1(\Omega)]\times K_1^2(\Omega).
\end{aligned}\right\} \quad \ldots\ldots \quad \text{(xv)}$$

Hence any differential equation, derived by differentiation and elimination from the equations (x) and (xi) of Art. 11, will subsist unchanged if we write simultaneously

$$\Phi\left(\frac{c-a\omega}{-d+b\omega}\right) \text{ for } \Phi(\omega),$$

$$\frac{\sqrt{n}}{-b\omega+d}\times K\left(\frac{c-a\omega}{-d+b\omega}\right) \text{ for } K(\omega),$$

and

$$\frac{\sqrt{n}}{-b\omega+d}\times J\left(\frac{c-a\omega}{-d+b\omega}\right)+\frac{bi\pi}{2\sqrt{n}}\div K\left(\frac{c-a\omega}{-d+b\omega}\right) \text{ for } J(\omega).$$

When $d\omega$ is the equicrescent differential, as it is in the equations (xii) and (xi), this observation enables us to assign the general solution of the differential equation when a particular solution is known. Thus the particular solution of the equation (xii), which we have found, is

$$z=\sqrt[4]{\pi}(1+2e^{i\pi\omega}+2e^{4i\pi\omega}+\ldots\ldots)=\sqrt[4]{\pi}\times\sqrt{\frac{2K}{\pi}}.$$

If then we put

$$\frac{a}{\sqrt{n}}=\delta,\ \frac{b}{\sqrt{n}}=-\beta,\ \frac{c}{\sqrt{n}}=-\gamma,\ \frac{d}{\sqrt{n}}=\alpha,\ \text{so that}\ \Omega=\frac{\gamma+\delta\omega}{\alpha+\beta\omega},\ \alpha\delta-\beta\gamma=1,$$

the general solution of the equation (xii), containing three arbitrary constants, is

$$z=\frac{\sqrt[4]{\pi}}{\sqrt{\alpha+\beta\omega}}(1+2e^{i\pi\Omega}+2e^{4i\pi\Omega}+\ \dots\ )=\sqrt[4]{\pi}\times\sqrt{\frac{2}{\pi}\frac{K\left(\frac{\gamma+\delta\omega}{\alpha+\beta\omega}\right)}{\alpha+\beta\omega}}.$$

Again, the equation (xi) or (xiii) remains unchanged when we take $d\omega$ as the equicrescent differential instead of $d\tau$; hence a particular solution of that equation being $y=\sqrt{\Phi(\omega)}$, its general solution is

$$y=\sqrt{\Phi\left(\frac{\gamma+\delta\omega}{\alpha+\beta\omega}\right)}.$$

13. *The Abelian Functions.*

Integrating the equation (iii) of Art. 10, between the limits 0 and x, and taking the exponential function of each member, we find

$$e^{-\frac{1}{2}\frac{J}{K}x^2}\times\frac{\vartheta_0\left(\frac{\pi x}{2K}\right)}{\vartheta_0(0)}=e^{-\int_0^x Z(x)dx};\ \dots\dots\dots\ \text{(i)}$$

in which the value of the right-hand side is independent of the track of the integration, because the different values of which $\int_0^x Z(x)dx$ is susceptible differ only by multiples of $2i\pi$. The expression for $\vartheta_0\left(\frac{\pi x}{2K}\right)$ at which we have thus arrived, is that by which Jacobi originally defined the function $\Theta(x)=\vartheta_0\left(\frac{\pi x}{2K}\right)$ in the 'Fundamenta Nova'; the importance of this expression in the theory will be seen from the following observations.

The quantity ω enters into the right-hand side of the equation (i) only in $k^2\sin^2\mathrm{am}\,x$; the equation (xxv) of Art. 7 shows that x can be developed in a series proceeding by powers of $u=\sin\mathrm{am}\,x$, of which the coefficients are rational and integral functions of k^2; again, this series gives by reversion a development of $\sin\mathrm{am}x$ in a series proceeding by powers of x, and having coefficients which are, in like manner, rational and integral functions

of k^2. The same thing is therefore true of $Z(x)$, of $\int_0^x Z(x)dx$, and, lastly, of $e^{-\int_0^x Z(x)\,dx}$. But whereas the developments of $\sin \mathrm{am}\, x$, $\sin^2 \mathrm{am}\, x$, $Z(x)$, and $\int_0^x Z(x)\,dx$ cannot be convergent for values of x of which the analytical modulus surpasses a certain limit, the function

$$= e^{-\int_0^x Z(x)\,dx}$$

gives rise to a development which is necessarily convergent for all finite values of x, real and imaginary. For if, in the cosine-development of $\vartheta_0\left(\frac{\pi x}{2K}\right)$ (see equation (iv), Art. 3), we expand each cosine in a series proceeding by powers of x, we obtain an equation which we may write in the form

$$\left.\begin{aligned}\frac{\vartheta_0\left(\frac{\pi x}{2K}\right)}{\vartheta_0(0)} &= 1 + \frac{1}{1.2}\frac{\vartheta_0''(0)}{\vartheta_0(0)}\left(\frac{\pi x}{2K}\right)^2 + \frac{1}{1.2.3.4}\frac{\vartheta_0^{iv}(0)}{\vartheta_0(0)}\left(\frac{\pi x}{2K}\right)^4 + \dots \\ &= 1 + \frac{1}{1.2}\frac{T_0'[\tau]}{T_0[\tau]}\frac{x^2}{K^2} + \frac{1}{1.2.3.4}\frac{T_0''[\tau]}{T_0[\tau]}\frac{x^4}{K^4} + \dots\end{aligned}\right\}, \quad . \quad \text{(ii)}$$

if we again put $\omega = i\pi\tau$, and denote the series

$$\vartheta_0[\omega] = \vartheta_0(0, q) = 1 - e^{-\pi^2\tau} + e^{-4\pi^2\tau} - e^{-9\pi^2\tau} + \dots$$

by $T_0[\tau]$. The expansion (ii) is convergent for all finite values of x; so is also the expansion of $e^{-\frac{1}{2}\frac{J}{K}x^2}$; and so, consequently, is the expansion of $e^{-\int_0^x Z(x)\,dx}$, which is obtained by multiplying the two together.

We shall now (after M. Weierstrass *) represent by $\mathrm{Al}_0(x)$ the function

$$\mathrm{Al}_0(x) = e^{-\frac{1}{2}\frac{J}{K}x^2} \times \frac{\vartheta_0\left(\frac{\pi x}{2K}\right)}{\sqrt{\frac{2k'K}{\pi}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

which appears in the left-hand member of the equation (i); we shall write

$$\mathrm{Al}_0(x) = 1 - A_1^0\frac{x^2}{1.2} + A_2^0\frac{x^4}{1.2.3.4} - \dots; \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

* Weierstrass, 'Theorie der Abel'schen Functionen,' Crelle's Journal, vol. lii.

and we shall show by a direct method that the coefficients A_1^0, A_2^0, A_3^0, ..., are rational and integral functions of k^2 having integral numerical coefficients. For this purpose we transform the partial differential equation (i) of Art. 9 into an equation involving $\mathrm{Al}_0(x)$ and k^2, instead of $\vartheta_0(x)$ and ω. The partial differential equation gives immediately

$$\left[\frac{d.\vartheta\left(\frac{\pi x}{2K}\right)}{d\tau}\right] = K^2 \frac{d^2\vartheta\left(\frac{\pi x}{2K}\right)}{dx^2},$$

where, on the left-hand side, we have included the partial differential coefficients in brackets, in order to indicate that K is regarded as a constant in the differentiation.

Substituting for $\left[\frac{d.\vartheta\left(\frac{\pi x}{2K}\right)}{d\tau}\right]$ its value $\frac{d.\vartheta\left(\frac{\pi x}{2K}\right)}{d\tau} - \frac{\pi x}{2}\vartheta'\left(\frac{\pi x}{2K}\right)\frac{d}{d\tau}\left(\frac{1}{K}\right)$, we find

$$\frac{d.\vartheta\left(\frac{\pi x}{2K}\right)}{d\tau} = -\frac{x}{K}\frac{d.\vartheta\left(\frac{\pi x}{2K}\right)}{dx}\frac{dK}{d\tau} + K^2\frac{d^2.\vartheta\left(\frac{\pi x}{2K}\right)}{dx^2}.$$

Writing, in this equation, for $\vartheta_0\left(\frac{\pi x}{2K}\right)$ the equivalent expression

$$\sqrt{\frac{2k'K}{\pi}} \times e^{\frac{1}{2}\frac{J}{K}x^2} \times \mathrm{Al}_0(x),$$

effecting the differentiations, introducing (from the equations (x) and (xi) of Art. 11) the values of $\frac{dK}{d\tau}$, $\frac{d.\sqrt{k'K}}{d\tau}$, $\frac{d}{d\tau}\left(\frac{J}{K}\right)$, and dividing the resulting equation by

$$\sqrt{\frac{2k'K}{\pi}} \times e^{\frac{1}{2}\frac{J}{K}x^2} \times K^2,$$

we obtain the partial differential equation required, viz.

$$\frac{d^2.\mathrm{Al}_0(x)}{dx^2} + 2k^2x\frac{d.\mathrm{Al}_0(x)}{dx} + 4k^2k'^2\frac{d.\mathrm{Al}_0(x)}{d.k^2} + k^2x^2\mathrm{Al}_0(x) = 0, \quad . \quad . \quad . \quad \text{(v)}$$

This equation gives, for the successive calculation of the coefficients

A_1^0, A_2^0, ... the following equation of mixed differences and differentials

$$A_{n+1}^0 - 4nk^2A_n^0 - 4k^2k'^2\frac{d.A_n^0}{d.k^2} + 2n(2n-1)k^2A_{n-1}^0 = 0, \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

whence $A_1^0 = 0$, $A_2^0 = -2k^2$, $A_3^0 = -8(k^2+k^4)$, &c.

The equation (vi) shows, (1) that the coefficients of the powers of k^2 in A_n^0 are integral numbers; (2) that, if $n>0$, A_n^0 contains no power of k^2 higher than $k^{2(n-1)}$ and no term free from k^2. Hence, if $n>1$, we may denote the coefficient of $k^{2(\mu+1)}$ in $A_{\mu+\nu+2}^0$ by $\alpha_{\mu,\nu}$, where $\mu \geqq 0$, $\nu \geqq 0$; and we then find from the equation (vi),

$$\alpha_{\mu,\nu} = 4(\mu+1)\alpha_{\mu,\nu-1} + 4(\nu+1)\alpha_{\mu-1,\nu} - 2(\mu+\nu+1)(2\mu+2\nu+1)\alpha_{\mu-1,\nu-1}. \quad . \quad \text{(vii)}$$

This equation of partial differences supplies a formula of reversion by which the values of the coefficients $\alpha_{\mu,\nu}$ may be successively calculated. It is symmetrical with regard to the two indices μ and ν; and, since the value of $A_3^0 = -8(k^2+k^4)$ shows that $\alpha_{0,1} = -8 = \alpha_{1,0}$, we have generally $\alpha_{\mu,\nu} = \alpha_{\nu,\mu}$. This implies that the coefficients A_n^0 are *reciprocal* functions of k^2 of the order $n-1$, or, which is the same thing, that

$$\mathrm{Al}_0\left(kx, \frac{1}{k}\right) = \mathrm{Al}_0(x, k).$$

The function $\mathrm{Al}_0(x)$, of which we have now obtained the development

$$\mathrm{Al}_0(x) = 1 + \Sigma(-)^{\mu+\nu}\alpha_{\mu,\nu}k^{2\mu+2}\frac{x^{2\mu+2\nu+4}}{\Pi(2\mu+2\nu+4)}, \quad . \quad . \quad . \quad . \quad \text{(viii)}$$

is one of the four Abelian functions of M. Weierstrass. The three others are defined by the equations

$$\left.\begin{aligned}
\mathrm{Al}_1(x) &= x - A_1'\frac{x^3}{1.2.3} + A_2'\frac{x^5}{1.2.3.4.5} - A_3'\frac{x^7}{1.2.3.4.5.6.7} \ldots = e^{-\frac{1}{2}\frac{J}{K}x^2} \times \frac{\vartheta_1\left(\frac{\pi x}{2K}\right)}{\sqrt{\frac{2kk'K}{\pi}}}, \\
\mathrm{Al}_2(x) &= 1 - A_1''\frac{x^2}{1.2} + A_2''\frac{x^4}{1.2.3.4} - A_3''\frac{x^6}{1.2.3.4.5.6} + \ldots = e^{-\frac{1}{2}\frac{J}{K}x^2} \times \frac{\vartheta_2\left(\frac{\pi x}{2K}\right)}{\sqrt{\frac{2kK}{\pi}}}, \\
\mathrm{Al}_3(x) &= 1 - A_1'''\frac{x^2}{1.2} + A_2'''\frac{x^4}{1.2.3.4} - A_3'''\frac{x^6}{1.2.3.4.5.6} + \ldots = e^{-\frac{1}{2}\frac{J}{K}x^2} \times \frac{\vartheta_3\left(\frac{\pi x}{2K}\right)}{\sqrt{\frac{2K}{\pi}}},
\end{aligned}\right\} \text{(ix)}$$

so that

$$\sin \operatorname{am} x = \frac{\mathrm{Al}_1(x)}{\mathrm{Al}_0(x)}, \qquad \cos \operatorname{am} x = \frac{\mathrm{Al}_2(x)}{\mathrm{Al}_0(x)}, \qquad \Delta \operatorname{am} x = \frac{\mathrm{Al}_3(x)}{\mathrm{Al}_0(x)}; \quad \ldots \quad \text{(x)}$$

and, if we use the notation of the Tables,

$$\mathrm{Al}_s(x) = e^{-\frac{1}{2}\frac{J}{K}x^2} \times \frac{\Theta_s(x)}{\sqrt{\frac{2k'K}{\pi}}}. \quad \ldots \quad \text{(xi)}$$

Since, for values of x which do not surpass a certain limit, the three elliptic functions $\sin \operatorname{am} x$, $\cos \operatorname{am} x$, $\Delta \operatorname{am} x$ are all developable in series proceeding by powers of x, of which the coefficients are rational and integral functions of k^2, it is evident *à priori* that the same thing must be true for the functions

$$\mathrm{Al}_1(x) = \mathrm{Al}_0(x) \times \sin \operatorname{am} x,$$
$$\mathrm{Al}_2(x) = \mathrm{Al}_0(x) \times \cos \operatorname{am} x,$$
$$\mathrm{Al}_3(x) = \mathrm{Al}_0(x) \times \Delta \operatorname{am} x;$$

while the equations (ix), by which we have defined these functions, show that their developments must be convergent for all finite values of x. To obtain these developments, we first deduce from the partial differential equation of Art. 9, by precisely the same process which we have indicated in the case of $\mathrm{Al}_0(x)$, a partial differential equation for each of the three functions, containing only k^2 and not ω. These equations are of the type

$$\frac{d^2.\mathrm{Al}_s(x)}{dx^2} + 2k^2x\frac{d.\mathrm{Al}_s(x)}{dx} + 4k^2k'^2\frac{d.\mathrm{Al}_s(x)}{d.k^2} + G_s \times \mathrm{Al}_s(x) = 0, \quad \ldots \quad \text{(xii)}$$

where $G_1 = k'^2 + k^2x^2$, $G_2 = 1 + k^2x^2$, $G_3 = k^2 + k^2x^2$. Treated in the same manner as equation (v) they lead to the developments

$$\left.\begin{aligned}
\mathrm{Al}_1(x) &= \Sigma(-)^{\mu+\nu} b_{\mu,\nu} k^{2\mu} \frac{x^{2\mu+2\nu+1}}{\Pi(2\mu+2\nu+1)}, \\
\mathrm{Al}_2(x) &= 1 - \Sigma(-)^{\mu+\nu} c_{\mu,\nu} k^{2\mu} \frac{x^{2\mu+2\nu+2}}{\Pi(2\mu+2\nu+2)}, \\
\mathrm{Al}_3(x) &= 1 - \Sigma(-)^{\mu+\nu} c_{\mu,\nu} k^{2\nu+2} \frac{x^{2\mu+2\nu+2}}{\Pi(2\mu+2\nu+2)},
\end{aligned}\right\} \quad \ldots \quad \text{(xiii)}$$

where

$$\left.\begin{aligned}
b_{\mu,\nu} &= (4\mu+1)b_{\mu,\nu-1} + (4\nu+1)b_{\mu-1,\nu} - (2\mu+2\nu-1)(2\mu+2\nu-2)b_{\mu-1,\nu-1}, \\
c_{\mu,\nu} &= (4\mu+1)c_{\mu,\nu-1} + (4\nu+1)c_{\mu-1,\nu} - (2\mu+2\nu)(2\mu+2\nu-1)c_{\mu-1,\nu-1},
\end{aligned}\right\} \quad \text{(xiv)}$$

so that $b_{\mu,\nu} = b_{\nu,\mu}$, or the coefficients of the powers of x in $\mathrm{Al}_1(x)$ are reciprocal functions of k^2, as in the case of $\mathrm{Al}_0(x)$, the coefficients of the powers of x in $\mathrm{Al}_2(x)$ and $\mathrm{Al}_3(x)$ being reciprocal to one another. We thus have the equations

$$\mathrm{Al}_1\left(kx, \frac{1}{k}\right) = \mathrm{Al}_1(x, k), \quad \mathrm{Al}_2\left(kx, \frac{1}{k}\right) = \mathrm{Al}_3(x, k), \quad \mathrm{Al}_3\left(kx, \frac{1}{k}\right) = \mathrm{Al}_2(x, k).$$

14. *Differential Coefficients of the Omega Functions expressed by means of the Abelian Coefficients.*

The differential coefficients of the four Omega functions

$$\sqrt{\frac{2k'K}{\pi}}, \quad \sqrt{\frac{2kk'K}{\pi}}, \quad \sqrt{\frac{2kK}{\pi}}, \quad \sqrt{\frac{2K}{\pi}},$$

can be expressed in terms of the coefficients of the powers of x in the expansions of the four Abelian functions.

We have

$$T_0[\tau] = \sqrt{\frac{2k'K}{\pi}} = \vartheta_0(0, q), \qquad T_0'[\tau] = T_0[\tau] \times KJ$$

(equation (xi) Art. 11); also, by a theorem in Art. 11, we may write

$$\frac{1}{K^{2n}} \frac{T_0^{(n)}[\tau]}{T_0[\tau]} = f_n\left(\frac{J}{K}\right), \quad \ldots \quad \text{(i)}$$

where $f_n\left(\frac{J}{K}\right)$ is a rational and integral function of $\frac{J}{K}$ and k^2, with integral numerical coefficients, and of the order n in $\frac{J}{K}$. The equations (ii), (iii), and (iv) of Art. 13 now give

$$\left.\begin{aligned} &1 + f_1\left(\frac{J}{K}\right)\frac{x^2}{1.2} + f_2\left(\frac{J}{K}\right)\frac{x^4}{1.2.3.4} + \ldots \\ &\quad = \left[1 + \tfrac{1}{2}\frac{J}{K}x^2 + \tfrac{1}{4}\frac{J^2}{K^2}\frac{x^4}{1.2} + \ldots\right] \times \left[1 - A_1^0\frac{x^2}{1.2} + A_2^0\frac{x^4}{1.2.3.4} - \ldots\right], \end{aligned}\right\} \quad \text{(ii)}$$

whence, on equating coefficients,

$$f_1\left(\frac{J}{K}\right) = \frac{J}{K} + A_1^0 = \frac{J}{K}, \quad \text{since } A_1^0 = 0;$$

$$f_2\left(\frac{J}{K}\right) = 1.3\frac{J^2}{K^2} + A_2^0 = 1.3\frac{J^2}{K^2} - 2k^2,$$

and, in general,

$$\left.\begin{aligned} f_n\left(\frac{J}{K}\right) = (2n-1)!\,\frac{J^n}{K^n} - (2n-3)!\,C_2^{2n}\,A_1^0\,\frac{J^{n-1}}{K^{n-1}} + (2n-5)!\,C_4^{2n}\,A_2^0\,\frac{J^{n-2}}{K^{n-2}} \\ - (2n-7)!\,C_6^{2n}A_3^0\,\frac{J^{n-3}}{K^{n-3}} + \ldots + (-)^{n-1}n(2n-1)A_{n-1}^0\,\frac{J}{K} + (-)^n A_n^0, \end{aligned}\right\} \text{(iii)}$$

where $(2r-1)!$ is the continued product of the uneven numbers $1, 3, 5, \ldots 2r-1$, and C_r^s is the coefficient of x^r in the expansion of $(1+x)^s$.

If in the right-hand side of the formula (iii) we take successively (instead of $A_1^0, A_2^0, \ldots$) the coefficients $A_1'', A_2'', \ldots A_1''', A_2''', \ldots$ in the expansions of $\mathrm{Al}_2(x)$ and $\mathrm{Al}_3(x)$, we obtain expressions for

$$\frac{1}{K^{2n}}\,\frac{T_2^{(n)}[\tau]}{T_2[\tau]} \text{ and } \frac{1}{K^{2n}}\,\frac{T_3^{(n)}[\tau]}{T_3[\tau]},$$

where $T_2[\tau] = \vartheta_2(0, q)$, and $T_3[\tau] = \vartheta_3(0, q)$.

The expansion of the uneven function $\mathrm{Al}_1(x)$ leads to a slightly different formula. Representing $\frac{1}{2K}\,\frac{T_1^{(n)}(\tau)}{T_1(\tau)}$, where $T_1(\tau) = \frac{2K}{\pi} \times \sqrt{\frac{2kk'K}{\pi}} = \vartheta_1'(0, q)$, by $f_n^1\left(\frac{J}{K}\right)$, we obtain by the same process as before

$$\left.\begin{aligned} f_n^1\left(\frac{J}{K}\right) = (2n-1)!\,C_1^{2n+1}\,\frac{J^n}{K^n} - (2n-3)!\,C_3^{2n+1}A_1'\,\frac{J^{n-1}}{K^{n-1}} + (2n-5)!\,C_5^{2n+1}A_2'\,\frac{J^{n-2}}{K^{n-2}} \\ - \ldots + (-)^{n-1}n\,(2n+1)A_{n-1}'\,\frac{J}{K} + (-)^n A_n' = 0. \end{aligned}\right\} \text{(iv)}$$

It will be seen that $(-)^n A_n^0$ is the term not containing $\frac{J}{K}$ in $f_n\left(\frac{J}{K}\right)$. Thus, without using the partial differential equation (v) of Art. 13, we may obtain the equation (vi) of that article by differentiating the equation

$$T_0^{(n)}[\tau] = K^{2n}T_0[\tau] \times f_n\left(\frac{J}{K}\right),$$

and putting $\frac{J}{K} = 0$ in the result. A similar remark applies to the coefficients A_n', A_n'', A_n'''.

Arts. 15–23. Arithmetical Theor of Binary Matrices.

The theory of the Transformation of the Theta functions and Omega functions depends in great measure upon the arithmetical theory of binary matrices, of which the constituents are integral numbers. We shall therefore give in this place an account of such properties of these matrices as we require for our present purpose. We omit many of the demonstrations, on account of the elementary nature of the subject.

15. *Composition of Matrices.*

$$\text{If} \quad |A'| = \begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix}, \text{ and } |A''| = \begin{vmatrix} a'', & b'' \\ c'', & d'' \end{vmatrix}$$

the matrix

$$|A| = \begin{vmatrix} a'a''+b'c'', & a'b''+b'd'' \\ c'a''+d'c'', & c'b''+d'd'' \end{vmatrix}$$

is said to be compounded of the matrices $|A|$ and $|A'|$; and this composition is expressed by the equation

$$|A| = |A'| \times |A''|.$$

The determinant of the compounded matrix is the product of the determinants of the component matrices.

It will be observed that the order of composition is not indifferent; or, which is the same thing, the matrices $|A''| \times |A'|$ and $|A'| \times |A''|$ are not, in general, identical. In the expression $|A'| \times |A''|$ the matrix $|A'|$ is said to be *postmultiplied* by the matrix $|A'|$, and $|A''|$ is said to be *premultiplied* by $|A'|$.

The composition of matrices differs from the multiplication of numbers in the respect just mentioned. But in the composition of matrices the components (as in the multiplication of numbers the factors) may be grouped together in any way we please, provided that the order in which they succeed one another be not changed; for example, $|A| \times |B| \times |C|$ is either $|A| \times |B|$ postmultiplied by $|C|$, or $|A|$ postmultiplied by $|B| \times |C|$.

16. *Unit Matrices, Primitive Matrices, Reciprocal Matrices.*

We shall have occasion to consider only those matrices of which the determinants are positive numbers different from zero.

A matrix of which the determinant is $+1$ is an *unit matrix*; a matrix of determinant n, of which the four constituents have no common divisor,

is a *primitive matrix* of determinant n. If n has no square divisors, every matrix of determinant n is necessarily primitive.

The matrix compounded of two primitive matrices, of which the determinants are relatively prime, is always a primitive matrix.

The matrices $\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right|$ and $\left|\begin{matrix} d, & -b \\ -c, & a \end{matrix}\right|$ are said to be *reciprocal* matrices; and the matrix reciprocal to the matrix $|A|$ is sometimes represented by the symbol $|A|^{-1}$. Reciprocal matrices have the same determinant. The result of compounding two reciprocal matrices is the same in whatever order the composition is effected; in fact

$$|A| \times |A|^{-1} = |A|^{-1} \times |A| = \left|\begin{matrix} n, & 0 \\ 0, & n \end{matrix}\right|,$$

n being the common determinant of the two matrices.

The matrix reciprocal to $|A| \times |B|$ is $|B|^{-1} \times |A|^{-1}$; for we have

$$|A| \times |B| \times |B|^{-1} \times |A|^{-1} = \left|\begin{matrix} ab, & 0 \\ 0, & ab \end{matrix}\right|$$

if a and b are the determinants of $|A|$ and $|B|$.

Theorem. If $|A|$ and $|B|$ are given matrices, of which the determinants are a and b respectively, the equations

$$|A| = |X| \times |B| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$
$$|A| = |B| \times |Y| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

are irresoluble when b is not a divisor of a, and may be either resoluble or irresoluble when b is a divisor of a; if either of them is resoluble, it admits of only one solution.

For, postmultiplying (i) by $(B)^{-1}$, and premultiplying (ii) by the same matrix, we obtain the equations

$$|X| = \frac{1}{b} \times |A| \times |B|^{-1},$$

$$|Y| = \frac{1}{b} \times |B|^{-1} \times |A|;$$

which completely determine the constituents of the matrices $|X|$ and $|Y|$; but these constituents are not necessarily integral.

Cor. If $|A| = \left|\begin{matrix} b & 0 \\ 0, & b \end{matrix}\right|$, the equations (i) and (ii) are both resoluble, and we have

$$|X| = |Y| = |B|^{-1}.$$

17. *Equivalence of Matrices.*

If $|A|$ and $|B|$ are any two matrices connected by the relation

$$|A| = |\epsilon| \times |B| \quad \ldots\ldots\ldots\ldots \text{(i)}$$

where $|\epsilon|$ is an unit matrix, $|A|$ and $|B|$ are said to be equivalent by pre-multiplication, or, for brevity, pre-equivalent; if, again, $|A|$ and $|B|$ are connected by the relation

$$|A| = |B| \times |\epsilon| \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

$|A|$ and $|B|$ are said to be equivalent by postmultiplication, or, for brevity, post-equivalent. The relation (i) may also be written

$$|B| = |\epsilon|^{-1} \times |A|;$$

and the relation (ii) may be written

$$|B| = |A| \times |\epsilon|^{-1}.$$

Equivalent matrices have the same determinant; and the greatest common divisor of their constituents is the same. In any matrix the greatest common divisor of any column is not altered by premultiplication with an unit matrix, nor the greatest common divisor of any row by postmultiplication with an unit matrix.

If two matrices are equivalent by premultiplication, their reciprocals are equivalent by postmultiplication, and *vice versâ*.

If two matrices are equivalent in the same way to a third matrix, they are equivalent in that way to one another.

18. *Systems of non-equivalent Matrices.*

Let n be any positive integer, and $\sigma(n)$ the sum of the divisors of n; every matrix of determinant n is equivalent by premultiplication to one, and only one, of a system of $\sigma(n)$ matrices.

Consider the system of matrices included in the formula

$$|G| = \begin{vmatrix} g, & 0 \\ h, & g' \end{vmatrix},$$

where g, g' is any pair of conjugate divisors of n (we may suppose these divisors taken positively), and h is any one of the g numbers $0, 1, 2, \ldots, g-1$. The number of the matrices $|G|$ is $\sigma(n)$; and it can be shown (1) that no two of them are pre-equivalent, (2) that every matrix of determinant n is pre-equivalent to one of them.

Similarly it may be proved that every matrix of determinant n is equivalent by postmultiplication to one, and only one, of the $\sigma(n)$ matrices

$$|G'| = \begin{vmatrix} g', & 0 \\ h, & g \end{vmatrix},$$

the signification of the letters g, g' being the same as before.

We may regard all the matrices pre-equivalent, or post-equivalent, to a given matrix as forming a *class*. For any determinant n, and for either kind of equivalence, the number of non-equivalent classes is $\sigma(n)$; and as representatives of these classes we may take any $\sigma(n)$ matrices we please, one out of each class. Thus the matrices $|G|$ form a representative system of the classes of determinant n, non-equivalent by premultiplication; so also, if $|\alpha|$ and $|\beta|$ are given unit matrices, do the systems of matrices

$$|\alpha| \times |G|, \qquad |G| \times |\beta|, \qquad |\alpha| \times |G| \times |\beta|.$$

For each of these systems consists of $\sigma(n)$ matrices of determinant n; and in the same system no two matrices are pre-equivalent. Similarly the system

$$|G'|, \qquad |\alpha| \times |G'|, \qquad |G'| \times |\beta|, \qquad |\alpha| \times |G'| \times |\beta|$$

are representative systems of the $\sigma(n)$ classes of determinant n, non-equivalent by postmultiplication. Any such system is termed a complete system of matrices of the determinant n; the special systems $|G|$ and $|G'|$ are said to be *reduced*.

If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ denote a complete system of matrices non-equivalent by premultiplication, the reciprocal system $\begin{vmatrix} d, & -b \\ -c, & a \end{vmatrix}$ is a complete system non-equivalent by postmultiplication; and *vice versâ*.

The $\sigma(n)$ classes are not all primitive, if n has square divisors. Let p', p'', p''' be the different primes of which the squares divide n; the number of primitive classes (in either classification) of matrices of determinant n is

$$\sigma'(n) = \sigma(n) - \Sigma\sigma\left(\frac{n}{p'^2}\right) + \Sigma\sigma\left(\frac{n}{p'^2 p''^2}\right) - \Sigma\sigma\left(\frac{n}{p'^2 p''^2 p'''^2}\right) + \ldots = n\,\Pi\left(1 + \frac{1}{p}\right),$$

the sign of multiplication Π extending to all the primes p which divide n.

19. *Composition of Systems of Matrices.*

Let $|A|$ and $|B|$ denote complete systems of matrices, non-equivalent by premultiplication, of the determinants a and b, supposed to be relatively prime. Then the system $|A| \times |B|$ is a complete system of matrices of the determinant ab. For (1) the number of matrices in this system is

$$\sigma(a) \times \sigma(b) = \sigma(ab);$$

and (2) no two of the matrices in the system are pre-equivalent. For, if possible, let

$$|A| \times |B| = |\alpha| \times |A_1| \times |B_1|,$$

$|\alpha|$ being an unit matrix. Let

$$|B| = |\beta| \times |G|, \qquad |B_1| = |\beta_1| \times |G_1|,$$

$|\beta|$ and $|\beta_1|$ being unit matrices, $|G|$ and $|G_1|$ reduced matrices of determinant b. Postmultiplying each side of the equation

$$|A| \times |\beta| \times |G| = |\alpha| \times |A_1| \times |\beta_1| \times |G_1|$$

by $|G|^{-1}$, we find

$$|A| \times |\beta| \times \begin{vmatrix} b, & 0 \\ 0, & b \end{vmatrix} = |\alpha| \times |A_1| \times |\beta_1| \times |G_1| \times |G|^{-1}.$$

Hence the constituents of the matrix on the right-hand side are divisible by b. But the determinant of $|\alpha| \times |A_1| \times |\beta|$ is a, which is prime to b; therefore the constituents of $|G_1| \times |G|^{-1}$ are divisible by b. It is found, on trial, that this is impossible unless $|G|$ and $|G_1|$ are identical; hence, finally, $|B|$, $|B_1|$ and therefore also $|A|$, $|A_1|$ are identical; which is contrary to the hypothesis.

We may add that if $|A|$ and $|B|$ denote complete systems of *primitive* matrices for the determinants a and b, $|A| \times |B|$ will denote a complete system of primitive matrices for the determinant ab.

It follows from the preceding theorem that if $a = P \times Q \times R \times \ldots$, where $P, Q, R, \ldots$, are powers of different primes, and if

$$|P|, \quad |Q|, \quad |R|, \quad \ldots$$

denote complete systems of matrices (or complete systems of primitive matrices) for the determinants $P, Q, R, \ldots$, respectively,

$$|P| \times |Q| \times |R| \ldots$$

(where the matrices are to be compounded in any definite order) will denote

a complete system of matrices (or a complete system of primitive matrices) for the determinant $a = P \times Q \times R \ldots$

When the two determinants a and b are not relatively prime, it will suffice for our purpose to consider the case in which one of them is the power p^μ of a prime and the other is that prime p itself. We shall attend only to primitive matrices.

First, let $\mu > 1$, and let $|p^\mu|$ and $|p|$ denote the reduced systems of primitive matrices of the determinants p^μ and p, so that $|p^\mu|$ consists of $p^{\mu-1}(1+p)$ matrices, and $|p|$ of $p+1$. It will be found that the system $|p^\mu| \times |p|$ contains (i) the primitive matrices (each occurring once) of determinant $p^{\mu+1}$; (ii) the matrices (each occurring p times) which are derived from the primitive matrices of determinant $p^{\mu-1}$ by multiplying each constituent by p. In all therefore the system $|p^\mu| \times |p|$ contains

$$p^\mu(1+p) + p \times p^{\mu-2}(1+p) = p^{\mu-1}(1+p)^2$$

matrices, as it ought to do.

Secondly, let $\mu = 1$: the system $|p| \times |p|$ consists (i) of the primitive matrices $|p^2|$ each once repeated; (ii) of $p+1$ matrices of determinant p^2 having their coefficients divisible by p: in all it contains $p(p+1)+p+1 = (p+1)^2$ matrices.

For brevity we have in this article considered systems of matrices non-equivalent by premultiplication; but the theorems which we have enunciated hold equally for systems non-equivalent by postmultiplication.

20. *Reduction of any two Primitive Matrices of the same Determinant to one another.*

If $|A_1|$ and $|A_2|$ are two primitive matrices of the same determinant n, we can always find two unit-matrices $|\alpha|$ and $|\beta|$ such that

$$|A_1| = |\alpha| \times |A_2| \times |\beta|.$$

To establish this it is sufficient to show that we can always satisfy the equation

$$|A| = |\alpha| \times \begin{vmatrix} n, & 0 \\ 0, & 1 \end{vmatrix} \times |\beta|$$

with two unit matrices $|\alpha|$ and $|\beta|$.

Let $|A| = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$, where $ad - bc = n$, and a, b, c, d have no common divisor.

The simultaneous congruences

$$a\zeta+b\eta\equiv 0,\quad c\zeta+d\eta\equiv 0,\ \mod n,$$

always admit of solution with relatively prime values of ζ and η (this may be seen by resolving n into a product of powers of different primes, and considering the congruences with regard to each factor separately). Let

$$\zeta\eta_1-\eta\zeta_1=1,\quad a\zeta+b\eta=n\lambda,\quad c\zeta+d\eta=n\mu,\quad a\zeta_1+b\eta_1=\lambda_1,\quad c\zeta_1+d\eta_1=\mu_1,$$

so that $\lambda\mu_1-\lambda_1\mu=1$; we find

$$|A|\times\begin{vmatrix}\zeta,&\zeta_1\\ \eta,&\eta_1\end{vmatrix}=\begin{vmatrix}n\lambda,&\lambda_1\\ n\mu,&\mu_1\end{vmatrix}=\begin{vmatrix}\lambda,&\lambda_1\\ \mu,&\mu_1\end{vmatrix}\times\begin{vmatrix}n,&0\\ 0,&1\end{vmatrix},$$

or

$$|A|=\begin{vmatrix}\lambda,&\lambda_1\\ \mu,&\mu_1\end{vmatrix}\times\begin{vmatrix}n,&0\\ 0,&1\end{vmatrix}\times\begin{vmatrix}\eta_1,&-\zeta_1\\ -\eta,&\zeta\end{vmatrix}.$$

21. *The six types of Matrices of an uneven Determinant.*

Every matrix of uneven determinant is, with regard to the modulus 2, of one of the six types

$$\begin{vmatrix}1,&0\\ 0,&1\end{vmatrix},\quad\begin{vmatrix}0,&1\\ -1,&0\end{vmatrix},\quad\begin{vmatrix}1,&0\\ 1,&1\end{vmatrix},\quad\begin{vmatrix}1,&1\\ 0,&1\end{vmatrix},\quad\begin{vmatrix}1,&1\\ -1,&0\end{vmatrix},\quad\begin{vmatrix}0,&1\\ -1,&-1\end{vmatrix},\qquad\text{(i)}$$

which we shall represent by the symbols

$$1,\quad\psi,\quad\sigma,\quad\tau,\quad\rho,\quad\rho^2;$$

giving rise to the system of congruences (mod 2):

$$\begin{aligned}
&\psi^2\equiv\sigma^2\equiv\tau^2\equiv 1;\quad \rho^3\equiv 1;\\
&\sigma\tau\equiv\tau\psi\equiv\psi\sigma\equiv\rho;\quad \tau\sigma\equiv\psi\tau\equiv\sigma\psi\equiv\rho^2;\\
&\psi\equiv\sigma\rho^2\equiv\tau\rho\equiv\rho\sigma\equiv\rho^2\tau\equiv\tau\sigma\tau\equiv\sigma\tau\sigma;\\
&\sigma\equiv\psi\rho\equiv\tau\rho^2\equiv\rho^2\psi\equiv\rho\tau\equiv\tau\psi\tau\equiv\psi\tau\psi;\\
&\tau\equiv\psi\rho^2\equiv\sigma\rho\equiv\rho\psi\equiv\rho^2\sigma\equiv\sigma\psi\sigma\equiv\psi\sigma\psi.
\end{aligned}$$

This system is symmetrical with regard to ψ, σ, τ and ρ, ρ^2; viz. it is not altered by a cyclic permutation of ψ, σ, τ, nor again by a simultaneous interchange of ρ, ρ^2 and of any two of the three ψ, σ, τ.

The relations between the six anharmonic functions

$$x,\quad 1-x,\quad\frac{x}{x-1},\quad\frac{1}{x},\quad\frac{1}{1-x},\quad\frac{x-1}{x},$$

are identical in form with the relations expressed by the foregoing congruences; for, representing these functions, taken in order, by

$$x, \quad \psi(x), \quad \sigma(x), \quad \tau(x), \quad \rho(x), \quad \rho^2(x),$$

we have the equations

$$\psi^2(x) = \sigma^2(x) = \tau^2(x) = x, \quad \rho^3(x) = x,$$
$$\sigma[\tau(x)] = \tau[\psi(x)] = \psi[\sigma(x)] = \rho(x),$$
$$\text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.}$$

If we represent the six unit matrices which we have chosen as representatives of the six types, by the symbols

$$|1|, \quad |\psi|, \quad |\sigma|, \quad |\tau|, \quad |\rho|, \quad |\rho^2|, \quad \ldots\ldots\ldots \text{(ii)}$$

we find

$$|\psi|^2 = -|1|, \quad |\sigma|^2 = P, \quad |\tau|^2 = Q, \quad |\rho|^3 = -|1|,$$

where we write $-|1|$, P, and Q for

$$\begin{vmatrix} -1, & 0 \\ 0, & -1 \end{vmatrix}, \quad \begin{vmatrix} 1, & 0 \\ 2, & 1 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} 1, & 2 \\ 0, & 1 \end{vmatrix}.$$

We have also

$$\left.\begin{aligned}
|\rho| &= |\sigma|^{-1} \times |\tau| = |\tau|^{-1} \times |\psi| = |\psi| \times |\sigma|, \\
-|\rho^2| &= |\tau|^{-1} \times |\sigma| = |\psi|^{-1} \times |\tau| = |\sigma|^{-1} \times |\psi|^{-1}, \\
|\psi| &= |\sigma|^{-1} \times |\tau| \times |\sigma|^{-1} = |\tau| \times |\sigma|^{-1} \times |\tau|, \\
|\sigma| &= |\psi|^{-1} \times |\tau|^{-1} \times |\psi| = |\tau| \times |\psi|^{-1} \times |\tau|, \\
|\tau| &= |\psi|^{-1} \times |\sigma|^{-1} \times |\psi| = |\sigma| \times |\psi| \times |\sigma|.
\end{aligned}\right\} \quad \ldots \text{(iii)}$$

The powers of $|\tau|$ and $|\sigma|$ are all different; but the matrices $\pm|\psi|$ are square roots of the unit matrix $-|1|$; and $|\rho|$ and $-|\rho^2|$ are cube roots of the same matrix. Thus each of the equations

$$|X|^2 = |1|, \quad |X|^3 = |1|, \quad |X|^4 = |1|, \quad |X|^6 = |1|,$$

admits of as many solutions as it has dimensions. No other equation included in the formula $|X|^n = |1|$ admits of any solution other than $|X| = \pm|1|$.

The following congruences are satisfied by the constituents of unit matrices $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ of the six types respectively; the modulus is 16 throughout.

$$\left.\begin{array}{ll} (1) & (a-d)(a+d\pm b)\equiv 0, \quad (a-d)(a+d\pm c)\equiv 0, \\ (\psi) & (b+c)(b-c\pm a)\equiv 0, \quad (b+c)(b-c\pm d)\equiv 0, \\ (\sigma) & (a+b)(a-d\pm b)\equiv 0, \\ (\tau) & (a+d)(a-d\pm c)\equiv 0, \\ (\rho) & (b-c)(b+c\pm d)\equiv 0, \\ (\rho^2) & (b-c)(b+c\pm a)\equiv 0. \end{array}\right\} \quad . \quad . \quad . \quad \text{(iv)}$$

22. *Primary Matrices and Primary Equivalence.*

A matrix of uneven determinant is said to be *primary* when it is of type (1) and satisfies the congruence $a\equiv 1$, mod 4. Thus $\begin{vmatrix} -7, & 4 \\ 2, & -5 \end{vmatrix}$ is a primary matrix of determinant 27; $\begin{vmatrix} -3, & 2 \\ -8, & 5 \end{vmatrix}$ is a primary unit matrix; if n is any uneven number, the matrices

$$|G| = \begin{vmatrix} (-)^{\frac{1}{2}(g-1)}g, & 0 \\ 2h, & (-)^{\frac{1}{2}(g-1)}g' \end{vmatrix}$$

(the signification of the symbols g, g', h being the same as in Art. 18) form a complete system of primary matrices non-equivalent by premultiplication. Similarly the matrices

$$|G'| = \begin{vmatrix} (-)^{\frac{1}{2}(g'-1)}g', & 0 \\ 2h, & (-)^{\frac{1}{2}(g'-1)}g \end{vmatrix}$$

form a complete system of primary matrices non-equivalent by postmultiplication.

If $|A|$ be any matrix of the uneven determinant n, and of the type (1), it is evident (1) that either $|A|$ or $-|A|$ is primary, (2) that, if $n\equiv 1$, mod 4, $|A|$ and $|A|^{-1}$ are either both primary or else both not primary, but that if $n\equiv 3$, mod 4, one of these matrices is primary and the other not.

Theorem I. "Every unit matrix can be represented in one way, and in one way only, in the form

$$\eta\times|\zeta|\times|\alpha|,$$

where $\eta=\pm 1$, $|\zeta|$ is one of the six typical unit matrices, and $|\alpha|$ is a primary unit matrix; similarly every unit matrix can be represented in one way, and in one way only, in the form

$$\eta\times|\alpha|\times|\zeta|."$$

Theorem II. "Every primary unit matrix $|A|$ can be represented in one way, and in one way only, in the form

$$P^{\lambda_1}\times Q^{\lambda_2}\times P^{\lambda_3}\times \ldots \times Q^{\lambda_{2s}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{2s}$ are positive or negative integers, of which the first and the last may be zero."

For, in the formula (i), if $\lambda_1 \leqq 0$, the two constituents of the lower row of $|A|$ are respectively greater in absolute magnitude than the corresponding constituents of the upper row; if $\lambda_1 = 0$, the contrary is the case. Hence the representation is possible in one way only, because an equation of the form

$$P^{\lambda_1} \times Q^{\lambda_2} \times P^{\lambda_3} \times \ldots = P^{\mu_1} \times Q^{\mu_2} \times P^{\mu_3} \times \ldots$$

implies the equations

$$\lambda_1 = \mu_1, \qquad \lambda_2 = \mu_2, \qquad \lambda_3 = \mu_3.$$

Again, if in $|A|$ the lower constituents are the greater, we can always determine an integer λ_1, such that in $P^{-\lambda_1} \times |A|$, which has the same upper constituents as $|A|$, the lower constituents shall be less than the upper; we can then determine an integer λ_2 such that in $Q^{-\lambda_2} \times P^{-\lambda_1} \times |A|$, which has the same lower constituents as $P^{-\lambda_1} \times |A|$, the upper constituents shall be less than the lower; by proceeding in this way, we shall finally arrive at an equation of the form

$$\ldots P^{-\lambda_3} \times Q^{-\lambda_2} \times P^{-\lambda_1} \times |A| = +|1|,$$

which coincides with the formula (i), and gives the required representation.

Employing the symbols $(q_1, q_2, \ldots, q_n)$ and $[q_1, q_2, \ldots, q_n]$ to represent respectively the numerator of the continued fraction

$$q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \ldots + \cfrac{1}{q_n}}},$$

and the continued fraction itself, we find

$$|A| = \begin{vmatrix} (2\lambda_2, \ldots, 2\lambda_{2s-1}), & (2\lambda_2, \ldots, 2\lambda_{2s}) \\ (2\lambda_1, \ldots, 2\lambda_{2s-1}), & (2\lambda_1, \ldots, 2\lambda_{2s}) \end{vmatrix};$$

and we may write the equation $\omega = |A| \times \Omega$ (see Art. 24) in the form

$$\omega = [2\lambda_1, 2\lambda_2, \ldots, 2\lambda_{2s}, \Omega].$$

If the equations

$$|A| = |\epsilon| \times |B|,$$
$$|A| = |B| \times |\epsilon|,$$

the unit $|\epsilon|$ is primary, the equivalence of the matrices $|A|$ and $|B|$ is said to be primary.

Theorem III. "If $|\Gamma|$ represents a complete system of matrices (or a complete system of primitive matrices) of determinant n, non-equivalent by premultiplication, the formula

$$\eta \times |\zeta| \times |\Gamma| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

represents a complete system of matrices (or of primitive matrices), non-equivalent by primary premultiplication."

For (1) if $|Z|$ be any given matrix of determinant n, let $|Z| = |A| \times |\Gamma_1|$, $|A|$ being an unit matrix, and $|\Gamma_1|$ being one of the matrices Γ. Let $|A| = \eta \times |\alpha| \times |\zeta|$, where $|\alpha|$ is primary; then

$$|Z| = \eta \times |\alpha| \times |\zeta| \times |\Gamma_1|.$$

Again, (2) if $\eta_1 \times |\zeta_1| \times |\Gamma_1|$ is primarily equivalent to $\eta_2 \times |\zeta_2| \times |\Gamma_2|$, we must have, first of all, $|\Gamma_1| = |\Gamma_2|$, or else the two matrices could not be equivalent at all; then the equation

$$\eta_1 \times |\zeta_1| \times |\Gamma_1| = \eta_2 \times |\beta| \times |\zeta_2| \times |\Gamma_1|,$$

in which $|\beta|$ is a primary unit matrix, gives by postmultiplication with $|\Gamma_1|^{-1}$

$$\eta_1 \times |\zeta_1| = \eta_2 \times |\beta| \times |\zeta_2|,$$

whence finally $|\beta| = 1$, $\eta_1 = \eta_2$, $\zeta_1 = \zeta_2$; *i.e.* two matrices included in the formula (ii) cannot be primarily equivalent without being identical.

Similarly the formula

$$\eta \times |\Gamma'| \times |\zeta|, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

where $|\Gamma'|$ represents a complete system of (primitive) matrices of determinant n, non-equivalent by postmultiplication, represents a complete system of (primitive) matrices of the same determinant non-equivalent by primary postmultiplication.

If N be the number of matrices (see Art. 18) in the systems Γ and Γ', $12N$ is the number of matrices in each of the systems (ii) and (iii).

When n is an uneven number these $12N$ matrices are equally distributed between the six types; *i.e.* there are $2N$ matrices, or rather N pairs of matrices, the two of each pair differing only in sign, which appertain to each of the six types. This may be seen by taking for Γ or Γ' the primary systems G or G' of Art. 18, since each matrix of Γ or Γ' gives rise to a pair of opposite matrices of each of the six types.

23. *The nine types of Primitive Matrices of an Even Determinant.*

The primitive matrices of any even determinant, considered with regard

to the modulus 2, are of one or other of nine different types, which are exhibited in the following scheme:

$$\begin{vmatrix} 1, & 1 \\ 1, & 1 \end{vmatrix}, \quad \begin{vmatrix} 1, & 0 \\ 1, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & 1 \\ 0, & 1 \end{vmatrix},$$

$$\begin{vmatrix} 0, & 0 \\ 1, & 1 \end{vmatrix}, \quad \begin{vmatrix} 0, & 0 \\ 1, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & 0 \\ 0, & 1 \end{vmatrix},$$

$$\begin{vmatrix} 1, & 1 \\ 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} 1, & 0 \\ 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & 1 \\ 0, & 0 \end{vmatrix},$$

and which we shall symbolize thus:

$$(C) \quad . \quad . \quad . \quad . \quad . \quad \begin{vmatrix} C_{1,1}, & C_{2,1}, & C_{3,1} \\ C_{1,2}, & C_{2,2}, & C_{3,2} \\ C_{1,3}, & C_{2,3}, & C_{3,3} \end{vmatrix}.$$

No two of these types are primarily equivalent, either by premultiplication or by postmultiplication; two types in the same row are absolutely non-equivalent by premultiplication; two types in the same column are absolutely non-equivalent by postmultiplication. This may be seen (Art. 17) by observing that, in any matrix, the greatest common divisor of a column is not altered by premultiplication with an unit, nor the greatest common divisor of a row by postmultiplication with an unit. The types $C_{1,1}$, $C_{2,2}$, $C_{3,3}$ are their own reciprocals; $C_{r,s}$ is the reciprocal of $C_{s,r}$. The types (C) are unchanged by premultiplication with the corresponding unit-types in the scheme

$$\begin{vmatrix} \psi, & \psi, & \psi \\ \sigma, & \sigma, & \sigma \\ \tau, & \tau, & \tau \end{vmatrix},$$

and by postmultiplication with the unit-types in the scheme

$$\begin{vmatrix} \psi, & \sigma, & \tau \\ \psi, & \sigma, & \tau \\ \psi, & \sigma, & \tau \end{vmatrix}$$

Again, the type matrices which lie in the same column of (C) are equivalent by premultiplication with the corresponding unit-types in either of the schemes

$$\begin{vmatrix} 1, & \tau, & \sigma \\ \tau, & 1, & \psi \\ \sigma, & \psi, & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1, & \rho, & \rho^2 \\ \rho^2, & 1, & \rho \\ \rho, & \rho^2, & 1 \end{vmatrix};$$

and the type matrices which lie in the same row of (C) are equivalent by postmultiplication with the same unit-types; thus, for example,

$$\tau \times C_{2,1} \equiv C_{2,2} \equiv \psi \times C_{2,3},$$

$$C_{1,1} = C_{2,1} \times \rho = C_{3,1} \times \rho^2.$$

If we select any one of the types (C), for example $C_{r,s}$, and denote by ζ_1, ζ_2 any two of the six unit-types, the expression $\zeta_1 \times C_{r,s} \times \zeta_2$ (which has $36 = 9 \times 4$ different values) represents the nine types (C) indifferently, *i.e.* each of them four times.

If n be any even number, and N the number of primitive matrices non-equivalent by premultiplication, the $12N$ matrices non-equivalent by primary premultiplication are equally distributed between the nine types (C). To establish this, we first consider the case in which $n = 2^\mu$ is a power of 2. The primitive matrices G of determinant 2^μ (see Art. 18) divide themselves into three groups:

$$\text{(i)} \quad . \quad . \quad . \quad . \quad . \quad \begin{vmatrix} 2^\mu, & 0 \\ h, & 1 \end{vmatrix},$$

h an uneven remainder of 2^μ;

$$\text{(ii)} \quad . \quad . \quad . \quad . \quad . \quad \begin{vmatrix} 2^{\mu-s}, & 0 \\ h, & 2^s \end{vmatrix},$$

$0 < s \leqq \mu$, and h an uneven remainder of $2^{\mu-s}$, except when $s = \mu$, in which case $h = 0$;

$$\text{(iii)} \quad . \quad . \quad . \quad . \quad . \quad \begin{vmatrix} 2^\mu, & 0 \\ h, & 1 \end{vmatrix},$$

h an even remainder of 2^μ.

Each of these groups contains $2^{\mu-1}$ matrices; hence the $N = 3 \times 2^{\mu-1}$ matrices of G are equally distributed between the three types $C_{1,2}$, $C_{2,2}$, $C_{3,2}$; except only that the matrix $\begin{vmatrix} 1, & 0 \\ 0, & 2^\mu \end{vmatrix}$ is of the type $C_{2,3}$ instead of $C_{2,2}$. Next let $n = 2^\mu \times m$, m being uneven, and let $|M|$ represent the general term of a system of primitive matrices of determinant m, of type (1), and non-equivalent by premultiplication. The matrices $|M| \times |G|$ are a complete system of primitive matrices of determinant $2^\mu \times m$; these matrices are equally distributed between the types of the first, second, and third columns of (C), because $|M| \times |G|$ and $|G|$ appertain to types occupying the same column of (C). Lastly, to obtain a system of primitive matrices, non-equivalent by

primary premultiplication, we have to premultiply each of the matrices $|M| \times |G|$ by the twelve unit matrices $\eta \times |\zeta|$; and it will be found, by means of the properties of the type-system (C), that four of these units do not alter the type of $|M| \times |G|$, and that this type is changed into each of the other types which occupy the same column by four of the remaining eight. Hence the

$$12 \times 3 \times 2^{\mu-1} \times m \Pi \left(1 + \frac{1}{p}\right)$$

matrices contained in any system of primitive matrices of determinant $n = 2^{\mu} \times m$, non-equivalent by primary premultiplication, consist of nine groups, each containing

$$2^{\mu} \times m \times \Pi \left(1 + \frac{1}{p}\right)$$

pairs of opposite matrices, and appertaining respectively to the nine types (C).

There is a corresponding theorem for primary equivalence by post-multiplication.

Arts. 24–33. The Transformation of the Theta and Omega Functions.

24. *Enunciation of the Problem of Transformation.*

Let ω and Ω be two complex quantities connected by the relation

$$\omega = \frac{c + d\Omega}{a + b\Omega}, \quad \ldots \ldots \ldots \ldots \quad \text{(i)}$$

the coefficients a, b, c, d being integral numbers relatively prime, and the determinant $ad - bc = \Delta$ being positive; so that, as we have already observed (Art. 12), the real part of $i\Omega$ is negative when the real part of $i\omega$ is negative, and *vice versâ*. We shall frequently write the equation (i) in the symbolic form

$$\omega = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times \Omega, \quad \ldots \ldots \ldots \quad \text{(ii)}$$

or in the equivalent form

$$\Omega = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix}^{-1} \times \omega = \begin{vmatrix} d, & -b \\ -c, & a \end{vmatrix} \times \omega. \quad \ldots \ldots \quad \text{(iii)}$$

The problem of Transformation is "to express the Theta and Omega functions containing Ω by means of the Theta and Omega functions containing ω;" and the general nature of the solution of which this problem is susceptible is indicated by the theorem:

"Any Theta function of the arguments $(a+b\Omega)\frac{\pi x}{h}$ and Ω can be expressed in the form

$$e^{-b(a+b\Omega)\frac{i\pi x^2}{h^2}} \times T,$$

where, if $\Delta > 1$, T is a homogeneous function of the order Δ of two of the Theta functions of the arguments $\frac{\pi x}{h}$ and ω; and if $\Delta = 1$, T is a multiple, by a coefficient not containing x, of one of the Theta functions of the arguments $\frac{\pi x}{h}$ and ω."

The demonstration of this theorem and the determination of the forms of the functions T may be obtained by a method which is due to M. Hermite, and which we proceed to explain.

25. *General Solution of the Problem of Transformation. Method of M. Hermite.*

Lemma. "Let the values of the complex variable x be represented in the usual manner by the points of a plane; and let $F(x)$ be a function of x, synectic throughout the whole plane, and having the period l, so that

$$F(x+l) = F(x);$$

$F(x)$ can always, and in one way only, be expressed by an exponential series convergent for every finite value of x, of the form

$$\sum_{-\infty}^{+\infty} A_m e^{\frac{2mi\pi x}{l}}." \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

This is, in fact, the theorem of Laurent; for, if $z = e^{\frac{2mi\pi x}{l}}$, $F(x)$ is a function of z, synectic throughout the whole plane upon which z is represented, except at the point $z = 0$; viz. the values of x corresponding to any given value of z are all included in the formula $x + sl$, where s is an indeterminate integer. But $F(x+sl) = F(x)$; *i.e.* $F(x)$ is a one-valued function of z, and is finite and continuous when z is finite and different from zero. Hence, by the theorem of Laurent, $F(x)$ is developable, and in one way only, in a series proceeding by ascending and descending powers of z, convergent for all values of z which lie between two circles having their common centre at the point $z = 0$, the radius of the inner circle being as small and that of the outer circle as

great as we please; *i. e.* $F(x)$ is developable, and in one way only, in a series of the form (i), convergent for all finite values of x.

Let us now represent by $F(x)$ a function of x, synectic throughout the whole plane, and satisfying the two equations

$$F(x+h) = (-)^m F(x), \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

$$F(x+h\omega) = (-)^n e^{-\Delta i\pi\left(\frac{2x}{h}+\omega\right)} \times F(x), \quad \ldots\ldots\ldots \text{(iii)}$$

m and n denoting any two integers (which, however, we may suppose, without loss of generality, to be either 0 or 1), and the symbols h, ω, Δ retaining the meanings attributed to them in Art. 24. We proceed to show that by the equations (ii) and (iii) $F(x)$ is determined as a function of x containing Δ arbitrary constants, which enter linearly into its expression. From the Lemma we infer that, to satisfy the condition (ii), we must have

$$F(x) = \sum_{s=-\infty}^{s=+\infty} A_s e^{(2s+m)\frac{i\pi x}{h}}; \quad \ldots\ldots\ldots \text{(iv)}$$

and again that, to satisfy the condition (iii), we must have

$$A_s \times e^{(2s+m)i\pi\omega} = (-)^n A_{s+\Delta} e^{-\Delta i\pi\omega}, \quad \ldots\ldots\ldots \text{(v)}$$

Let j be one of the numbers 0, 1, 2, ..., $\Delta-1$, and let j' be any number included in the formula $r\Delta+j$, where r is an indeterminate integer; we find from (v)

$$A_{j'} = (-)^m e^{\frac{i\pi\omega}{4\Delta}[(2j'+m)^2-(2j+m)^2]} \times A_j = (-)^m e^{\frac{i\pi\omega}{4\Delta}(2j'+m)^2} \times \mathrm{a}_j,$$

where

$$\mathrm{a}_j = e^{-\frac{i\pi\omega}{4\Delta}(2j+m)^2} \times A_j.$$

If then for brevity we write

$$\Sigma_j(x) = \sum_{r=-\infty}^{r=+\infty} (-)^m e^{(2r\Delta+2j+m)^2\frac{i\pi\omega}{4\Delta}+(2r\Delta+2j+m)\frac{i\pi x}{h}},$$

we obtain, finally, the equation

$$F(x) = \mathrm{a}_0\Sigma_0(x) + \mathrm{a}_1\Sigma_1(x) + \ldots + \mathrm{a}_{\Delta-1}\Sigma_{\Delta-1}(x); \quad \ldots\ldots \text{(vi)}$$

which shows that any function, synectic throughout the whole plane, and satisfying the equations (ii) and (iii), can be expressed as a linear function of the Δ functions $\Sigma_j(x)$.

From this result we infer that if $F_0, F_1, \ldots, F_{\Delta-1}$ are any Δ synectic functions of x, satisfying the equations (ii) and (iii), and independent of one another (*i. e.* not satisfying any linear homogeneous relation of the type

$$C_0F_0 + C_1F_1 + C_2F_2 + \ldots + C_{\Delta-1}F_{\Delta-1} = 0,$$

where $C_1, C_2, \ldots$, do not contain x), all the functions F, synectic throughout the whole plane, which satisfy those two equations, may be expressed as linear and homogeneous functions with constant coefficients of $F_0, F_1, F_2, \ldots, F_{\Delta-1}$.

If we suppose that $F(x)$, besides satisfying the equations (ii) and (iii), also satisfies the equation

$$F(x) = (-)^{\sigma} F(-x) \quad \ldots\ldots\ldots \quad \text{(vii)}$$

(*i.e.* if we suppose that $F(x)$ is either an even or an uneven function), the number of arbitrary constants in the general expression (vi) is reduced by about one half. Substituting in the equation (vii) for $F(x)$ the equivalent expression (vi), and denoting by j_1, j_2, two of the numbers j (the same or different), we find that if

$$j_1 + j_2 + m \equiv 0, \mod \Delta, \quad \ldots\ldots\ldots \quad \text{(viii)}$$

we have necessarily

$$\mathrm{a}_{j_1} = (-)^{\sigma+\sigma'} \mathrm{a}_{j_2}, \quad \ldots\ldots\ldots \quad \text{(ix)}$$

where

$$\sigma' = \frac{n}{\Delta}(j_1 + j_2 + m).$$

We now distinguish between the cases in which Δ is uneven and in which Δ is even. (i) Let Δ be uneven: the equation (ix) shows that the coefficients $\pm a_j$ are equal in pairs—one of them, of which the index j is determined by the congruence

$$2j + m \equiv 0, \mod \Delta,$$

pairing with itself. If $nm + \sigma$ is uneven, the equation (ix) shows that this coefficient is zero, and that there are only $\frac{1}{2}(\Delta - 1)$ independent coefficients; if $nm + \sigma$ is even, the odd coefficient is not determined by the equation (ix), and there are $\frac{1}{2}(\Delta + 1)$ independent coefficients. We shall presently see that the case in which $nm + \sigma$ is uneven does not present itself in the theory of the transformation of the Theta functions. (ii) Let Δ be even. And (α) let $m = 1$ be uneven: we then have $j_1 + j_2 + 1 = \Delta$, $\sigma' = n$, $\mathrm{a}_{j_1} = (-1)^{n+\sigma} \mathrm{a}_{j_2}$; *i.e.* the coefficients $\pm \mathrm{a}_j$ are equal in pairs, and there are $\frac{1}{2}\Delta$ independent coefficients. (β) Let $m = 0$: of the coefficients $\pm \mathrm{a}_j$, $\Delta - 2$ are equal in pairs, each of the

two coefficients a_0 and $a_{\frac{1}{2}\Delta}$ pairing with itself; for these two coefficients the equation (ix) becomes

$$a_0 = (-)^\sigma a_0, \qquad a_{\frac{1}{2}\Delta} = (-)^{n+\sigma} a_{\frac{1}{2}\Delta};$$

and thus there are $\frac{1}{2}\Delta+1$, or $\frac{1}{2}\Delta$, or $\frac{1}{2}\Delta-1$ independent coefficients, according as σ and $n+\sigma$ are both even, or one even and one uneven, or both uneven: the last case does not present itself in the theory of transformation.

To apply the preceding theory to the Theta functions, we introduce an auxiliary function Π defined by the equation

$$\Pi_{m,n}\left(\frac{x}{h}\right) = \vartheta_{\mu\nu}\left([\alpha+b\Omega]\frac{\pi x}{h},\ \Omega\right) \times e^{b(a+b\Omega)\frac{i\pi x^2}{h^2}}, \quad \ldots \quad \text{(x)}$$

where m and n are integral numbers defined by the equations

$$\left.\begin{aligned} m &= a\mu + b\nu + ab, \\ n &= c\mu + d\nu + cd. \end{aligned}\right\} \quad \ldots \quad \text{(xi)}$$

From the elementary properties of the Theta functions (see Art. 1, equations vi, vii, viii) it follows that the function Π verifies the equations

$$\left.\begin{aligned} \Pi_{m,n}\left(\frac{x}{h}+1\right) &= (-)^m \Pi_{m,n}\left(\frac{x}{h}\right), \\ \Pi_{m,n}\left(\frac{x}{h}+\omega\right) &= (-)^n e^{-\Delta i\pi\left(\frac{2x}{h}+\omega\right)} \times \Pi_{m,n}\left(\frac{x}{h}\right), \\ \Pi_{m,n}\left(-\frac{x}{h}\right) &= (-)^{\mu\nu}\Pi_{m,n}\left(\frac{x}{h}\right); \end{aligned}\right\} \quad \ldots \quad \text{(xii)}$$

which, if we put $\sigma = \mu\nu$, coincide in form with the equations (ii), (iii), and (vii).

We now distinguish between the cases in which the determinant is uneven, and in which it is even.

I. *The determinant is uneven.* In this case the equations (xi) give rise to the congruences

$$\left.\begin{aligned} m-1 &\equiv a(\mu-1)+b(\nu-1) \\ n-1 &\equiv c(\mu-1)+d(\nu-1) \end{aligned}\right\}, \ \text{mod } 2; \quad \ldots \quad \text{(xiii)}$$

from which we infer that if μ, ν are both uneven, m, n are both uneven, and *vice versâ*. Hence $mn+\mu\nu = mn+\sigma$ is always even; and every synectic function which satisfies the equations (xii) can be linearly expressed by means of any $\frac{1}{2}(\Delta+1)$ independent functions which satisfy that system. Let ρ represent any one of the $\frac{1}{2}(\Delta+1)$ *even* numbers $0, 2, 4, \ldots \Delta-1$; and let $\vartheta\left(\frac{\pi x}{h}, \omega\right)$ denote one of

the three Theta functions other than $\vartheta_{m,n}\left(\frac{\pi x}{h}, \omega\right)$; the $\frac{1}{2}(\Delta+1)$ functions

$$\left[\vartheta_{m,n}\left(\frac{\pi x}{h}, \omega\right)\right]^{\Delta-\rho} \times \left[\vartheta\left(\frac{\pi x}{h}, \omega\right)\right]^{\rho}, \quad \ldots\ldots \text{(xiv)}$$

are independent, and satisfy the equations (xii). We have therefore the equation

$$\vartheta_{\mu,\nu}\left([a+b\Omega]\frac{\pi x}{h}, \Omega\right) = e^{-i\pi b(a+b\Omega)\frac{x^2}{h^2}} \times \vartheta_{m,n}\left(\frac{\pi x}{h}, \omega\right) \times T, \quad \ldots \text{(xv)}$$

where T is a homogeneous function of the order $\frac{1}{2}(\Delta-1)$ of the squares of $\vartheta_{m,n}$ and ϑ. This equation establishes the theorem of Art. 24, for the case in which Δ is uneven.

The following Table gives, for each of the six types (Art. 21) to which an uneven matrix may appertain, the values (mod 2) of m and n corresponding to given values of μ and ν.

	1	ψ	σ	τ	ρ	ρ^2
$m \equiv$	μ	ν	μ	$\mu+\nu+1$	$\mu+\nu+1$	ν
$n \equiv$	ν	μ	$\mu+\nu+1$	ν	μ	$\mu+\nu+1$

II. *The determinant is even.* In this case the theorem of Art. 24 may be established in the same manner; and the general character of the result is the same, though there is a little more variety in its form. The number of functions linearly independent of one another which satisfy the equations (xii) is always either $\frac{1}{2}\Delta+1$ or $\frac{1}{2}\Delta$, but never $\frac{1}{2}\Delta-1$; for, since by hypothesis the matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ is primitive, we infer from the equations (xi) that we cannot have simultaneously $m \equiv n \equiv 0$, $\mu \equiv \nu \equiv 1$; *i.e.* these equations exclude the case in which the solution of the system (xii) contains only $\frac{1}{2}\Delta-1$ arbitrary constants. According as that system (xii) admits of $\frac{1}{2}\Delta+1$, or $\frac{1}{2}\Delta$ linearly independent solutions the expression for $\vartheta_{\mu,\nu}\left([a+b\Omega]\frac{\pi x}{h}, \Omega\right)$ is of the type

$$e^{-i\pi b(a+b\Omega\,\frac{x^2}{h^2}} \times A, \quad \ldots\ldots \text{(xvi)}$$

or

$$e^{-i\pi b(a+b\Omega)\frac{x^2}{h^2}} \times B \times \vartheta_\alpha\left(\frac{\pi x}{h}, \omega\right) \times \vartheta_\beta\left(\frac{\pi x}{h}, \omega\right), \quad \ldots \text{(xvii)}$$

where A and B are homogeneous functions of the squares of any two of the Theta functions of the arguments $\frac{\pi x}{h}$ and u, and are respectively of the orders $\frac{1}{2}\Delta$ and $\frac{1}{2}\Delta - 1$; ϑ_α and ϑ_β are two Theta functions of which the indices depend on μ, ν and on the type $C_{r,s}$ (see Art. 23) of the even matrix of the transformation.

The following Tables show how to determine, for any transformation of an even determinant, the form of the expression for $\vartheta_{\mu,\nu}\left([a+b\Omega]\frac{\pi x}{h}, \Omega\right)$.

TABLE I.

	$C_{1,1}$	$C_{2,1}$	$C_{2,1}$
$m \equiv$	$\mu+\nu+1$	μ	ν
$n \equiv$	$\mu+\nu+1$	μ	ν
	$C_{1,2}$	$C_{2,2}$	$C_{2,2}$
$m \equiv$	0	0	0
$n \equiv$	$\mu+\nu+1$	μ	ν
	$C_{1,2}$	$C_{2,3}$	$C_{3,3}$
$m \equiv$	$\mu+\nu+1$	μ	ν
$n \equiv$	0	0	ν

TABLE II.

	$\mu\nu$	m	n	ϑ_α ; ϑ_β	
B	0	1	1	$\vartheta_{1,0}$; $\vartheta_{0,1}$	i
B	1	1	1	$\vartheta_{1,1}$; $\vartheta_{0,0}$	ii
B	0	1	0	$\vartheta_{1,0}$; $\vartheta_{0,0}$	iii
B	1	1	0	$\vartheta_{1,1}$; $\vartheta_{0,1}$	iv
B	0	0	1	$\vartheta_{0,1}$; $\vartheta_{0,0}$	v
B	1	0	1	$\vartheta_{1,1}$; $\vartheta_{1,0}$	vi
A	0	0	0		

TABLE III.

	$C_{1,1}$	$C_{2,1}$	$C_{3,1}$
ϑ_0	A	A	B, i
ϑ_1	B, ii	B, ii	B, ii
ϑ_2	A	B, i	A
ϑ_3	B, i	A	A
	$C_{1,2}$	$C_{2,2}$	$C_{3,2}$
ϑ_0	A	A	B, v
ϑ_1	B, vi	B, vi	B, vi
ϑ_2	A	B, v	A
ϑ_3	B, v	A	A
	$C_{1,3}$	$C_{2,3}$	$C_{3,3}$
ϑ_0	A	A	B, iii
ϑ_1	B, iv	B, iv	B, iv
ϑ_2	A	B, iii	A
ϑ_3	B, iii	A	A

Table I. gives the values, for the modulus 2, of the indices m and n, corresponding to given values of μ and ν for each of the nine types of the matrix of the transformation.

Table II. gives the form of the expression equivalent to

$$\vartheta_{\mu,\nu}\left([a+b\Omega]\frac{\pi x}{h}, \Omega\right)$$

for each admissible combination of the values (mod 2) of the indices $\mu \times \nu$, m, and n. Thus, if $\mu \times \nu \equiv 0$, $m \equiv n \equiv 1$, mod 2, that expression is of the type (xvii); and the Theta functions designated by ϑ_α and ϑ_β are $\vartheta_{1,0}$ and $\vartheta_{0,1}$. This assertion is proved by observing that the $\frac{1}{2}\Delta$ functions included in the formula

$$\left[\vartheta_{1,0}\left(\frac{\pi x}{h}, \omega\right)\right]^{\Delta-\rho} \times \left[\vartheta_{0,1}\left(\frac{\pi x}{h}, \omega\right)\right]^{\rho}$$

(where ρ is any one of the $\frac{1}{2}\Delta$ *uneven* numbers 1, 3, ..., $\Delta - 1$) are independent,

and satisfy the equations (xii), if we attribute in those equations to $\mu \times \nu$, m, and n values satisfying the congruences $\mu \times \nu \equiv 0$, $m \equiv n \equiv 1$, mod 2.

Table III. refers throughout to Table II., and is founded on the Tables I. and II. It gives the form of the expression for

$$\vartheta_s\left([a+b\Omega]\frac{\pi x}{h}, \Omega\right),$$

answering to each of the four values of the index s, and to each of the nine types of the matrix of transformation. Thus, if $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ be of the type $C_{2,1}$, *i. e.* if $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = \begin{vmatrix} 1, & 0 \\ 1, & 0 \end{vmatrix}$, mod 2, the expression of

$$\vartheta_2\left([a+b\Omega]\frac{\pi x}{h}, \Omega\right)$$

is of the type (B, i) in Table II., so that we have the equation

$$\vartheta_2\left([a+b\Omega]\frac{\pi x}{h}, \Omega\right) = e^{-i\pi b(a+b\Omega)\frac{x^2}{h^2}} \times \vartheta_0\left(\frac{\pi x}{h}, \omega\right) \times \vartheta_2\left(\frac{\pi x}{h}, \omega\right) \times B,$$

where B is a homogeneous function of the order $\frac{1}{2}\Delta - 1$ of the squares of any two of the Theta functions $\vartheta\left(\frac{\pi x}{h}, \omega\right)$.

26. *The Multiplier.*

Since $\omega = i\frac{K'(\omega)}{K(\omega)}$, $\Omega = i\frac{K'(\Omega)}{K(\Omega)}$ the equation (i) of Art. 24 may be written in the form

$$i\frac{K'(\omega)}{K(\omega)} = \frac{cK(\Omega) + diK'(\Omega)}{aK(\Omega) + biK'(\Omega)}, \quad \ldots\ldots\ldots \text{(i)}$$

or, which is the same thing,

$$\left.\begin{aligned} \frac{1}{M} \times K(\omega) &= aK(\Omega) + b.iK'(\Omega), \\ \frac{1}{M} \times iK'(\omega) &= cK(\Omega) + d.iK'(\Omega). \end{aligned}\right\} \quad \ldots\ldots \text{(ii)}$$

Either of these equations may be regarded as defining, for any given matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$, the quantity M (which is termed the *multiplier*) as a one-valued function

of ω. We observe that if the sign of the matrix be changed, the sign of the multiplier is also changed. If in the equations xv–xvii of Art. 25 we put $h = 2K(\omega) = 2K$, and write for brevity $\Lambda = K(\Omega)$, these equations assume the form

$$\vartheta_{\mu,\nu}\left(\frac{\pi x}{2M\Lambda}, \Omega\right) = T \times e^{-i\pi b \frac{x^2}{4K\Lambda M}}, \quad \ldots\ldots \quad \text{(iii)}$$

where T is a homogeneous function of order Δ of two of the Theta functions $\vartheta\left(\frac{\pi x}{2K}, \omega\right)$.

We have supposed in Art. 24 that the transformation $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ is primitive; but, by introducing the multiplier, we can adapt the formulæ to the case of a non-primitive transformation. Let

$$\begin{vmatrix} a_1, & b_1 \\ c_1, & d_1 \end{vmatrix} = g \times \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = \begin{vmatrix} g\alpha, & gb \\ gc, & gd \end{vmatrix},$$

where $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ is primitive and g is any positive or negative integer; and let M, M_1 be the multipliers appertaining to the transformations $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$, $\begin{vmatrix} a_1, & b_1 \\ c_1, & d_1 \end{vmatrix}$. From the definition of the multiplier we have $M - gM_1$; and, hence, if we write gx for x, the equation (iii) becomes

$$\vartheta_{\mu,\nu}\left(\frac{\pi x}{2M_1\Lambda}, \Omega\right) = T_1 \times e^{-i\pi b_1 \frac{x^2}{4K\Lambda M_1}},$$

where T_1 is deduced from T by writing gx for x—the indices of the Theta functions which occur in the expression of T_1 (equations xv–xvii, Art. 25) being still determined by the equations (xi) of Art. 25, and not (at least when g is even) by the equations

$$m = a_1\mu + b_1\nu + a_1b_1, \qquad n = c_1\mu + d_1\nu + a_2b_2.$$

27. *Composition of Transformations.*

Theorem I. "If the *primitive* matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ is compounded of any number of other matrices, so that

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = \begin{vmatrix} a_1, & \beta_1 \\ \gamma_2, & \delta_1 \end{vmatrix} \times \begin{vmatrix} a_2, & \beta_2 \\ \gamma_2, & \delta_2 \end{vmatrix} \times \ldots \times \begin{vmatrix} a_s, & \beta_s \\ \gamma_s, & \delta_s \end{vmatrix},$$

the transformation $\omega = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times \Omega$ results from the successive application of the transformations

$$\omega_{s-1} = \begin{vmatrix} a_s, & \beta_s \\ \gamma_s, & \delta_s \end{vmatrix} \times \Omega, \qquad \omega_{s-2} = \begin{vmatrix} a_{s-1}, & \beta_{s-1} \\ \gamma_{s-1}, & \delta_{s-1} \end{vmatrix} \times \omega_{s-1}, \ \dots, \ \omega = \begin{vmatrix} a_1, & 1 \\ \gamma_1, & \delta_1 \end{vmatrix} \times \omega_1.$$

Thus we may first express the Theta functions containing Ω by means of the Theta functions containing ω_{s-1}, these by means of the Theta functions containing ω_{s-2}, and so on continually until we have expressed the Theta functions containing Ω in terms of those containing ω. Since the equation $\omega = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times \Omega$ can be written in the form

$$\Omega = \begin{vmatrix} \delta_s, & -\beta_s \\ -\gamma_s, & a_s \end{vmatrix} \times \begin{vmatrix} \delta_{s-1}, & -\beta_{s-1} \\ -\gamma_{s-1}, & a_{s-1} \end{vmatrix} \times \dots \times \begin{vmatrix} \delta_1, & -\beta_1 \\ -\gamma_1, & a_1 \end{vmatrix} \times \omega,$$

we can also by a converse process express the Theta functions of ω in terms of the Theta functions containing Ω.

Theorem II. "If any number of transformations be compounded, the multiplier appertaining to the resulting transformation is the product of the multipliers appertaining to the components."

If, for example, $\omega = \begin{vmatrix} a_1, & \beta_1, \\ \gamma_1, & \delta_1, \end{vmatrix} \times \omega_1$, $\omega_1 = \begin{vmatrix} a_2, & \beta_2 \\ \gamma_2, & \delta_2 \end{vmatrix} \times \omega_2$, and M_1, M_2, M are the multipliers respectively appertaining to these transformations, and to the transformation compounded of them, we have

$$\frac{K(\omega)}{M_1} = a_1 K(\omega_1) + \beta_1 iK'(\omega_1), \qquad \frac{iK'(\omega)}{M_1} = \gamma_1 K(\omega_1) + \delta_1 iK'(\omega_1),$$

and

$$\frac{K(\omega_1)}{M_2} = a_2 K(\omega_2) + \beta_2 iK'(\omega_2), \qquad \frac{iK'(\omega_1)}{M_2} = \gamma_2 K(\omega_2) + \beta_2 iK'(\omega_2);$$

whence, eliminating $K(\omega_1)$ and $iK'(\omega_1)$, we find

$$\left.\begin{aligned} \frac{K(\omega)}{M_1 M_2} &= (a_1 a_2 + \beta_1 \gamma_2) K(\omega_2) + (a_1 \beta_2 + \beta_1 \delta_2) iK'(\omega_2), \\ \frac{iK'(\omega)}{M_1 M_2} &= (\gamma_1 a_2 + \delta_1 \gamma_2) K(\omega_2) + (\gamma_1 \beta_2 + \delta_1 \delta_2) iK(\omega_2); \end{aligned}\right\}$$

and these equations imply that $M = M_1 \times M_2$.

In the first theorem of this article we have expressly supposed the resultant matrix (and consequently the component matrices) to be primitive. In the second theorem the matrices compounded may be any whatever.

When we compound two primitive matrices, of which the determinants are not relatively prime, the resultant matrix is not necessarily primitive. Let

$$\begin{vmatrix} g\alpha, & g\beta \\ g\gamma, & g\delta \end{vmatrix} = \begin{vmatrix} \alpha_1, & \beta_1 \\ \gamma_1, & \delta_1 \end{vmatrix} \times \begin{vmatrix} \alpha_2, & \beta_2 \\ \gamma_2, & \delta_2 \end{vmatrix}, \quad \ldots\ldots\ldots \text{(i)}$$

where g is some integral number different from unity; and let Δ, Δ_1, Δ_2 be the determinants of

$$\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}, \quad \begin{vmatrix} \alpha_1, & \beta_1 \\ \gamma_1, & \delta_1 \end{vmatrix}, \quad \begin{vmatrix} \alpha_2, & \beta_2 \\ \gamma_2, & \delta_2 \end{vmatrix}.$$

The transformations

$$\omega = \begin{vmatrix} g\alpha, & g\beta \\ g\gamma, & g\delta \end{vmatrix} \times \Omega \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

and

$$\omega = \begin{vmatrix} \alpha_1, & \beta_1 \\ \gamma_1, & \delta_1 \end{vmatrix} \times \begin{vmatrix} \alpha_2, & \beta_2 \\ \gamma_2, & \delta_2 \end{vmatrix} \times \Omega \quad \ldots\ldots\ldots \text{(iii)}$$

give different results; viz. if M be the multiplier appertaining to the transformation $\omega = \begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \times \Omega$, the transformation (ii) expresses $\vartheta\left(\frac{g\pi x}{2M\Lambda}, \Omega\right)$ as a homogeneous function of order Δ of the Theta functions $\vartheta\left(\frac{g\pi x}{2K}, \omega\right)$; but the compound transformation (iii) expresses $\vartheta\left(\frac{g\pi x}{2M\Lambda}, \Omega\right)$ as a homogeneous function of the order $g^2\Delta = \Delta_1\Delta_2$ of the Theta functions $\vartheta\left(\frac{\pi x}{2K}, \omega\right)$. If, in particular, $\begin{vmatrix} \alpha_1, & \beta_1 \\ \gamma_2, & \delta_2 \end{vmatrix}$ and $\begin{vmatrix} \alpha_2, & \beta_2 \\ \gamma_2, & \delta_2 \end{vmatrix}$ are reciprocal, we have $g = \Delta_1 = \Delta_2$, $\Delta = 1$, $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} = \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}$, $\omega = \Omega$, $\Lambda = K$, $M = 1$: the transformation (ii) becomes an identity; while the transformation (iii) expresses $\vartheta\left(\frac{g\pi x}{2K}, \omega\right)$ as a homogeneous function of order g^2 of the Theta functions $\vartheta\left(\frac{\pi x}{2K}, \omega\right)$. We are thus led to the theorem of Jacobi: "The composition of two reciprocal transformations of determinant g serves to express the Theta functions of the argument gx by means of the Theta functions of the argument x."

This theorem is, in fact, a corollary from the Theorem II., which shows

that the product of the multipliers appertaining to two reciprocal transformations of determinant g is $\frac{1}{g}$.

Since every matrix can be expressed as a product of matrices of which the determinants are primes (Art. 19), it appears from the Theorem I. that in the theory of transformation it is sufficient to consider transformations of a prime order.

Again, if we regard the theory of linear transformations (*i. e.* of transformations of which the matrix is an unit), as known, we need only consider, for any determinant Δ, $\sigma'(\Delta)$ different transformations. For (Art. 18) every primitive transformation of determinant Δ is included in the formula

$$\omega = |G'| \times |\alpha| \times \Omega,$$

where $|\alpha|$ is an unit matrix and $|G'|$ is a system of $\sigma'(\Delta)$ primitive matrices of determinant Δ, non-equivalent by postmultiplication. Or, again, we might employ the formula

$$\omega = |\alpha| \times |G| \times \Omega,$$

where $|G|$ is a system of $\sigma'(\Delta)$ primitive matrices non-equivalent by premultiplication.

If $\left|\begin{matrix} \alpha, & \beta \\ \gamma, & \delta \end{matrix}\right|$ is an unit matrix of type (1), the equations (xi) of Art. 25 show that $m \equiv \mu$, mod 2, $n \equiv \nu$, mod 2; and we shall presently see that $M = \pm 1$; hence by a linear transformation of type (1) the Theta functions are unchanged, and only acquire an exponential factor. It is therefore convenient, instead of the $\sigma'(\Delta)$ transformations $|G|$ or $|G'|$, to consider the $6\sigma'(\Delta)$ pairs of transformations included in either of the formulæ

$$\omega = |\Gamma'| \times |\alpha| \times \Omega,$$

$$\omega = |\alpha| \times |\Gamma| \times \Omega,$$

where $|\alpha|$ is a primary unit, and $|\Gamma|$, $|\Gamma'|$ are systems of primary matrices non-equivalent by primary postmultiplication, and by primary premultiplication respectively.

Lastly, since every primitive matrix of determinant Δ can be exhibited in the form

$$|\alpha| \times \left|\begin{matrix} \Delta, & 0 \\ 0, & 1 \end{matrix}\right| \times |\beta|$$

(Art. 20), it is possible, in the theory of the transformations of order Δ, to confine

ourselves to the consideration of the single matrix $\begin{vmatrix} \Delta, & 0 \\ 0, & 1 \end{vmatrix}$, and to obtain the results relative to the other transformations of determinant Δ by compounding this transformation with linear transformations.

28. *Linear Transformation of the Theta Functions.*

When $\Delta = 1$, T is a constant, and the formula (xv) of Art. 25 becomes

$$\vartheta_{\mu,\nu}\left[(a+b\Omega)\frac{\pi x}{h}, \Omega\right] = C \times e^{-i\pi b(a+b\Omega)\frac{x^2}{h^2}} \times \vartheta_{m,n}\left(\frac{\pi x}{h}, \omega\right) \quad . \quad . \quad . \quad \text{(i)}$$

where

$$\omega = \frac{c+d\Omega}{a+b\Omega}, \qquad ad-bc=1,$$

$$m = a\mu + b\nu + ab, \qquad n = c\mu + d\nu + cd.$$

The determination of the constant C, which depends on ω, on the indices μ and ν, and on the integral numbers a, b, c, d, has been effected by M. Hermite in the following manner *. We shall at first suppose that b is different from zero, and positive.

Putting $h=1$, we have from (i)

$$C \times i^{mn} \sum_{r=-\infty}^{r=+\infty} (-)^{nr} e^{i\pi[(2r+m)x + \frac{1}{4}(2r+m)^2\omega]}$$

$$= i^{\mu\nu} \sum_{r=-\infty}^{r=+\infty} (-)^{\nu r} e^{i\pi[b(a+b\Omega)x^2 + (2r+\mu)(a+b\Omega)x + \frac{1}{4}(2r+\mu)^2\Omega]}.$$

Denoting the right-hand member of this equation by $i^{\mu\nu} \times S$, multiplying by $e^{-mi\pi x}dx$, and integrating from $x=0$ to $x=1$, we find

$$C = i^{\mu\nu - mn} e^{-\frac{1}{4}i\pi m^2\omega} \times \int_0^1 e^{-mi\pi x} S dx.$$

Let

$$f(x, r) = b(a+b\Omega)x^2 + [(2r+\mu)(a+b\Omega) - m]x + \tfrac{1}{4}(2r+\mu)^2\Omega - \nu r,$$

so that

$$e^{-mi\pi x} \times S = \sum_{r=-\infty}^{r=+\infty} e^{i\pi f(x, r)};$$

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* Liouville's Journal, 2nd series, vol. iii. p. 26.

we find immediately

$$f(x+1, r) - f(x, r+b) = 2ra, \qquad e^{i\pi f(x+1, r)} = e^{i\pi f(x, r+b)},$$

whence we infer

$$\int_0^1 e^{-mi\pi x} S dx = \sum_{s=0}^{s=b-1} \int_{-\infty}^{+\infty} e^{i\pi f(x, s)} dx.$$

By a formula due to Cauchy,

$$\int_{-\infty}^{+\infty} e^{i\pi(Px^2 + 2Qx + R)} dx = \frac{1}{\sqrt{-iP}} \times e^{i\pi \frac{PR - Q^2}{P}},$$

the coefficient of i in the imaginary part of P, and the real part of the radical $\sqrt{-iP}$, being each of them positive and different from zero. Evaluating the integrals

$$\int_{-\infty}^{+\infty} e^{i\pi f(x, s)} dx$$

by this formula, we obtain for C the expression

$$C = \delta_{\mu, \nu} \times H \times \frac{1}{\sqrt{-ib(a+b\Omega)}}, \qquad \text{(ii)}$$

where

$$\left.\begin{aligned} \delta_{\mu, \nu} &= i^{\mu\nu - mn} \times e^{-\frac{i\pi}{4b}(a\mu^2 - 2\mu m + dm^2 - ab^2)} \\ &= i^{\mu\nu - mn} \times e^{-\frac{1}{4}i\pi(ac\mu^2 + 2bc\mu\nu + bd\nu^2 + 2abc\mu + 2abd\nu + ab^2c)} \\ &= (-)^{\frac{1}{2}abcd} \times e^{\frac{1}{4}i\pi(ac\mu^2 + 2bc\mu\nu + bd\nu^2 + 2acd\mu + 2bcd\nu - ab^2c)}, \end{aligned}\right\} \qquad \text{(iii)}$$

$$H = \sum_{s=0}^{s=b-1} e^{-\frac{ai\pi}{b}(s - \frac{1}{2}b)^2}, \qquad \text{(iv)}$$

the real part of the radical $\sqrt{-ib(a+b\Omega)}$ being positive.

To sum the series (iv) we employ the formulæ of Gauss:

$$\left.\begin{aligned} &\sum_{s=0}^{s=n-1} e^{s^2 \frac{2i\pi m}{n}} = \left(\frac{m}{n}\right) i^{\frac{1}{4}(n-1)^2} \sqrt{n} \text{ when } n \text{ is uneven;} \\ &\sum_{s=0}^{s=n-1} e^{s^2 \frac{2i\pi m}{n}} = 0 \text{ when } n \text{ is unevenly even;} \\ &\sum_{s=0}^{s=n-1} e^{s^2 \frac{2i\pi m}{n}} = \left(\frac{n}{m}\right)(1+i^m)\sqrt{n}, = \left(\frac{2n}{m}\right) \times i^{\frac{1}{2}m} \times \sqrt{2n} \text{ when } n \text{ is evenly even.}^* \end{aligned}\right\} \qquad \text{(v)}$$

* The symbol $i^{\frac{1}{2}m}$ stands for $e^{\frac{1}{4}mi\pi}$, and therefore presents no ambiguity of sign. A similar abbreviation is occasionally used in the sequel.

In these formulæ n is a positive integer, m a positive or negative integer prime to n; the symbols $\left(\frac{m}{n}\right)$, $\left(\frac{n}{m}\right)$, $\left(\frac{2n}{m}\right)$ are the quadratic symbols of Legendre, as generalized by Jacobi; $\sqrt{n}$ and $\sqrt{2n}$ are the positive square roots of n and $2n$.

To apply these formulæ to the evaluation of the series (iv), we consider separately three cases.

(1) Let $a = 2a'$ be even, so that b is uneven; we have

$$H = e^{-\frac{1}{4}abi\pi} \times \sum_{s=0}^{s=b-1} \epsilon^{-s^2 \frac{2i\pi a'}{b}} = e^{-\frac{1}{4}abi\pi} \times \left(\frac{-a'}{b}\right) \times i^{\frac{1}{4}(b-1)^2} \sqrt{b}$$

$$= \left(\frac{a}{b}\right) \times i^{\frac{1}{2}(b-1)} \times i^{-\frac{1}{2}ab} \times \sqrt{b},$$

observing that

$$\left(\frac{-2}{b}\right) = (-1)^{\frac{1}{2}(b-1)+\frac{1}{8}(b^2-1)}.$$

If $a = 0$, $b = 1$, the method by which this formula has been obtained is inapplicable; but if we attribute to the symbol $\left(\frac{0}{1}\right)$ the value $+1$, the formula gives the true value of the series (iv), viz. $H = 1$.

(2) Let a and b be both uneven; we find

$$H = e^{-\frac{1}{4}abi\pi} \times \sum_{s=0}^{s=b-1} (-)^s e^{-s^2 \frac{i\pi a}{b}} = e^{-\frac{1}{4}abi\pi} \times \sum_{s=0}^{s=b-1} e^{s^2 \frac{i\pi(b-a)}{b}} = \left(\frac{a}{b}\right) \times i^{\frac{1}{2}(b-1)} \times i^{-\frac{1}{2}ab} \sqrt{b},$$

the form of the result being the same as in case (1).

(3) Let a be uneven, b even; then

$$H = e^{-\frac{1}{4}abi\pi} \times \sum_{s=0}^{s=b-1} (-)^s e^{-s^2 \frac{i\pi a}{b}} = \tfrac{1}{2} e^{-\frac{1}{4}abi\pi} \times \sum_{s=0}^{s=2b-1} e^{s^2 \frac{2i\pi}{2b}(b-a)}$$

$$= e^{-\frac{1}{4}abi\pi} \times \left(\frac{b}{b-a}\right) \times i^{\frac{1}{2}(b-a)} \times \sqrt{b}.$$

But

$$\left(\frac{b}{b-a}\right) = (-)^{\frac{1}{4}(a-1)b} \times \left(\frac{b}{a}\right),$$

as will be found by applying the laws of quadratic reciprocity; hence

$$H = \left(\frac{b}{a}\right) i^{-\frac{1}{2}a} \sqrt{b}.$$

This formula is also applicable to the case (2); in fact when a and b are both uneven

$$\left(\frac{b}{a}\right) i^{-\frac{1}{2}a} = (-1)^{\frac{1}{2}(a-1)\cdot\frac{1}{2}(b-1)} i^{-\frac{1}{2}a} \left(\frac{a}{b}\right) = \left(\frac{a}{b}\right) \times i^{\frac{1}{2}(b-1)} \times i^{-\frac{1}{4}ab}.$$

The value of C is therefore

$$C = \frac{\delta_{\mu,\nu} \times H(a, b)}{\sqrt{-i(a+b\Omega)}}, \quad \ldots\ldots\ldots\ldots \text{(vi)}$$

where the real part of the radical is positive, and where $\delta_{\mu,\nu}$ and $H(a, b) = \dfrac{H}{\sqrt{b}}$ are eighth roots of unity, of which $\delta_{\mu,\nu}$ is determined by the equations (iii), and $H(a, b)$ by the formulæ

$$\left.\begin{aligned} H(a, b) &= \left(\frac{b}{a}\right) i^{-\frac{1}{2}a}, \text{ if } a \text{ is uneven,} \\ &= \left(\frac{a}{b}\right) i^{-\frac{1}{2}a} \times i^{-\frac{1}{4}(a-1)(b-1)}, \text{ if } b \text{ is uneven.} \end{aligned}\right\} \quad \ldots \text{(vii)}$$

If b is negative, and different from zero, the value of C is given by the equation

$$C = \frac{\delta_{\mu,\nu} \times H(-a, -b)}{\sqrt{i(a+b\Omega)}}, \quad \ldots\ldots\ldots\ldots \text{(viii)}$$

the real part of the radical having a positive sign.

If b is zero, we have $a = d = \pm 1$, $\omega = \pm c + \Omega = ac + \Omega$; and the determination of $\vartheta_{\mu,\nu}\left(\dfrac{a\pi x}{h}, \Omega\right)$ is, in this case, obtained immediately from the equations defining the Theta series; we have, in fact,

$$\vartheta_{\mu,\nu}\left(\frac{a\pi x}{h}, \Omega\right) = i^{\mu c} \times e^{\frac{1}{4}\mu^2 ac i\pi} \times \vartheta_{m,n}\left(\frac{\pi x}{h}, \omega\right). \quad \ldots\ldots \text{(ix)}$$

This result is in accordance with the general formulæ (vi), (vii), if in them we attribute to the indeterminate symbol $\left(\dfrac{b}{a}\right) = \left(\dfrac{0}{\pm 1}\right)$ the value $+1$.

29. *Linear Transformation.—Determination of the Multiplier.*

Putting in the formulæ (i), (iii), (vi), (ix), $x = 0$, $\mu = \nu = 0$, we have

$$\sqrt{\frac{2\Lambda}{\pi}} = \sqrt{\frac{2K(\Omega)}{\pi}} = \vartheta_{0,0}[\Omega] = \frac{\delta_{0,0} \times H(a,b)}{\sqrt{-i(a+b\Omega)}} \times \vartheta_{ab,cd}[\omega], \quad \ldots \quad \text{(i)}$$

if b is positive, and different from zero; and if $b=0$, $a=d=\pm 1$, then

$$\sqrt{\frac{2\Lambda}{\pi}} = \vartheta_{0,c}[\omega]. \quad \ldots \quad \text{(ii)}$$

Hence

$$\frac{1}{M} = \frac{\Lambda}{K}(a+b\Omega) = i \times \delta_{0,0}^2 \times H^2(a,b) \times \frac{\vartheta_{ab,cd}^2[\omega]}{\vartheta_{0,0}^2[\omega]}, \quad \ldots \quad \text{(iii)}$$

if b is not zero; and

$$\frac{1}{M} = (-)^{\frac{1}{2}(a-1)} \frac{\vartheta_{0,c}^2[\omega]}{\vartheta_{0,0}^2[\omega]}, \quad \ldots \quad \text{(iv)}$$

if b is zero, or any even number.

30. *Linear Transformation of* $\phi(\omega)$ and $\psi(\omega)$.

Since (Art. 2, equation (i))

$$\phi^2(\omega) = \frac{\vartheta_2[\omega]}{\vartheta_3[\omega]}, \qquad \psi^2(\omega) = \frac{\vartheta_0[\omega]}{\vartheta_3[\omega]}, \quad \ldots \quad \text{(i)}$$

we can employ the general formulæ of linear transformation (Art. 28, equations (i), (iii), (vi), (vii)) to express $\phi^2(\Omega)$ and $\psi^2(\Omega)$, in terms of $\phi^2(\omega)$ and $\psi^2(\omega)$. By extracting the square root we obtain, but with an ambiguous sign, the corresponding expressions for $\phi(\Omega)$ and $\psi(\Omega)$ in terms of $\phi(\omega)$ and $\psi(\omega)$: this ambiguity can be avoided by employing the equations of Jacobi (Art. 4, I (*b*), (*d*); II (*c*), (*d*)):—

$$\left.\begin{aligned} \phi(\omega) &= \sqrt{2}\,\frac{\vartheta_2[\omega]}{\vartheta_2[\frac{1}{2}\omega]} = \frac{1}{\sqrt{2}}\,\frac{\vartheta_2[\frac{1}{2}\omega]}{\vartheta_3[\omega]}, \\ \psi(\omega) &= \frac{\vartheta_0[\omega]}{\vartheta_0[2\omega]} = \frac{\vartheta_0[2\omega]}{\vartheta_3[\omega]}, \end{aligned}\right\} \quad \ldots \quad \text{(ii)}$$

The resulting formulæ are contained in the following Table; they were first given in a complete form by M. Hermite. The symbol η is written for $e^{\frac{1}{8}i\pi}$.

Table of the Linear Transformations of $\phi(\omega)$ and $\psi(\omega)$.

$\begin{vmatrix} a, b \\ c, d \end{vmatrix} \equiv$	$\phi(\Omega)$.	$\phi(\omega)$.	$\psi(\Omega)$.	$\psi(\omega)$.	$\frac{1}{M}$.	λ^2.	λ'^2.
$\begin{vmatrix} 1, 0 \\ 0, 1 \end{vmatrix}$ or $\lvert 1 \rvert$	$\left(\frac{2}{a}\right)\eta^{-ac}\phi(\omega)$ $\left(\frac{2}{d}\right)\eta^{-dc}\phi(\omega)$	$\left(\frac{2}{a}\right)\eta^{ac}\phi(\Omega)$ $\left(\frac{2}{d}\right)\eta^{dc}\phi(\Omega)$	$\left(\frac{2}{a}\right)\eta^{ab}\psi(\omega)$ $\left(\frac{2}{d}\right)\eta^{bd}\psi(\omega)$	$\left(\frac{2}{a}\right)\eta^{-ab}\psi(\Omega)$ $\left(\frac{2}{d}\right)\eta^{-bd}\psi(\Omega)$	$(-)^{\frac{1}{2}(a-1)}1$	k^2	k^2
$\begin{vmatrix} 0, 1 \\ -1, 0 \end{vmatrix}$ or $\lvert \psi \rvert$	$\left(\frac{2}{b}\right)\eta^{-ab}\psi(\omega)$ $\left(\frac{2}{c}\right)\eta^{ac}\psi(\omega)$	$\left(\frac{2}{c}\right)\eta^{-cd}\psi(\Omega)$ $\left(\frac{2}{b}\right)\eta^{bd}\psi(\Omega)$	$\left(\frac{2}{b}\right)\eta^{-db}\phi(\omega)$ $\left(\frac{2}{c}\right)\eta^{dc}\phi(\omega)$	$\left(\frac{2}{b}\right)\eta^{ab}\phi(\Omega)$ $\left(\frac{2}{c}\right)\eta^{-ac}\phi(\Omega)$	$(-)^{\frac{1}{2}(b-1)}i$	k'^2	k^2
$\begin{vmatrix} 1, 0 \\ 1, 1 \end{vmatrix}$ or $\lvert \sigma \rvert$	$\eta^{-ac}\frac{\phi(\omega)}{\psi(\omega)}$	$\eta^{ca}\frac{\phi(\Omega)}{\psi(\Omega)}$	$\left(\frac{2}{d}\right)\eta^{-db}\frac{1}{\psi(\omega)}$ $\left(\frac{2}{a}\right)\eta^{ab}\frac{1}{\psi(\omega)}$	$\left(\frac{2}{a}\right)\eta^{ab}\frac{1}{\psi(\Omega)}$ $\left(\frac{2}{d}\right)\eta^{-bd}\frac{1}{\psi(\Omega)}$	$(-)^{\frac{1}{2}(a-1)}k'$	$-\frac{k^2}{k'^2}$	$\frac{1}{k'^2}$
$\begin{vmatrix} 1, 1 \\ 0, 1 \end{vmatrix}$ or $\lvert \tau \rvert$	$\left(\frac{2}{d}\right)\eta^{-cd}\frac{1}{\phi(\omega)}$ $\left(\frac{2}{a}\right)\eta^{ac}\frac{1}{\phi(\omega)}$	$\left(\frac{2}{d}\right)\eta^{-cd}\frac{1}{\phi(\Omega)}$ $\left(\frac{2}{a}\right)\eta^{ac}\frac{1}{\phi(\Omega)}$	$\eta^{db}\frac{\psi(\omega)}{\phi(\omega)}$	$\eta^{-ab}\frac{\psi(\Omega)}{\phi(\Omega)}$	$(-)^{\frac{1}{2}(a+c-1)}k$	$\frac{1}{k^2}$	$-\frac{k'^2}{k^2}$
$\begin{vmatrix} 1, 1 \\ -1, 0 \end{vmatrix}$ or $\lvert \rho \rvert$	$\eta^{ac}\frac{\psi(\omega)}{\phi(\omega)}$	$\left(\frac{2}{c}\right)\eta^{cd}\frac{1}{\psi(\Omega)}$ $\left(\frac{2}{b}\right)\eta^{bd}\frac{1}{\psi(\Omega)}$	$\left(\frac{2}{b}\right)\eta^{db}\frac{1}{\phi(\omega)}$ $(-)\eta^{dc}\frac{1}{\phi(\omega)}$	$\eta^{ab}\frac{\phi(\Omega)}{\psi(\Omega)}$	$(-)^{\frac{1}{2}(c+1)}ik$	$-\frac{k'^2}{k^2}$	$\frac{1}{k^2}$
$\begin{vmatrix} 0, 1 \\ -1, -1 \end{vmatrix}$ or $\lvert \rho^2 \rvert$	$\left(\frac{2}{c}\right)\eta^{-ac}\frac{1}{\psi(\omega)}$ $\left(\frac{2}{b}\right)\eta^{-ab}\frac{1}{\psi(\omega)}$	$\eta^{-cd}\frac{\psi(\Omega)}{\phi(\Omega)}$	$\eta^{-db}\frac{\phi(\omega)}{\psi(\omega)}$	$\left(\frac{2}{b}\right)\eta^{-ab}\frac{1}{\phi(\Omega)}$ $\left(\frac{2}{c}\right)\eta^{-ac}\frac{1}{\psi(\Omega)}$	$(-)^{\frac{1}{2}(b-1)}ik'$	$\frac{1}{k'^2}$	$-\frac{k^2}{k'^2}$

The matrix of the linear transformation being $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$, so that

$$\omega = \frac{c + d\Omega}{a + b\Omega}, \qquad ad - bc = 1,$$

the Table gives, for each of the six types of the unit matrix, the values of $\phi(\Omega)$ and $\psi(\Omega)$ in terms of $\phi(\omega)$ and $\psi(\omega)$; and also conversely the values of $\phi(\omega)$ and $\psi(\omega)$ in terms of $\phi(\Omega)$ and $\psi(\Omega)$. This second set of determinations is implicitly included in the first, and is given for convenience only; for the same reason, a column is added giving the values of $\Phi(\Omega) = \lambda^2$, $\Psi(\Omega) = \lambda'^2 = 1 - \lambda^2$, in terms of $\Phi(\omega) = k^2$, and $\Psi(\omega) = k'^2 = 1 - k^2$. The value of the multiplier, corresponding to each transformation, is also given (Art. 29, equations (iii) and (iv)).

To demonstrate the formulæ of the Table, we first consider the case in which $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ is of the type 1. We have

$$\phi(\Omega) = \sqrt{2}\,\frac{\vartheta_2[\Omega]}{\vartheta_2[\frac{1}{2}\Omega]}$$

and

$$\omega = \frac{c + d\Omega}{a + b\Omega}, \qquad \tfrac{1}{2}\omega = \frac{\frac{1}{2}c + d \times \frac{1}{2}\Omega}{a + 2b \times \frac{1}{2}\Omega}.$$

Hence, by the formulæ of Art. 28,

$$\vartheta_{1,0}[\Omega] = e^{-\frac{1}{4}i\pi ac}\left(\frac{a}{b}\right) i^{-\frac{1}{2}a} \times \frac{1}{\sqrt{[-i(a + b\Omega)]}}\,\vartheta_{1,0}[\omega],$$

$$\vartheta_{1,0}[\tfrac{1}{2}\Omega] = e^{-\frac{1}{8}i\pi ac}\left(\frac{2b}{a}\right) i^{-\frac{1}{8}a} \times \frac{1}{\sqrt{[-i(a + b\Omega)]}}\,\vartheta_{1,0}[\tfrac{1}{2}\omega].$$

The combination of these equations gives

$$\phi(\Omega) = \left(\frac{2}{a}\right) e^{-\frac{1}{8}i\pi ac}\,\phi(\omega) = \left(\frac{2}{d}\right) e^{-\frac{1}{8}i\pi dc}\,\phi(\omega). \quad \ldots\ldots \text{(iii)}$$

Again,

$$\psi(\Omega) = \frac{\vartheta_0[\Omega]}{\vartheta_0[2\Omega]}, \qquad \omega = \frac{c + d\Omega}{a + b\Omega}, \qquad 2\omega = \frac{2c + d \times 2\Omega}{a + \frac{1}{2}b \times 2\Omega}$$

whence

$$\psi(\Omega) = e^{\frac{1}{8}i\pi ab} \times \left(\frac{2}{a}\right) \frac{\vartheta_0[\omega]}{\vartheta_0[2\omega]},$$

or

$$\psi(\Omega) = \left(\frac{2}{a}\right) e^{\frac{1}{8}i\pi ab} \times \psi(\omega) \quad \ldots\ldots\ldots \text{(iv)}$$

Next, let $\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right|$ be of the type ψ; we have

$$2\omega = \frac{c+2d \times \frac{1}{2}\Omega}{\frac{1}{2}a+b \times \frac{1}{2}\Omega} \qquad \tfrac{1}{2}\omega = \frac{c+\frac{1}{2}d \times 2\Omega}{2a+b \times 2\Omega},$$

and hence

$$\phi(\Omega) = \sqrt{2} \times \frac{\vartheta_{1,0}[\Omega]}{\vartheta_{1,0}[\frac{1}{2}\Omega]} = \left(\frac{2}{b}\right) e^{-\frac{1}{8}i\pi ab} \times \frac{\vartheta_{0,1}[\omega]}{\vartheta_{0,1}[2\omega]},$$

$$\phi(\Omega) = \left(\frac{2}{b}\right) e^{-\frac{1}{8}i\pi ab} \times \psi(\omega); \quad \ldots\ldots\ldots\ldots \text{(v)}$$

and also

$$\psi(\Omega) = \frac{\vartheta_{0,1}[\Omega]}{\vartheta_{0,1}[2\Omega]} = \left(\frac{2}{b}\right) e^{-\frac{1}{8}i\pi bd} \times \frac{\sqrt{2}\,\vartheta_{1,0}[\omega]}{\vartheta_{1,0}[\frac{1}{2}\omega]},$$

or

$$\psi(\Omega) = \left(\frac{2}{b}\right) e^{-\frac{1}{8}i\pi bd} \times \phi(\omega). \quad \ldots\ldots\ldots\ldots \text{(vi)}$$

Each of the formulæ (v) and (vi) involves the other.

The cases in which $\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right|$ is of one of the types σ, τ, ρ, ρ^2 may be treated similarly, but the formulæ relating to them may also be obtained as follows:

Putting $\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right| = \left|\begin{matrix} 0, & 1 \\ -1, & 0 \end{matrix}\right| = |\psi|$ in (v) and (vi), we find

$$\phi\left(-\frac{1}{\omega}\right) = \psi(\omega), \qquad \psi\left(-\frac{1}{\omega}\right) = \phi(\omega). \quad \ldots\ldots \text{(vii)}$$

Again, if $\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right| = \left|\begin{matrix} 1, & 0 \\ 1, & 1 \end{matrix}\right| = |\sigma|$, we have from the equations of definition of $\phi(\omega)$ and $\psi(\omega)$, Art. 2, equation (iv),

$$\phi(\Omega) = \phi(\omega-1) = e^{-\frac{1}{8}i\pi}\frac{\phi(\omega)}{\psi(\omega)}, \qquad \psi(\Omega) = \psi(\omega-1) = \frac{1}{\psi(\omega)}. \quad \ldots \text{(viii)}$$

The formulæ relating to the unit-matrices $|\tau|$, $|\rho|$, $|\rho^2|$ may be obtained by combining the formulæ (vii) and (viii). If, for example,

$$\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right| = |\tau| = |\sigma| \times |\psi| \times |\sigma|,$$

let

$$\omega = |\sigma| \times \omega_1, \qquad \omega_1 = |\psi| \times \omega_2, \qquad \omega_2 = |\sigma| \times \Omega,$$

we find successively

$$\left.\begin{aligned} \phi(\Omega) &= e^{-\frac{1}{2}i\pi}\frac{\phi(\omega_2)}{\psi(\omega_2)} = e^{-\frac{1}{2}i\pi}\frac{\psi(\omega_1)}{\phi(\omega_1)} = \frac{1}{\phi(\omega)}, \\ \psi(\Omega) &= \frac{1}{\psi(\omega_2)} = \frac{1}{\phi(\omega_1)} = e^{\frac{1}{2}i\pi}\frac{\psi(\omega)}{\phi(\omega)}. \end{aligned}\right\} \quad \ldots\ldots \text{(ix)}$$

Similarly, since

$$|\rho| = |\psi| \times |\sigma|, \qquad \rho^2 = |\sigma|^{-1} \times |\psi|,$$

we have, if $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = |\rho|$,

$$\phi(\Omega) = e^{-\frac{1}{2}i\pi}\frac{\psi(\omega)}{\phi(\omega)}, \quad \psi(\Omega) = \frac{1}{\phi(\omega)}, \quad \ldots\ldots\ldots \text{(x)}$$

and, if $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = |\rho^2|$,

$$\phi(\Omega) = \frac{1}{\psi(\omega)}, \qquad \psi(\Omega) = e^{\frac{1}{2}i\pi}\frac{\phi(\omega)}{\psi(\omega)}. \quad \ldots\ldots\ldots \text{(xi)}$$

Combining the special formulæ (viii) to (xi) with the general formulæ (iii) to (vi), we obtain the formulæ of the Table. A single example of the process will suffice. Let $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ be of the type ρ^2, and let us write

$$\omega = |\rho^2| \times \omega_1, \qquad \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = |\rho^2| \times \begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix}, \qquad \omega_1 = \begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix} \times \Omega,$$

the matrix $\begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix}$ being of type 1.

From (iii) and (ix) we have

$$\phi(\Omega) = \left(\frac{2}{a'}\right) e^{-\frac{1}{2}i\pi a'c'} \times \phi(\omega_1), \qquad \phi(\omega_1) = \frac{1}{\psi(\omega)}.$$

But

$$\begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix} = -|\rho| \times \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = \begin{vmatrix} -a-c, & -b-d \\ a, & b \end{vmatrix},$$

whence, by Art. 21, equation (iv),

$$\left(\frac{2}{a'}\right) e^{-\frac{1}{2}i\pi a'c'} = \left(\frac{2}{c}\right) e^{-\frac{1}{2}i\pi ac} = \left(\frac{2}{b}\right) e^{-\frac{1}{2}i\pi ab},$$

and finally

$$\phi(\Omega) = \left(\frac{2}{c}\right) \eta^{-ac} \frac{1}{\psi(\omega)} = \left(\frac{2}{b}\right) \eta^{-ab} \frac{1}{\psi(\omega)},$$

in accordance with the Table.

31. *Linear Transformation of* $\chi(\omega)$.

The formulæ for the linear transformation of the functions $\chi(\omega)$, which were first given by M. Hermite, have been presented by M. Schlaefli in the following form.

Let $\zeta = e^{\frac{1}{24}i\pi}$, and distinguishing the six cases in which the matrix of transformation $\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right|$ is of the type $1, \psi, \sigma, \tau, \rho, \rho^2$ respectively, let C be an integral number, defined for the modulus 48 by the congruences

$$\left.\begin{aligned} &C \equiv (a+d)(b-acd)\,; \text{ cases } \psi, \rho^2, \\ &C \equiv (b-c)(bcd-a)\,; \text{ cases } 1, \sigma, \\ &C \equiv (b-c)(abc-d)\,; \text{ cases } 1, \tau, \\ &C \equiv (a+d)(abd-c)\,; \text{ cases } \psi, \rho. \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(i)}$$

We then have

$$\left.\begin{aligned} &\chi(\omega) = \zeta^c \times \chi(\Omega)\,; \text{ cases } 1, \psi, \\ &\chi(\omega) = \zeta^c \times \frac{\chi(\Omega)}{\psi(\Omega)}\,; \text{ cases } \rho, \sigma, \\ &\chi(\omega) = \zeta^c \times \frac{\chi(\Omega)}{\psi(\Omega)}\,; \text{ cases } \rho^2, \tau. \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(ii)}$$

To establish these formulæ we have from Art. 3, equation (xv),

$$\chi(1+\omega) = \zeta \times \frac{\chi(\omega)}{\psi(\omega)};$$

and

$$\chi^3\left(-\frac{1}{\omega}\right) = \phi\left(-\frac{1}{\omega}\right) \times \psi\left(-\frac{1}{\omega}\right) = \psi(\omega) \times \phi(\omega) = \chi^3(\omega).$$

Hence $\chi\left(-\frac{1}{\omega}\right) \div \chi(\omega)$ is a cube root of unity; but this cube root is $+1$; for, since $\chi(\omega)$ and $\chi\left(-\frac{1}{\omega}\right)$ are one-valued and continuous functions of ω, the cube root of unity must be the same for all values of ω, and cannot be imaginary when the real part of ω vanishes, in which case $\chi(\omega)$ and $\chi\left(-\frac{1}{\omega}\right)$ are both real.

From the two equations

$$\left.\begin{aligned} &\chi(\omega) = \chi(\Omega), \qquad \omega = |\psi| \times \Omega, \\ &\chi(\omega) = \zeta \times \frac{\chi(\Omega)}{\phi(\Omega)}, \qquad \omega = |\sigma| \times \Omega, \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(iii)}$$

we find, attending to the relations

$$|\tau| = |\sigma| \times |\psi| \times |\sigma|, \quad |\rho| = |\psi| \times |\sigma|, \quad |\rho|^2 = |\sigma|^{-1} \times |\psi|,$$

$$\left.\begin{aligned} \chi(\omega) &= \zeta^{-1} \times \frac{\chi(\Omega)}{\phi(\Omega)}; \quad \omega = |\tau| \times \Omega, \\ \chi(\omega) &= \zeta \times \frac{\chi(\Omega)}{\phi(\Omega)}; \quad \omega = |\rho| \times \Omega, \\ \chi(\omega) &= \zeta^{-1} \times \frac{\chi(\Omega)}{\phi(\Omega)}; \quad \omega = |\rho|^2 \times \Omega. \end{aligned}\right\} \quad \ldots\ldots \quad \text{(iv)}$$

We have also

$$\left.\begin{aligned} \chi(\omega) &= \zeta^2 \times \chi(\Omega); \quad \omega = |\sigma|^2 \times \Omega, \\ \chi(\omega) &= \zeta^{-2} \times \chi(\Omega); \quad \omega = |\tau|^2 \times \Omega. \end{aligned}\right\} \quad \ldots\ldots \quad \text{(v)}$$

Let, as in Art. 22,

$$\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right| = |\sigma|^{2\mu_1} \times |\tau|^{2\nu_1} \times |\sigma|^{2\mu_2} \times \ldots \times |\lambda|,$$

where $|\lambda|$ is one of the six unit matrices

$$|1|, \quad |\psi|, \quad |\sigma|, \quad |\tau|, \quad |\rho|, \quad |\rho|^2;$$

the formulæ (ii) are verified, if in them we attribute to the exponent C the values

$$\left.\begin{aligned} C &= \Sigma\mu - \Sigma\nu; && \text{cases } 1, \psi, \\ C &= \Sigma\mu - \Sigma\nu + 1; && \text{cases } \rho, \sigma, \\ C &= \Sigma\mu - \Sigma\nu - 1; && \text{cases } \rho^2, \tau. \end{aligned}\right\} \quad \ldots\ldots \quad \text{(vi)}$$

These values of C coincide with the congruential values given by the formulæ (i) when $\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right|$ is one of the six simple unit matrices; to establish the coincidence generally it is sufficient to show that

$$C' \equiv C + 2\mu, \quad C'' \equiv C - 2\nu, \text{ mod } 48, \quad \ldots\ldots \quad \text{(vii)}$$

C' and C'' denoting the values assumed by the expressions in the right-hand members of the congruences (i), when $|\sigma|^{2\mu} \times \left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right|$ and $|\tau|^{2\nu} \times \left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right|$ respectively are substituted for $\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right|$. The congruences (vii) are readily verified in the several cases; for example, if $\left|\begin{matrix} a, & b \\ c, & d \end{matrix}\right|$ is of the type ψ, so that

a, d are even, b, c uneven, we have

$$C \equiv (a+d)(b-acd) \equiv \quad b(a+d), \text{ mod } 16,$$
$$C \equiv (a+d)(abd-c) \equiv -c(a+d), \text{ mod } 16,$$

the two values coinciding because

$$(b+c)(a+d) \equiv 0, \qquad \text{mod } 16.$$

Hence
$$C' \equiv b(a+d+2\mu b) \equiv C+2\mu, \text{ mod } 16,$$
$$C'' \equiv -c(a+2\nu c+d) \equiv C-2\nu, \text{ mod } 16.$$

Again, the four values of C in the congruences (i) are all congruous to one another for the modulus 3; as may be seen by distinguishing the three cases $ad \equiv 1$, $bc \equiv 0$, mod 3; $ad \equiv 0$, $bc \equiv -1$, mod 3; $ad \equiv -bc \equiv -1$, mod 3. And using either of the last two values of C we may immediately verify the congruence $C' \equiv C+2\mu$, mod 3, while either of the first two values will serve to establish the congruence $C'' \equiv C-2\nu$, mod 3.

It will be noticed that the formulæ (vi) are obtained by direct reasoning; but that the congruential values (i) (which may be exhibited in many different forms) are established only by an *à posteriori* verification.

32. *Linear Transformation of the Elliptic Functions.*

For the linear transformation of the elliptic functions the determination of the constants C of Art. 28 is not necessary; and it is sufficient to assign the ratio of any two of them to one another. This may be done by means of the formula (see Art. 33, equation (x))

$$\left.\begin{aligned} &\vartheta_{\mu+\mu_1,\,\nu+\nu_1}\left[(a+b\Omega)\frac{\pi x}{h},\ \Omega\right] \\ &= C \times e^{\frac{1}{4}i\pi(m_1 n_1 - \mu_1\nu_1)} \times e^{\frac{1}{2}i\pi(m_1 n - \mu_1 \nu)} \times \vartheta_{m+m_1,\,n+n_1}\left(\frac{\pi x}{h},\ \omega\right) \times e^{-i\pi b(a+b\Omega)\frac{x^2}{h^2}}, \end{aligned}\right\}$$

in which μ_1, ν_1 are any two integral numbers, and m_1, n_1 are determined by the equations

$$m_1 = a\mu_1 + b\nu_1, \qquad n_1 = c\mu_1 + d\nu_1,$$

the other symbols having the same signification as in equation (i) Art. 28.

In the following tables, however, we have given, for convenience, the values of the transformed Theta functions, as well as those of the transformed elliptic functions; the latter set of values being derived from the former by writing

$h = 2K(\omega)$, $\dfrac{a+b\Omega}{h} = \dfrac{1}{2MK(\Omega)}$, and forming the quotients in accordance with the equations of definition (Art. 7, equations (xiii), (xiv), (xv)). We consider the general transformation of type 1, subject only to the restriction that b is not to be negative, but only the simplest transformations of the types ψ, σ, τ, ρ, ρ^2; the real parts of the radicals $\surd(a+b\Omega)$, $\sqrt{\Omega}$, $\surd(\Omega \pm 1)$ are supposed to be positive, and η denotes, as before, $e^{\frac{1}{4}i\pi}$.

I.

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}, \text{ mod } 2, \qquad \omega = \frac{c+d\Omega}{a+b\Omega}, \qquad P = \frac{i^{-\frac{1}{2}(a-1)} e^{i\pi b(a+b\Omega)\frac{x^2}{h^2}}}{\surd(a+b\Omega)},$$

$$\vartheta_0\left((a+b\Omega)\frac{\pi x}{h}, \Omega\right) = i^{\frac{1}{2}bd} \times P \times \vartheta_0\left(\frac{\pi x}{h}\right),$$

$$\vartheta_1\left((a+b\Omega)\frac{\pi x}{h}, \Omega\right) = (-)^{\frac{1}{2}(a-1)} i^{\frac{1}{2}(bd-ac)} \times P \times \vartheta_1\left(\frac{\pi x}{h}\right),$$

$$\vartheta_2\left((a+b\Omega)\frac{\pi x}{h}, \Omega\right) = i^{-\frac{1}{2}ac} \times P \times \vartheta_2\left(\frac{\pi x}{h}\right),$$

$$\vartheta_3\left((a+b\Omega)\frac{\pi x}{h}, \Omega\right) = P \times \vartheta_3\left(\frac{\pi x}{h}\right),$$

$$\phi(\Omega) = \left(\frac{2}{a}\right)\eta^{-ac}\phi(\omega), \qquad \psi(\Omega) = \left(\frac{2}{a}\right)\eta^{ab}\psi(\omega),$$

$$\lambda^2 = k^2, \qquad \frac{1}{M} = (-1)^{\frac{1}{2}(a-1)}.$$

The elliptic functions are unchanged; viz.

$$\operatorname{sn}\operatorname{am}\left(\frac{x}{M}, \lambda^2\right) = \operatorname{sn}\operatorname{am}\left((-)^{\frac{1}{2}(a-1)}x, k^2\right) = (-)^{\frac{1}{2}(a-1)}\operatorname{sn}\operatorname{am}x,$$

$$\operatorname{cn}\operatorname{am}\left(\frac{x}{M}, \lambda^2\right) = \operatorname{cn}\operatorname{am}x, \qquad \Delta\operatorname{am}\left(\frac{x}{M}, \lambda^2\right) = \Delta\operatorname{am}x.$$

II.

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = \begin{vmatrix} 0, & 1 \\ -1, & 0 \end{vmatrix}, \qquad \omega = -\frac{1}{\Omega}, \qquad P = \frac{e^{-i\pi\Omega\frac{x^2}{h^2}}}{\sqrt{\Omega}},$$

$$\vartheta_0\left(\frac{\pi x}{h}, \Omega\right) = \eta^2 P \times \vartheta_2\left(\frac{\pi x}{h}\right),$$

$$\vartheta_1\left(\frac{\pi x}{h}, \Omega\right) = -\eta^{-2} P \times \vartheta_1\left(\frac{\pi x}{h}\right),$$

$$\vartheta_2\left(\frac{\pi x}{h}, \Omega\right) = \quad \eta^2 P \times \vartheta_0\left(\frac{\pi x}{h}\right),$$

$$\vartheta_3\left(\frac{\pi x}{h}, \Omega\right) = \quad \eta^2 P \times \vartheta_3\left(\frac{\pi x}{h}\right),$$

$$\phi(\Omega) = \psi(\omega), \qquad \psi(\Omega) = \phi(\omega), \qquad \lambda^2 = k'^2, \qquad \frac{1}{M} = i,$$

$$\sin \operatorname{am}(ix, k'^2) = i \frac{\sin \operatorname{am} x}{\cos \operatorname{am} x}, \qquad \cos \operatorname{am}(ix, k'^2) = \frac{1}{\cos \operatorname{am} x},$$

$$\Delta \operatorname{am}(ix, k'^2) = \frac{\Delta \operatorname{am} x}{\cos \operatorname{am} x}.$$

III.

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = \begin{vmatrix} 1, & 0 \\ \pm 1, & 1 \end{vmatrix}, \qquad \omega = \pm 1 + \Omega,$$

$$\vartheta_0\left(\frac{\pi x}{h}, \Omega\right) = \quad \vartheta_3\left(\frac{\pi x}{h}\right),$$

$$\vartheta_1\left(\frac{\pi x}{h}, \Omega\right) = \eta^{\mp 2} \vartheta_1\left(\frac{\pi x}{h}\right),$$

$$\vartheta_2\left(\frac{\pi x}{h}, \Omega\right) = \eta^{\mp 2} \vartheta_2\left(\frac{\pi x}{h}\right),$$

$$\vartheta_3\left(\frac{\pi x}{h}, \Omega\right) = \quad \vartheta_0\left(\frac{\pi x}{h}\right),$$

$$\phi(\Omega) = \eta^{\mp 1} \frac{\phi(\omega)}{\psi(\omega)}, \qquad \psi(\Omega) = \frac{1}{\psi(\omega)},$$

$$\lambda^2 = \frac{k^2}{k'^2}, \qquad \frac{1}{M} = k',$$

$$\sin \operatorname{am}\left(k'x, \frac{k^2}{k'^2}\right) = k' \frac{\sin \operatorname{am} x}{\Delta \operatorname{am} x}, \qquad \cos \operatorname{am}\left(k'x, \frac{k^2}{k'^2}\right) = \frac{\cos \operatorname{am} x}{\Delta \operatorname{am} x},$$

$$\Delta \operatorname{am}\left(k'x, \frac{k^2}{k'^2}\right) = \frac{1}{\Delta \operatorname{am} x}.$$

IV.

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = \begin{vmatrix} \pm 1, & 1 \\ 0, & \pm 1 \end{vmatrix}, \qquad \omega = \frac{\pm\Omega}{\pm 1 + \Omega}, \quad P = \frac{e^{-i\pi(\Omega \pm 1)\frac{x^2}{h^2}}}{\sqrt{(\Omega \pm 1)}},$$

$$\vartheta_0\left(\frac{\pi x}{h}(\pm 1+\Omega), \Omega\right) = \quad \eta^2 P \times \vartheta_0\left(\frac{\pi x}{h}\right),$$

$$\vartheta_1\left(\frac{\pi x}{h}(\pm 1+\Omega), \Omega\right) = \pm\eta^2 P \times \vartheta_1\left(\frac{\pi x}{h}\right),$$

$$\vartheta_2\left(\frac{\pi x}{h}(\pm 1+\Omega), \Omega\right) = \eta^{2\mp 2} P \times \vartheta_3\left(\frac{\pi x}{h}\right),$$

$$\vartheta_3\left(\frac{\pi x}{h}(\pm 1+\Omega), \Omega\right) = \eta^{2\mp 2} P \times \vartheta_2\left(\frac{\pi x}{h}\right),$$

$$\phi(\Omega) = \frac{1}{\phi(\omega)}, \qquad \psi(\Omega) = \eta^{\pm 1}\frac{\psi(\omega)}{\phi(\omega)},$$

$$\lambda^2 = \frac{1}{k^2}, \qquad \frac{1}{M} = \pm k,$$

$$\sin\mathrm{am}\left(kx, \frac{1}{k}\right) = k \sin\mathrm{am} x, \qquad \cos\mathrm{am}\left(kx, \frac{1}{k}\right) = \Delta\,\mathrm{am} x,$$

$$\Delta\,\mathrm{am}\left(kx, \frac{1}{k}\right) = \cos\mathrm{am} x.$$

V.

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = \begin{vmatrix} \pm 1, & 1 \\ -1, & 0 \end{vmatrix}, \qquad \omega = \frac{-1}{\pm 1 + \Omega}, \qquad P = \frac{e^{-i\pi(\Omega \pm 1)\frac{x^2}{h^2}}}{\sqrt{(\Omega \pm 1)}}$$

$$\vartheta_0\left(\frac{\pi x}{h}(\pm 1+\Omega), \Omega\right) = \quad \eta^2 P \times \vartheta_3\left(\frac{\pi x}{h}\right),$$

$$\vartheta_1\left(\frac{\pi x}{h}(\pm 1+\Omega), \Omega\right) = i\eta^{2\mp 2} P \times \vartheta_1\left(\frac{\pi x}{h}\right),$$

$$\vartheta_2\left(\frac{\pi x}{h}(\pm 1+\Omega), \Omega\right) = \eta^{2\mp 2} P \times \vartheta_0\left(\frac{\pi x}{h}\right),$$

$$\vartheta_3\left(\frac{\pi x}{h}(\pm 1+\Omega), \Omega\right) = \quad \eta^2 P \times \vartheta_2\left(\frac{\pi x}{h}\right),$$

$$\phi(\Omega) = \eta^{\mp 1}\frac{\psi(\omega)}{\phi(\omega)}, \qquad \psi(\Omega) = \frac{1}{\psi(\omega)},$$

$$\lambda^2 = -\frac{k'^2}{k^2}, \qquad \frac{1}{M} = ik,$$

$$\sin\mathrm{am}\left(ikx, -\frac{k'^2}{k^2}\right) = ik\frac{\sin\mathrm{am}x}{\Delta\mathrm{am}x}, \qquad \cos\mathrm{am}\left(ikx, -\frac{k'^2}{k^2}\right) = \frac{1}{\Delta\mathrm{am}x},$$

$$\Delta\,\mathrm{am}\left(ikx, -\frac{k'^2}{k^2}\right) = \frac{\cos\mathrm{am}x}{\Delta\,\mathrm{am}x}.$$

VI.

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = \begin{vmatrix} 0, & 1 \\ -1, & \mp 1 \end{vmatrix}, \qquad \omega = -\frac{1 \pm \Omega}{\Omega}, \qquad P = \frac{e^{i\pi\Omega\frac{x^2}{h^2}}}{\sqrt{\Omega}},$$

$$\vartheta_0\left(\frac{\pi x\Omega}{h}, \Omega\right) = \eta^{2\pm 2}P \times \vartheta_2\left(\frac{\pi x}{h}\right),$$

$$\vartheta_1\left(\frac{\pi x\Omega}{h}, \Omega\right) = i\eta^{2\pm 2}P \times \vartheta_1\left(\frac{\pi x}{h}\right),$$

$$\vartheta_2\left(\frac{\pi x\Omega}{h}, \Omega\right) = \eta^2 P \times \vartheta_3\left(\frac{\pi x}{h}\right),$$

$$\vartheta_3\left(\frac{\pi x\Omega}{h}, \Omega\right) = \eta^2 P \times \vartheta_0\left(\frac{\pi x}{h}\right),$$

$$\phi(\Omega) = \frac{1}{\psi(\omega)}, \qquad \psi(\Omega) = \eta^{\pm 1}\frac{\phi(\omega)}{\psi(\omega)},$$

$$\lambda^2 = \frac{1}{k'^2}, \qquad \frac{1}{M} = ik',$$

$$\sin\mathrm{am}\left(ik'x, \frac{1}{k'^2}\right) = ik'\frac{\sin\mathrm{am}x}{\cos\mathrm{am}x}, \qquad \cos\mathrm{am}\left(ik'x, \frac{1}{k'^2}\right) = \frac{\Delta\,\mathrm{am}x}{\cos\mathrm{am}x},$$

$$\Delta\,\mathrm{am}\left(ik'x, \frac{1}{k'^2}\right) = \frac{1}{\cos\mathrm{am}x}.$$

33. *Transformation of any uneven Order.—Development of the Solution.* We have now to determine the function T in the equation (xv) of Art. 25. The zeros of $\vartheta_{\mu,\nu}\left[(a+b\Omega)\frac{\pi x}{h}, \Omega\right]$ are, by Art. 3, equation (vii),

$$\tfrac{1}{2}h \times \frac{1}{a+b\Omega} \times [2r+\nu-1+(2s+\mu-1)\Omega], \quad \ldots\ldots \quad \text{(i)}$$

where r and s are any two integers whatever. For given values of r and s, the equations

$$\left.\begin{aligned} pa+qc &= (2r+\nu-1)\Delta, \\ pb+qd &= (2s+\mu-1)\Delta, \end{aligned}\right\} \quad \ldots\ldots \quad \text{(ii)}$$

are always resoluble in integral numbers p and q, in such a manner that to every different pair of values of r and s there corresponds a different pair of values of p and q. When we attribute to r and s all pairs of values in succession, the pairs of values assumed by p and q are, in their totality, the same as the pairs of values satisfying the simultaneous congruences

$$\left.\begin{aligned} pa+qc &\equiv \nu-1, \\ pb+qd &\equiv \mu-1, \end{aligned}\right\} \text{ mod } 2, \quad \ldots\ldots \quad \text{(iii)}$$

$$\left.\begin{aligned} pa+qc &\equiv 0, \\ pb+qd &\equiv 0, \end{aligned}\right\} \text{ mod } \Delta. \quad \ldots\ldots \quad \text{(iv)}$$

The solution of the congruences (iii) is (see Art. 25, equation (xiii))

$$p \equiv n-1, \qquad q \equiv m-1, \qquad \text{mod } 2\,; \quad \ldots\ldots \quad \text{(v)}$$

the congruences (iv) admit of Δ sets of incongruous solutions

$$p \equiv 0, \quad q \equiv 0\,; \qquad p \equiv \pm p_j, \quad q \equiv \pm q_j, \quad \text{mod}\,\Delta, \quad \ldots\ldots \quad \text{(vi)}$$

where $j = 1, 2, \ldots, \frac{1}{2}(\Delta-1)$, and where we may suppose that p_j, q_j satisfy the congruences (v). Thus all the numbers p and q satisfying the simultaneous congruences (iii) and (iv) are included in the formulæ

$$[(2r'+n-1)\Delta, \quad (2s'+m-1)\Delta], \quad [2r'\Delta \pm p_j, \quad 2s'\Delta \pm q_j], \quad \ldots \quad \text{(vii)}$$

r' and s' denoting any two integral numbers whatever. Since a, b, c, d have no common divisor, a certain number of the pairs $[p_j, q_j]$ (viz. as many as there are numbers less than Δ and prime to it), have no common divisor with Δ. If $[p, q]$ is such a pair, the $\Delta-1$ pairs may be all represented by the formula $\pm[jp, jq]$, in which j is any one of the uneven numbers 1, 3, ..., $\Delta-2$; or, again (when $\mu \equiv \nu \equiv 1$, mod 2) any one of the even numbers 2, 4, 6, ..., $\Delta-1$. The zeros of $\vartheta_{\mu,\nu}[(a+b\Omega)\frac{\pi x}{h}, \Omega]$ are accordingly

$$\left.\begin{array}{ll} (a) & \frac{1}{2}h\,[2r'+n-1+(2s'+m-1)\omega], \\ (b) & \frac{1}{2}h\,[2r'+2s'\omega \pm j\frac{p+q\omega}{\Delta}], \end{array}\right\} \quad \ldots\ldots \text{(viii)}$$

By Art. 3, equation (vii), the formula (a) represents the zeros of $\vartheta_{m,n}\left(\frac{\pi x}{h}, \omega\right)$.

Again, if $\eta = \frac{p+q\omega}{2\Delta}$, the $\frac{1}{2}(\Delta-1)$ formulæ (b) represent the zeros of the $\frac{1}{2}(\Delta-1)$ functions

$$\vartheta^2_{\alpha,\alpha'}\left(\frac{\pi x}{h}\right)\vartheta^2_{\beta,\beta'}(\pi j\eta) - \vartheta^2_{\beta,\beta'}\left(\frac{\pi x}{h}\right)\vartheta^2_{\alpha,\alpha'}(\pi j\eta),$$

$\vartheta^2_{\alpha,\alpha'}$ and $\vartheta^2_{\beta,\beta'}$ denoting any two different Theta functions of the arguments $\frac{\pi x}{h}$ and ω. For the periods of the even, uniform, and doubly periodic function $\vartheta^2_{\alpha,\alpha'}\left(\frac{\pi x}{h}\right) \div \vartheta^2_{\beta,\beta'}\left(\frac{\pi x}{h}\right)$ being h and $h\omega$, and one solution of the equation

$$\frac{\vartheta^2_{\alpha,\alpha'}\left(\frac{\pi x}{h}\right)}{\vartheta^2_{\beta,\beta'}\left(\frac{\pi x}{h}\right)} = \frac{\vartheta^2_{\alpha,\alpha'}(\pi j\eta)}{\vartheta^2_{\beta,\beta'}(\pi j\eta)}$$

being $x = jh\eta$, all the solutions of that equation are included in the corresponding formula (b). Thus the equation (xv) of Art. 25 becomes

$$\left.\begin{array}{c} \vartheta_{\mu,\nu}\left[(a+b\Omega)\frac{\pi x}{h}, \Omega\right] \\ = C\times\vartheta_{m,n}\left(\frac{\pi x}{h}, \omega\right)\times\Pi_j\left[\vartheta^2_{\alpha,\alpha'}\left(\frac{\pi x}{h}\right)\vartheta^2_{\beta,\beta'}(\pi j\eta) - \vartheta^2_{\beta,\beta'}\left(\frac{\pi x}{h}\right)\vartheta^2_{\alpha,\alpha'}(\pi j\eta)\right] \\ \times e^{-i\pi b(a+b\Omega)\frac{x^2}{h^2}}, \end{array}\right\} \quad . \quad \text{(ix)}$$

C denoting a constant which can always be determined by putting $x=0$, when the matrix of the transformation and the indices μ, ν; α, α'; β, β' are given. The constants C, appertaining to the same matrix, but to different values of these indices, are not independent of one another. To establish the relations between them, let μ_1, ν_1 be any two integral numbers; let also

$$m_1 = a\mu_1 + b\nu_1,$$
$$n_1 = c\mu_1 + d\nu_1;$$

and let us write in the equation (ix) $x-\frac{1}{2}h(m_1\omega-n_1)$ for x, and consequently

$$(a+b\Omega)\frac{x}{h}-\tfrac{1}{2}(\mu_1\Omega-\nu_1)\Delta \text{ for } (a+b\Omega)\frac{x}{h}.$$

Changing the indices of the Theta functions in (ix) by the equation (ix) of Art. 1, we find, after some reductions,

$$\left.\begin{aligned}&\vartheta_{\mu+\Delta\mu_1,\,\nu+\Delta\nu_1}\Big[(a+b\Omega)\frac{\pi x}{h},\,\Omega\Big]\\ &=C\times e^{\frac{1}{4}i\pi(\Delta m_1 n_1-\mu_1\nu_1)}\times e^{\frac{1}{2}i\pi(m_1 n-\Delta\mu_1\nu)}\times(-1)^{\frac{1}{2}(\Delta-1)m_1\alpha'}\times\vartheta_{m+n_1,n+n_1}\Big(\frac{\pi x}{h},\,\omega\Big)\\ &\times\Pi_j\Big[\vartheta^2_{a+m_1,\alpha'+n_1}\Big(\frac{\pi x}{h}\Big)\vartheta^2_{\beta,\beta'}(\pi j\eta)-(-1)^{m_1(\alpha'-\beta')}\times\vartheta^2_{\beta+m_1,\beta'+n_1}\Big(\frac{\pi x}{h}\Big)\vartheta^2_{a,\alpha'}(\pi j\eta)\Big]\\ &\quad\times e^{-i\pi b(a+b\Omega)\frac{x^2}{h^2}}.\end{aligned}\right\}\quad(x)$$

The formulæ (ix) and (x) complete the general solution of the problem of transformation of the Theta functions when Δ is uneven. The following special cases are convenient in the theory of elliptic functions. In the matrix $\left|\begin{matrix}a,&b\\c,&d\end{matrix}\right|$ let $b\equiv c\equiv 0$, mod 8, $a\equiv 1$, mod 4; this supposition is admissible if we regard the theory of linear transformations as known (see Art. 27); also, in the formulæ (ix) and (x) let

$$\mu=\nu=1,\qquad \alpha=\alpha'=1,\qquad \beta=0,\qquad \beta'=1,$$

whence

$$m\equiv a,\quad n\equiv d,\quad \text{mod } 8;\qquad p\equiv q\equiv 0,\quad \text{mod } 2;$$

so that we may attribute to the index j the series of even values $2, 4, 6, \ldots, \Delta-1$.

Writing Q for $e^{-i\pi b(a+b\Omega)\frac{x^2}{h^2}}$, we obtain from (ix)

$$\left.\begin{aligned}\vartheta_1\Big[(a+b\Omega)\frac{\pi x}{h},\,\Omega\Big]&=(-)^{\frac{1}{2}(\Delta-1)}CQ\times\vartheta_1\Big(\frac{\pi x}{h}\Big)\\ &\times\Pi_j\Big[\vartheta_1^2\Big(\frac{\pi x}{h}\Big)\vartheta_0^2(\pi j\eta)-\vartheta_0^2\Big(\frac{\pi x}{h}\Big)\vartheta_1^2(\pi j\eta)\Big];\end{aligned}\right\}\quad\ldots\quad(xi)$$

and from (x), in which we write successively,

(A) $\mu_1=1,\quad \nu_1=0,\quad m_1\equiv a,\quad n_1\equiv 0,\quad \text{mod } 8,$

(B) $\mu_1=0,\quad \nu_1=1,\quad m_1\equiv 0,\quad n_1\equiv d,\quad \text{mod } 8,$

(C) $\mu_1=\nu_1=1,\quad m_1\equiv a,\quad n_1\equiv d,\quad \text{mod } 8;$

$$\left.\begin{aligned}\vartheta_0\Big[(a+b\Omega)\frac{\pi x}{h},\ \Omega\Big] &= (-)^{\frac{1}{2}(\Delta-1)}CQ\times\vartheta_0\Big(\frac{\pi x}{h}\Big)\\ &\times\Pi_j\Big[\vartheta_0^2\Big(\frac{\pi x}{h}\Big)\vartheta_0^2(\pi j\eta)-\vartheta_1^2\Big(\frac{\pi x}{h}\Big)\vartheta_1^2(\pi j\eta)\Big],\end{aligned}\right\}\quad\text{(xii)}$$

$$\left.\begin{aligned}\vartheta_2\Big[(a+b\Omega)\frac{\pi x}{h},\ \Omega\Big] &= (-)^{\frac{1}{2}(\Delta-1)}CQ\times\vartheta_2\Big(\frac{\pi x}{h}\Big)\\ &\times\Pi_j\Big[\vartheta_2^2\Big(\frac{\pi x}{h}\Big)\vartheta_0^2(\pi j\eta)-\vartheta_3^2\Big(\frac{\pi x}{h}\Big)\vartheta_1^2(\pi j\eta)\Big],\end{aligned}\right\}\quad\text{(xiii)}$$

$$\left.\begin{aligned}\vartheta_3\Big[(a+b\Omega)\frac{\pi x}{h},\ \Omega\Big] &= (-)^{\frac{1}{2}(\Delta-1)}CQ\times\vartheta_3\Big(\frac{\pi x}{h}\Big)\\ &\times\Pi_j\Big[\vartheta_3^2\Big(\frac{\pi x}{h}\Big)\vartheta_0^2(\pi j\eta)-\vartheta_2^2\Big(\frac{\pi x}{h}\Big)\vartheta_1^2(\pi j\eta)\Big].\end{aligned}\right\}\quad\text{(xiv)}$$

Let $h=2K$; $\zeta=\dfrac{pK+iqK'}{2\Delta}=K\eta$; introducing the elliptic and modular functions, and attributing to j the values 1, 2, ..., $\frac{1}{2}(\Delta-1)$, we find from (xii), (xiii), (xiv),

$$\sqrt{\lambda}=\phi^2(\Omega)=\frac{\vartheta_2[\Omega]}{\vartheta_3[\Omega]}=k^{\frac{1}{2}\Delta}\times\Pi\frac{\cos^2\mathrm{am}\,4j\zeta}{\Delta^2\mathrm{am}\,4j\zeta}=k^{\frac{1}{2}\Delta}\times\Pi\sin^2\mathrm{coam}\,4j\zeta;\quad\text{(xv)}$$

$$\sqrt{\lambda'}=\psi^2(\Omega)=\frac{\vartheta_0[\Omega]}{\vartheta_3[\Omega]}=\frac{k'^{\frac{1}{2}\Delta}}{\Pi.\Delta^2\mathrm{am}\,4j\zeta}=k'\times\frac{\Pi.\Delta^2\mathrm{coam}\,4j\zeta}{k'^{\frac{1}{2}\Delta}}.\quad\text{(xvi)}$$

Again, dividing (xi), (xiii), (xiv) respectively by (xii), and attending to (xv) and (xvi), we have

$$\sin\mathrm{am}\Big(\frac{x}{M},\lambda\Big)=\frac{\sin\mathrm{am}\,x}{M}\times\Pi\frac{1-\dfrac{\sin^2\mathrm{am}\,x}{\sin^2\mathrm{am}\,4j\zeta}}{1-k^2\sin^2\mathrm{am}\,4j\zeta\sin^2\mathrm{am}\,x},\quad\text{(xvii)}$$

$$\cos\mathrm{am}\Big(\frac{x}{M},\lambda\Big)=\cos\mathrm{am}\,x\times\Pi\frac{1-\dfrac{\sin^2\mathrm{am}\,x}{\sin^2\mathrm{coam}\,4j\zeta}}{1-k^2\sin^2\mathrm{am}\,4j\zeta\sin^2\mathrm{am}\,x},\quad\text{(xviii)}$$

$$\Delta\,\mathrm{am}\Big(\frac{x}{M},\lambda\Big)=\Delta\,\mathrm{am}\,x\times\Pi\frac{1-k^2\sin^2\mathrm{coam}\,4j\zeta\sin^2\mathrm{am}\,x}{1-k^2\sin^2\mathrm{am}\,4j\zeta\sin^2\mathrm{am}\,x},\quad\text{(xix)}$$

$$\frac{1}{M}=(a+b\Omega)\frac{K(\Omega)}{K(\omega)}=(-)^{\frac{1}{2}(\Delta-1)}\frac{k^{\frac{1}{2}\Delta}}{\sqrt{\lambda}}\times\Pi\sin^2\mathrm{am}\,4j\zeta=(-)^{\frac{1}{2}(\Delta-1)}\frac{\Pi\sin^2\mathrm{am}\,4j\zeta}{\Pi\sin^2\mathrm{coam}\,4j\zeta}.\quad\text{(xx)}$$

The equations (xvii), (xviii), (xix) have been transformed by Jacobi as

follows. The equations for the addition of the elliptic functions (equations (xxii)–(xxiv) of Art. 7), give immediately

$$\left.\begin{aligned}
\sin\mathrm{am}(x+y)\times\sin\mathrm{am}(x-y) &= \frac{\sin^2\mathrm{am}\,x-\sin^2\mathrm{am}\,y}{1-k^2\sin^2\mathrm{am}\,x\sin^2\mathrm{am}\,y},\\
\cos\mathrm{am}(x+y)\times\cos\mathrm{am}(x-y) &= \frac{\cos^2\mathrm{am}\,y-\Delta^2\mathrm{am}\,y\sin^2\mathrm{am}\,x}{1-k^2\sin^2\mathrm{am}\,x\sin^2\mathrm{am}\,y},\\
\Delta\mathrm{am}(x+y)\times\Delta\mathrm{am}(x-y) &= \frac{\Delta^2\mathrm{am}\,y+k^2\cos^2\mathrm{am}\,y\sin^2\mathrm{am}\,x}{1-k^2\sin^2\mathrm{am}\,x\sin^2\mathrm{am}\,y}.
\end{aligned}\right\} \quad . \quad . \quad \text{(xxi)}$$

Employing these formulæ to transform the type factors in (xvii), (xviii), (xix), and attending to the values of $\sqrt{\lambda}$, $\sqrt{\lambda'}$, $\frac{1}{M}$, we obtain the equations

$$\sin\mathrm{am}\left(\frac{x}{M},\lambda\right) = \frac{k^{\frac{1}{2}\Delta}}{\lambda^{\frac{1}{4}}}\times\Pi\sin\mathrm{am}(x+4j''\zeta), \quad . \quad . \quad . \quad . \quad \text{(xxii)}$$

$$\cos\mathrm{am}\left(\frac{x}{M},\lambda\right) = \frac{\lambda'^{\frac{1}{4}}k^{\frac{1}{2}\Delta}}{\lambda^{\frac{1}{4}}k'^{\frac{1}{2}\Delta}}\times\Pi\cos\mathrm{am}(x+4j''\zeta), \quad . \quad . \quad . \quad . \quad \text{(xxiii)}$$

$$\Delta\mathrm{am}\left(\frac{x}{M},\lambda\right) = \frac{\lambda'^{\frac{1}{4}}}{k'^{\frac{1}{2}\Delta}}\times\Pi\,\Delta\mathrm{am}(x+4j''\zeta), \quad . \quad . \quad . \quad . \quad \text{(xxiv)}$$

in which j'' is to receive all values from 0 to $\Delta-1$ inclusive.

A third set of expressions for the transformed elliptic functions has also been given by Jacobi. Comparing the equations (xvii) and (xxii) we see that the roots of the equation

$$z\Pi[z^2-\sin^2\mathrm{am}\,4j'\zeta]-\frac{\lambda}{M}\sin\mathrm{am}\left(\frac{x}{M},\lambda\right)\times\Pi\left[z^2-\frac{1}{k^2\sin^2\mathrm{am}\,4j'\zeta}\right]=0, \quad . \quad \text{(xxv)}$$

which is of order Δ and in which $j=1, 2, \ldots, \frac{1}{2}(\Delta-1)$, are all comprised in the formula

$$z=\sin\mathrm{am}(x+4j\zeta), \quad j=0,1,2,\ldots,\Delta-1.$$

Hence, equating the sum of the roots to the coefficient of $-z^{\Delta-1}$, we find

$$\sin\mathrm{am}\left(\frac{x}{M},\lambda\right)=\frac{kM}{\lambda}\Sigma\sin\mathrm{am}(x+4j''\zeta); \quad . \quad . \quad . \quad . \quad \text{(xxvi)}$$

and similarly, from a comparison of (xviii) with (xxiii), and of (xix) with (xxiv),

$$\cos\mathrm{am}\left(\frac{x}{M},\lambda\right)=(-1)^{\frac{1}{2}(\Delta-1)}\frac{kM}{\lambda}\Sigma\cos\mathrm{am}\,(x+4j''\zeta),\quad\ldots\quad\text{(xxvii)}$$

$$\Delta\,\mathrm{am}\left(\frac{x}{M},\lambda\right)=(-1)^{\frac{1}{2}(\Delta-1)}\;M\Sigma\Delta\,\mathrm{am}\,(x+4j''\zeta),\quad\ldots\quad\text{(xxviii)}$$

the sign of summation Σ in the three equations extending to the values $0, 1, \ldots, \Delta-1$ of j''.

34.] *Quadratic Transformations of the Elliptic, Theta, and Modular Functions.*

The matrices of determinant 2 (see Art. 23) are of nine different types; and if we regard matrices differing only in sign as identical, two classes of matrices non-equivalent by primary postmultiplication, and only two, appertain to each of the nine types. There are thus nine pairs of different quadratic transformations; but, so far as the elliptic functions are concerned, the difference between the two transformations of any one pair is not essential. As representatives of the eighteen classes of matrices we take the matrices $C_{\alpha,\beta}$, $\Gamma_{\alpha,\beta}$ exhibited in the annexed schemes; the first index referring to the columns, the second to the rows.

$$(C)\qquad\left\|\begin{array}{ccc}\begin{vmatrix}1,&1\\-1,&1\end{vmatrix}, & \begin{vmatrix}1,&0\\1,&2\end{vmatrix}, & \begin{vmatrix}0,&1\\-2,&-1\end{vmatrix}\\ \begin{vmatrix}0,&2\\-1,&-1\end{vmatrix}, & \begin{vmatrix}2,&2\\-1,&0\end{vmatrix}, & \begin{vmatrix}2,&0\\0,&1\end{vmatrix}\\ \begin{vmatrix}1,&-1\\0,&2\end{vmatrix}, & \begin{vmatrix}1,&2\\-2,&-2\end{vmatrix}, & \begin{vmatrix}2,&1\\-2,&0\end{vmatrix}\end{array}\right\|;\qquad(\Gamma)\qquad\left\|\begin{array}{ccc}\begin{vmatrix}-1,&1\\-1,&-1\end{vmatrix}, & \begin{vmatrix}1,&2\\-1,&0\end{vmatrix}, & \begin{vmatrix}2,&1\\0,&1\end{vmatrix}\\ \begin{vmatrix}2,&0\\-1,&1\end{vmatrix}, & \begin{vmatrix}0,&2\\-1,&-2\end{vmatrix}, & \begin{vmatrix}2,&2\\-2,&-1\end{vmatrix}\\ \begin{vmatrix}1,&1\\-2,&0\end{vmatrix}, & \begin{vmatrix}1,&0\\0,&2\end{vmatrix}, & \begin{vmatrix}0,&1\\-2,&-2\end{vmatrix}\end{array}\right\|.$$

It will be seen (1) that $C_{\alpha,\beta}$ and $\Gamma_{\alpha,\beta}$ are of the same type, (2) that $C_{\alpha,\beta}$ and $\Gamma_{\beta,\alpha}$ are reciprocal, (3) that the matrices C, and also the matrices Γ, are transformed into one another (irrespectively of sign) by premultiplication and postmultiplication with the matrices of the scheme

$$\left|\begin{array}{ccc}1,&\rho,&\rho^2\\ \rho^2,&1,&\rho\\ \rho,&\rho^2,&1\end{array}\right|,$$

in the manner explained in Art. 23.

In the following Table I. of the eighteen quadratic transformations of the elliptic functions, M is the multiplier, λ is the transformed modulus, s, c, d are respectively

$$\sin\operatorname{am}(x, k^2), \qquad \cos\operatorname{am}(x, k^2), \qquad \Delta\operatorname{am}(x, k^2);$$

the columns $(M \times S)$, (C), (Δ) exhibit the numerators of the expressions for

$$M \times \sin\operatorname{am}\left(\frac{x}{M}, \lambda^2\right), \quad \cos\operatorname{am}\left(\frac{x}{M}, \lambda^2\right), \quad \Delta\operatorname{am}\left(\frac{x}{M}, \lambda^2\right)$$

respectively; the common denominator being placed in the column (D). Thus the first line of the Table supplies the three equations

$$\sin\operatorname{am}\left(2\sqrt{ikk'}\,x, \quad \left[\frac{k'+ik}{2\sqrt{ikk'}}\right]^2\right) = \frac{2\sqrt{ikk'}\sin\operatorname{am}x \times \Delta\operatorname{am}x}{1-k(k-ik')\sin^2\operatorname{am}x},$$

$$\cos\operatorname{am}\left(2\sqrt{ikk'}\,x, \quad \left[\frac{k'+ik}{2\sqrt{ikk'}}\right]^2\right) = \frac{1-k(k+ik')\sin^2\operatorname{am}x}{1-k(k-ik')\sin^2\operatorname{am}x},$$

$$\Delta\operatorname{am}\left(2\sqrt{ikk'}\,x, \quad \left[\frac{k'+ik}{2\sqrt{ikk'}}\right]^2\right) = \frac{\cos\operatorname{am}x}{1-k(k-ik')\sin^2\operatorname{am}x},$$

and indicates that they arise from the transformation $\omega = \dfrac{-1+\Omega}{1+\Omega}$, or, more generally, from any transformation of the type

$$\omega = \begin{vmatrix} 1, & 1 \\ -1, & 1 \end{vmatrix} \times |\alpha| \times \Omega,$$

where $|\alpha|$ is a primary unit matrix. It will be observed that if in the Table I. we substitute for one of the matrices C or Γ any other matrix equivalent to it by primary postmultiplication, the signs of M and λ may change; but this does not affect the elliptic transformation.

When the index β is 1 or 2, $\sigma^2 \times \Gamma_{\alpha,\beta}$ is primarily equivalent by postmultiplication to $C_{\alpha,\beta}$; the same relation holds between $\tau^2 \times \Gamma_{\alpha,\beta}$ and $C_{\alpha,\beta}$ when $\beta = 1$ or 3. This observation explains why it is that the formulæ for $\Gamma_{\alpha,\beta}$ may be obtained from those for $C_{\alpha,\beta}$ by changing the sign of *either* k *or* k' when $\beta = 1$, by changing the sign of k' when $\beta = 2$, and of k when $\beta = 3$.

Again, the matrices of the nine pairs $C_{3,\beta}, \Gamma_{3,\beta}$; $C_{1,\beta}, \Gamma_{2,\beta}$; $C_{2,\beta}, \Gamma_{1,\beta}$; $\beta = 1, 2, 3$ are connected by a relation of the type

$$|\tau| \times |C| = |\Gamma| \times |\alpha|$$

where $|a|$ is a primary unit matrix. Thus the transformations corresponding to the two matrices of any one of these pairs may be reduced to one another by means of the linear transformation IV. of Art. 32.

The two reciprocal transformations $C_{3,2}$ and $\Gamma_{2,3}$ are of special importance, because they alone, of all the eighteen, transform real elliptic functions having moduli less than unity into functions of the same kind. The former, which is the transformation of Landen, is also termed the transformation "*à majori ad minorem*," because it diminishes the modulus; the latter, which increases the modulus, is the transformation "*à minori ad majorem*," and is sometimes called the transformation of Gauss, though it is apparently due to Lagrange.

If we employ the notation of Legendre for elliptic integrals of the first species, viz.

$$F(\phi, k) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 x}},$$

the transformations of Landen and Gauss respectively give rise to the equations

$$F(\phi, k) = \frac{1}{1+k'} F\left(\psi, \frac{1-k'}{1+k'}\right), \quad \tan(\psi - \phi) = k' \tan\phi, \quad . \quad . \quad . \quad \text{(A)}$$

$$F(\phi, k) = \frac{2}{1+k} F\left(\psi, \frac{2\sqrt{k}}{1+k}\right), \quad \sin(2\psi - \phi) = k \sin\phi, \quad . \quad . \quad . \quad \text{(B)}$$

which supply expeditious methods, known as the descending and ascending modular scales of Legendre, for computing the values of elliptic integrals by successive transformations. In these formulæ k and k' are positive and less than unity, and only the real value of the integral is considered; (A) is derived from $C_{3,2}$ in Table I. by writing

$$\phi = \mathrm{am}(x, k), \qquad \psi = \mathrm{am}\left[(1+k')x, \frac{1-k'}{1+k'}\right];$$

and (B) from $(\Gamma_{2,3})$ by writing $\phi = \mathrm{am}[2x, k]$, or

$$\sin\phi = \frac{2scd}{1 - k^2 s^4}, \text{ and } \psi = \mathrm{am}\left[n(1+k)x, \frac{2\sqrt{k}}{1+k}\right].$$

The formulæ A and B are not essentially different; in fact, if in either of them we call the transformed modulus k, and interchange ϕ and ψ, we obtain the other. As before η denotes $e^{\frac{1}{2}i\pi}$.

TABLE I.—QUADRATIC TRANSFORMATIONS OF THE ELLIPTIC FUNCTIONS.

Matrix.	$\frac{1}{M}$	$\frac{\lambda}{M}$	M×S	C	Δ	D
$C_{1,1}$	$2\eta^2\sqrt{kk'}$	$k'+ik$	sd	$1-k(k+ik')s^2$	c	$1-k(k-ik')s^2$
$\Gamma_{1,1}$	$2\eta^{-2}\sqrt{kk'}$	$k'-ik$	sd	$1-k(k-ik')s^2$	c	$1-k(k+ik')s^2$
$C_{2,1}$	$k'-ik$	$2\eta^{-2}\sqrt{kk'}$	sd	c	$1-k(k-ik')s^2$	$1-k(k+ik')s^2$
$\Gamma_{2,1}$	$k'+ik$	$2\eta^2\sqrt{kk'}$	sd	c	$1-k(k+ik')s^2$	$1-k(k-ik')s^2$
$C_{3,1}$	$k-ik'$	$k+ik'$	sd	$1-k(k-ik')s^2$	$1-k(k+ik')s^2$	c
$\Gamma_{3,1}$	$k+ik'$	$k-ik'$	sd	$1-k(k+ik')s^2$	$1-k(k-ik')s^2$	c
$C_{1,2}$	$2i\sqrt{k'}$	$(i1+k')$	sc	$1-(1-k')s^2$	d	$1-(1+k')s^2$
$\Gamma_{1,2}$	$2\sqrt{k'}$	$i(1-k')$	sc	$1-(1+k')s^2$	d	$1-(1-k')s^2$
$C_{2,2}$	$i(1-k')$	$2\sqrt{k'}$	sc	d	$1-(1+k')s^2$	$1-(1-k')s^2$
$\Gamma_{2,2}$	$i(1+k')$	$2i\sqrt{k'}$	sc	d	$1-(1-k')s^2$	$1-(1+k')s^2$
$C_{3,2}$	$1+k'$	$1-k'$	sc	$1-(1+k')s^2$	$1-(1-k')s^2$	d
$\Gamma_{3,2}$	$1-k'$	$1+k'$	sc	$1-(1-k')s^2$	$1-(1+k')s^2$	d
$C_{1,3}$	$2\sqrt{k}$	$1+k$	s	$1-ks^2$	cd	$1+ks^2$
$\Gamma_{1,3}$	$2i\sqrt{k}$	$1-k$	s	$1+ks^2$	cd	$1-ks^2$
$C_{2,3}$	$1-k$	$2i\sqrt{k}$	s	cd	$1+ks^2$	$1-ks^2$
$\Gamma_{2,3}$	$1+k$	$2\sqrt{k}$	s	cd	$1-ks^2$	$1+ks^2$
$C_{3,3}$	$i(1+k)$	$i(1-k)$	s	$1+ks^2$	$1-ks^2$	cd
$\Gamma_{3,3}$	$i(1-k)$	$i(1+k)$	s	$1-ks^2$	$1+ks^2$	cd

Every matrix of determinant 2 is comprised in one of the eighteen formulæ

$$\| C \| \times \| \alpha \|, \quad \| \Gamma \| \times \| \alpha \|, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where $\| \alpha \|$ is a primary unit matrix; or, again, in one of the three formulæ

$$\begin{Vmatrix} 1, 0 \\ 1, 2 \end{Vmatrix} \times \| \alpha \|, \quad \begin{Vmatrix} 1, 0 \\ 0, 2 \end{Vmatrix} \times \| \alpha \|, \quad \begin{vmatrix} 2, 0 \\ 0, 1 \end{vmatrix} \times \| \alpha \|, \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

where $\| \alpha \|$ is any unit matrix whatever.

The quadratic transformations of the Theta functions are of eighteen different types, corresponding to the eighteen formulæ (i); but the formulæ (ii)

are more convenient for the determination of the transformed Theta functions corresponding to any given matrix of determinant 2. For, if we were to use the formulæ (i), it would be necessary, whenever the second element is different from zero in each of the two compounded matrices, to multiply together two radical quantities and to determine the sign of the product; viz. if the transformation

$$\omega = \frac{C + D\Omega}{A + B\Omega}$$

is regarded as compounded of the quadratic transformation

$$\omega = \frac{c + d\omega_1}{a + b\omega_1},$$

and the linear transformation

$$\omega_1 = \frac{\gamma + \delta\Omega}{\alpha + \beta\Omega},$$

we have to consider the product

$$\surd(A + B\Omega) = \surd(a + b\omega_1) \times \surd(\alpha + \beta\Omega),$$

where the signs of the radicals $\surd(a + b\omega_1)$ and $\surd(\alpha + \beta\Omega)$ can be determined by the rule that their real parts are to be positive; but the sign of $\surd(A + B\Omega)$, which alone appears in the final result, depends on the signs of its two factors, and cannot be determined by inspection, or (as it would seem) by any simple rule, without first resolving it into its factors. The inconvenience thus arising is obviated if we use the formulæ (ii) in which the second coefficients of the matrices of determinant 2 are zero; and it is accordingly preferable in the theory of the Theta functions to consider only the three transformations $C_{2,1}$, $C_{3,2}$, $\Gamma_{2,3}$.

In the following Table II. $e^{\frac{1}{8}i\pi}$ is still represented by η, and ϑ is written

and or $\vartheta\left(\frac{\pi x}{h}, \omega\right)$.

BLE II.—QUADRATIC TRANSFORMATIONS OF THE THETA FUNCTIONS.

(i.)

The formula
we call the t
other. As bef

$$\omega = C_{2,1} \times \Omega = \begin{vmatrix} 1, & 0 \\ 1, & 2 \end{vmatrix} \times \Omega = 1 \times 2\Omega.$$

$$\vartheta_0\left(\frac{\pi x}{h}, \Omega\right) = B_1(\vartheta_0^2 + i\vartheta_2^2),$$

$$\vartheta_1\left(\frac{\pi x}{h}, \Omega\right) = A_1 \vartheta_1 \times \vartheta_3,$$

$$\vartheta_2\left(\frac{\pi x}{h}, \Omega\right) = A_1 \vartheta_2 \times \vartheta_0,$$

$$\vartheta_3\left(\frac{\pi x}{h}, \Omega\right) = B_1 (\vartheta_3^2 + i\vartheta_1^2),$$

$$A_1 = \frac{\vartheta_2[\Omega]}{\vartheta_2[\omega]\vartheta_0[\omega]} = \frac{\eta^{-1}\sqrt{2}}{\phi(\omega)\psi(\omega)} \times \sqrt{\frac{\pi}{2K}},$$

$$B_1 = \frac{\vartheta_3[\Omega]}{\vartheta_3^2[\omega]} = \frac{1}{\psi(\frac{1}{2}\omega - \frac{1}{2})} \times \sqrt{\frac{\pi}{2K}},$$

$$\phi^2(\Omega) = \eta^{-1}\sqrt{2}\phi(\omega)\psi(\omega)\psi(\Omega),$$

$$\psi^2(\Omega) = \psi^4(\omega) + i\phi^4(\omega), \quad \phi^4(\Omega) = \frac{2\eta^2\phi^2(\omega)\psi^2(\omega)}{\phi^4(\omega) + i\psi^4(\omega)}.$$

(ii.)

$$\omega = C_{3,2} \times \Omega = \begin{vmatrix} 2, 0 \\ 0, 1 \end{vmatrix} \times \Omega = \tfrac{1}{2}\Omega.$$

$$\vartheta_0\left(\frac{2\pi x}{h}, \Omega\right) = A_2 \times \vartheta_0 \times \vartheta_3,$$

$$\vartheta_1\left(\frac{2\pi x}{h}, \Omega\right) = A_2 \times \vartheta_1 \times \vartheta_2,$$

$$\vartheta_2\left(\frac{2\pi x}{h}, \Omega\right) = B_2 \times (\vartheta_3^2 - \vartheta_0^2),$$

$$\vartheta_3\left(\frac{2\pi x}{h}, \Omega\right) = B_2 \times (\vartheta_1^2 + \vartheta_2^2),$$

$$A_2 = \frac{\vartheta_0[\Omega]}{\vartheta_0[\omega]\vartheta_3[\omega]} = \frac{1}{\psi(\omega)} \times \sqrt{\frac{\pi}{2K}},$$

$$B_2 = \frac{\vartheta_3[\Omega]}{\vartheta_2^2[\omega]} = \frac{1}{\sqrt{2}\phi(\Omega)\phi^2(\omega)} \times \sqrt{\frac{\pi}{2K}},$$

$$\psi^2(\Omega)\phi^2(\omega) = \sqrt{2}\psi(\omega)\phi(\Omega),$$

$$\phi^2(\Omega) = \frac{\phi^4(\omega)}{1 + \psi^4(\omega)}, \quad \psi^4(\Omega) = \frac{2\psi^2(\omega)}{1 + \psi^4(\omega)}.$$

(iii.)

$$\omega = \Gamma_{2,3} \times \Omega = \begin{vmatrix} 1, & 0 \\ 0, & 2 \end{vmatrix} \times \Omega = 2\Omega,$$

$$\vartheta_0\left(\frac{\pi x}{h}, \Omega\right) = B_3 \times (\vartheta_3^2 - \vartheta_2^2),$$

$$\vartheta_1\left(\frac{\pi x}{h}, \Omega\right) = A_3 \times \vartheta_1 \times \vartheta_0,$$

$$\vartheta_2\left(\frac{\pi x}{h}, \Omega\right) = A_3 \times \vartheta_2 \times \vartheta_3,$$

$$\vartheta_3\left(\frac{\pi x}{h}, \Omega\right) = B_3 \times (\vartheta_0^2 + \vartheta_1^2),$$

$$A_3 = \frac{\vartheta_2[\Omega]}{\vartheta_2[\omega]\vartheta_3[\omega]} = \frac{\sqrt{2}}{\phi(\omega)} \times \sqrt{\frac{\pi}{2K}},$$

$$B_3 = \frac{\vartheta_3[\Omega]}{\vartheta_0^2[\omega]} = \frac{1}{\psi(\Omega)\,\psi^2(\omega)} \times \sqrt{\frac{\pi}{2K}},$$

$$\phi^2(\Omega)\psi^2(\omega) = \sqrt{2}\psi(\Omega)\,\phi(\omega),$$

$$\psi^2(\Omega) = \frac{\psi^4(\omega)}{1+\phi^4(\omega)}, \quad \phi^4(\Omega) = \frac{2\phi^2(\omega)}{1+\phi^4(\omega)}.$$

The equations in the preceding Table connecting the given and transformed modular functions are included in a complete system exhibited in Table III.

TABLE III.—QUADRATIC TRANSFORMATIONS OF THE MODULAR FUNCTIONS.

A.

Matrix.	Equation.	X.
$C_{1,1}$; $\Gamma_{1,1}$	$1 = \sqrt{2}\phi(\omega)\,\psi(\omega)\,\phi(\Omega)\,\psi(\Omega)$,	$a(c-b)$
$C_{2,1}$; $\Gamma_{2,1}$	$\phi^2(\Omega) = \eta^{-1}\sqrt{2}\phi(\omega)\,\psi(\omega)\,\psi(\Omega)$,	$-a(b+2c)$
$C_{3,1}$; $\Gamma_{3,1}$	$\psi^2(\Omega) = \eta\sqrt{2}\phi(\omega)\,\psi(\omega)\,\phi(\Omega)$,	$a(2b+c)$
$C_{1,2}$; $\Gamma_{1,2}$	$\phi^2(\omega) = \eta^{-1}\sqrt{2}\,\psi(\omega)\,\phi(\Omega)\,\psi(\Omega)$,	$a(c-b)$
$C_{2,2}$; $\Gamma_{2,2}$	$\phi^2(\omega)\,\phi^2(\Omega) = \eta^{-2}\sqrt{2}\psi(\omega)\,\psi(\Omega)$,	$-a(b+2c)$
$C_{3,2}$; $\Gamma_{3,2}$	$\phi^2(\omega)\,\psi^2(\Omega) = \sqrt{2}\psi(\omega)\,\phi(\Omega)$,	$a(2b+c)$
$C_{1,3}$, $\Gamma_{1,3}$	$\psi^2(\omega) = \eta\sqrt{2}\phi(\omega)\,\phi(\Omega)\,\psi(\Omega)$,	$a(c-b)$
$C_{2,3}$; $\Gamma_{2,3}$	$\phi^2(\Omega)\,\psi^2(\omega) = \sqrt{2}\psi(\Omega)\,\phi(\omega)$,	$-a(b+2c)$
$C_{3,3}$; $\Gamma_{3,3}$	$\psi^2(\omega)\,\psi^2(\Omega) = \eta^2\sqrt{2}\,\phi(\omega)\,\phi(\Omega)$.	$a(2b+c)$

B.

Matrix.	Equation.	X.
$C_{1,1}$	$\frac{\psi^2(\Omega)}{\phi^2(\Omega)} = \eta^{-2}[\phi^4(\omega)+i\psi^4(\omega)]$,	$2a(b+c)$
$\Gamma_{1,1}$	$\frac{\phi^2(\Omega)}{\psi^2(\Omega)} = \eta^{-2}[\phi^4(\omega)+i\psi^4(\omega)]$,	$-2a(b+c)$
$C_{2,1}$	$\frac{1}{\psi^2(\Omega)} = \eta^{-4}[\phi^4(\omega)+i\psi^4(\omega)]$,	$-2ab$
$\Gamma_{2,1}$	$\psi^2(\Omega) = \phi^4(\omega)+i\psi^4(\omega)$,	$2ab$
$C_{3,1}$	$\phi^2(\Omega) = \eta^{-4}[\phi^4(\omega)+i\psi^4(\omega)]$,	$-2ac$
$\Gamma_{3,1}$	$\frac{1}{\phi^2(\Omega)} = \phi^4(\omega)+i\psi^4(\omega)$,	$2ac$
$C_{1,2}$	$\frac{\psi^2(\Omega)}{\phi^2(\Omega)} = \frac{\eta^2\phi^4(\omega)}{1+\psi^4(\omega)}$,	$2a(b+c)$
$\Gamma_{1,2}$	$\frac{\phi^2(\Omega)}{\psi^2(\Omega)} = \frac{\eta^2\phi^4(\omega)}{1+\psi^4(\omega)}$,	$-2a(b+c)$
$C_{2,2}$	$\frac{1}{\psi^2(\Omega)} = \frac{\phi^4(\omega)}{1+\psi^4(\omega)}$,	$-2ab$
$\Gamma_{2,2}$	$\psi^2(\Omega) = \frac{\eta^4\phi^4(\omega)}{1+\psi^4(\omega)}$,	$2ab$
$C_{3,2}$	$\phi^2(\Omega) = \frac{\phi^4(\omega)}{1+\psi^4(\omega)}$,	$-2ac$
$\Gamma_{3,2}$	$\frac{1}{\phi^2(\Omega)} = \frac{\eta^4\phi^4(\omega)}{1+\psi^4(\omega)}$,	$2ac$
$C_{1,3}$	$\frac{\psi^2(\Omega)}{\phi^2(\Omega)} = \frac{\eta^{-2}\psi^4(\omega)}{1+\phi^4(\omega)}$,	$2a(b+c)$
$\Gamma_{1,3}$	$\frac{\phi^2(\Omega)}{\psi^2(\Omega)} = \frac{\eta^{-2}\psi^4(\omega)}{1+\phi^4(\omega)}$,	$-2a(b+c)$
$C_{2,3}$	$\frac{1}{\psi^2(\Omega)} = \frac{\eta^{-4}\psi^4(\omega)}{1+\phi^4(\omega)}$,	$-2ab$
$\Gamma_{2,3}$	$\psi^2(\Omega) = \frac{\psi^4(\omega)}{1+\phi^4(\omega)}$,	$2ab$

Matrix.	Equation.	X.
$C_{3,3}$	$\phi^2(\Omega) = \dfrac{\eta^{-4}\psi^4(\omega)}{1+\phi^4(\omega)}$,	$-2ac$
$\Gamma_{3,3}$	$\dfrac{1}{\phi^2(\omega)} = \dfrac{\psi^4(\omega)}{1+\phi^4(\omega)}$.	$2ac$

To obtain the corresponding equations between $\phi(\Omega)$ and $\phi(\omega)$, proper to any transformation

$$\omega = \| C \| \times \| a \| \times \Omega, \qquad \omega = \| \Gamma \| \times \| a \| \times \Omega,$$

where $\| a \| = \begin{vmatrix} a, b \\ c, d \end{vmatrix}$ is a primary unit matrix, we have to multiply the right-hand side of the equations in the Table by η^X.

Table III. contains the complete theory of the quadratic transformations of the modular functions $\phi(\omega)$ and $\psi(\omega)$; viz. given any equation whatever of the type

$$\omega = \frac{c+d\Omega}{a+b\Omega}, \qquad ad-bc=2,$$

no rational relations subsist between $\phi(\omega)$, $\psi(\omega)$, and $\phi(\Omega)$, $\psi(\Omega)$ which are not included in the formulæ furnished by the Table. The two formulæ (A) and (B) corresponding to any given matrix are not rationally deducible from one another; but they coincide with one another when rationalized, so as to contain only $\Phi(\omega)$ and $\Phi(\Omega)$. To effect this rationalization we raise each member of the equations in Table A to the power 8; in Table B each member of the equations is to be raised to the power 4, and the two equations $(C_{i,j})$ and $(\Gamma_{i,j})$ are to be multiplied together. The nine equations thus obtained are the nine modular equations of order 2.

It only remains to show how the formulæ of the Tables I., II., and III. are to be verified.

In Table II. the form of the expression of the Theta functions containing Ω, in terms of the Theta functions containing ω, is determined by the general theory of Art. 25; the reduction of the six coefficients, indicated by that theory, to only two, A and B, is effected by substituting $x+\frac{1}{2}h$, $x+\frac{1}{2}h\omega$, $x+\frac{1}{2}h+\frac{1}{2}h\omega$ for x, and employing the formulæ (ix) of Art. 1; the values of A and B, which are found by putting $x=0$, are conveniently expressed in terms of the modular

functions, and $\sqrt{\frac{2K}{\pi}}$ by means of the equations of Jacobi (Art. 4, I. and II.), which give immediately

$$(1)\quad \vartheta_2[\tfrac{1}{2}\omega-\tfrac{1}{2}] = \sqrt{2}\,\phi(\omega-1)\times\vartheta_3[\omega-1] = \frac{\sqrt{2}\,\vartheta_2[\omega-1]}{\phi(\omega-1)},$$

or, by Art. 32, III.,

$$\vartheta_2[\tfrac{1}{2}\omega-\tfrac{1}{2}] = \eta^{-1}\sqrt{2}\,\frac{\psi(\omega)}{\phi(\omega)}\times\vartheta_2[\omega] = \eta^{-1}\sqrt{2}\,\phi(\omega)\psi(\omega)\vartheta_3[\omega];$$

$$(2)\quad \vartheta_3[\tfrac{1}{2}\omega-\tfrac{1}{2}] = \frac{\vartheta_0[\omega-1]}{\psi(\frac{1}{2}\omega-\frac{1}{2})};$$

or, by Art. 32, III.,

$$\vartheta_3[\tfrac{1}{2}\omega-\tfrac{1}{2}] = \frac{\vartheta_3[\omega]}{\psi(\frac{1}{2}\omega-\frac{1}{2})},$$

$$(3)\quad \vartheta_0[2\omega] = \frac{\vartheta_0[\omega]}{\psi(\omega)};$$

$$(4)\quad \vartheta_3[2\omega] = \frac{1}{\sqrt{2}}\frac{\vartheta_2[\omega]}{\phi(2\omega)};$$

$$(5)\quad \vartheta_2[\tfrac{1}{2}\omega] = \sqrt{2}\,\phi(\omega)\vartheta_3[\omega];$$

$$(6)\quad \vartheta_3[\tfrac{1}{2}\omega] = \frac{\vartheta_0[\omega]}{\psi(\frac{1}{2}\omega)}.$$

The equations of Jacobi (Art. 4, I., d, and II., d) also give by multiplication

$$(c)\quad \phi^2(\tfrac{1}{2}\omega)\psi^2(\omega) = \sqrt{2}\,\phi(\omega)\psi(\tfrac{1}{2}\omega),$$

whence, writing 2ω for ω,

$$(b)\quad \phi^2(\omega)\phi^2(2\omega) = \sqrt{2}\,\phi(2\omega)\phi(\omega);$$

and again, writing $\omega-1$ for ω, and substituting for $\phi(\omega-1)$, $\psi(\omega-1)$ their values in terms of $\phi(\omega)$, $\psi(\omega)$,

$$(a)\quad \phi^2(\tfrac{1}{2}\omega-\tfrac{1}{2}) = \eta^{-1}\sqrt{2}\,\phi(\omega)\psi(\omega)\psi(\tfrac{1}{2}\omega-\tfrac{1}{2}).$$

The equations (a), (b), (c) are included in the Table II.; the values of $\psi^2(\frac{1}{2}\omega-\frac{1}{2})$, of $\phi^2(2\omega)$, and of $\psi^2(\frac{1}{2}\omega)$, which are given in the same Table, for the cases (i), (ii), and (iii) respectively, are obtained by forming the quotient of the two equations which contain the coefficient B, and putting $x=0$. The values of $\frac{1}{M}$ in the same three cases are, by the definition of the multiplier,

$$\frac{\vartheta_3^2[\frac{1}{2}\omega-\frac{1}{2}]}{\vartheta_3^2[\omega]},\quad 2\frac{\vartheta_3^2[2\omega]}{\vartheta_3^2[\omega]},\quad \frac{\vartheta_3^2[\frac{1}{2}\omega]}{\vartheta_3^2[\omega]};$$

and then, attending to the equations (2), (4), (6) of this article, become

$$\frac{1}{\psi^2\left(\frac{1}{2}\omega-\frac{1}{2}\right)} = \psi^4(\omega) - i\phi^4(\omega), \qquad \frac{\phi^4(\omega)}{\phi^2(2\omega)} = 1+\phi^4(\omega), \qquad \frac{\psi^4(\omega)}{\psi^2\left(\frac{1}{2}\omega\right)} = 1+\phi^4(\omega),$$

in accordance with Table I.

The formulæ in Table I., relating to the transformations $C_{2,1}$, $C_{3,2}$, $\Gamma_{2,3}$, result immediately from the equations defining the elliptic functions (Art. 7) and from the formulæ of Table II. Lastly, the remaining formulæ of Table I. and the corresponding formulæ of Table III. are to be deduced by linear transformation from the formulæ appertaining to the three fundamental transformations $C_{2,1}$, $C_{3,2}$, $\Gamma_{2,3}$. To facilitate these verifications, and the determination of the transformed Theta functions corresponding to any of the matrices C or Γ, we add a scheme of the postmultiplying unit matrices, which serve to reduce the matrices C and Γ to the three fundamental matrices $C_{2,1}$, $C_{3,2}$, $\Gamma_{2,3}$.

(i.)

$$C_{s,1} = C_{2,1} \times X_{s,1}, \qquad C_{s,2} = C_{3,2} \times X_{s,2}, \qquad C_{s,3} = \Gamma_{2,3} \times X_{s,3},$$

$$X_{s,t} = \begin{vmatrix} \begin{vmatrix} 1, & 1 \\ -1, & 0 \end{vmatrix}, & \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}, & \begin{vmatrix} 0, & 1 \\ -1, & -1 \end{vmatrix} \\ \begin{vmatrix} 0, & 1 \\ -1, & -1 \end{vmatrix}, & \begin{vmatrix} 1, & 1 \\ -1, & 0 \end{vmatrix}, & \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix} \\ \begin{vmatrix} 1, & -1 \\ 0, & 1 \end{vmatrix}, & \begin{vmatrix} 1, & 2 \\ -1, & -1 \end{vmatrix}, & \begin{vmatrix} 2, & 1 \\ -1, & 0 \end{vmatrix} \end{vmatrix}.$$

(ii.)

$$\Gamma_{s,1} = C_{2,1} \times Y_{s,1}, \qquad \Gamma_{s,2} = C_{3,2} \times Y_{s,2}, \qquad \Gamma_{s,3} = \Gamma_{2,3} \times Y_{s,3},$$

$$Y_{s,t} = \begin{vmatrix} \begin{vmatrix} -1, & 1 \\ 0, & -1 \end{vmatrix}, & \begin{vmatrix} 1, & 2 \\ -1, & -1 \end{vmatrix}, & \begin{vmatrix} 2, & 1 \\ -1, & 0 \end{vmatrix} \\ \begin{vmatrix} 1, & 0 \\ -1, & 1 \end{vmatrix}, & \begin{vmatrix} 0, & 1 \\ -1, & -2 \end{vmatrix}, & \begin{vmatrix} 1, & 1 \\ -2, & -1 \end{vmatrix} \\ \begin{vmatrix} 1, & 1 \\ -1, & 0 \end{vmatrix}, & \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}, & \begin{vmatrix} 0, & 1 \\ -1, & 1 \end{vmatrix} \end{vmatrix}.$$

Arts. 35–45.] GEOMETRICAL REPRESENTATION OF BINARY QUADRATIC FORMS.

Art. 35.] *Quadratic Forms of a Negative Determinant.*

Adopting the usual conventions, we represent the complex quantity $\omega = x + iy$ by a vector in a plane; the extremity of this vector we term the point ω. As y

is essentially positive, we have only to consider the region which lies above the axis of x; by the plane (xy) we shall understand this region only.

Every complex quantity ω is the root of a quadratic equation of which the coefficients are real and the determinant negative; the ratios of the coefficients are given when ω is given; for, if $\omega = x + iy$ satisfy the equation $a + 2b\omega + c\omega^2 = 0$, we have

$$\frac{a}{x^2+y^2} = \frac{b}{-x} = \frac{c}{1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Vice versâ every binary quadratic form of a negative determinant is represented in the plane (xy) by a point; for we may regard the equation $a + 2b\omega + c\omega^2 = 0$ as corresponding to the quadratic form (a, b, c) of determinant $-\Delta$ (viz. we have $\Delta = ac - b^2$), and the point $\omega = \dfrac{-b - i\sqrt{\Delta}}{c}$ as representing the quadratic form. This representation is admissible whatever real values the quantities a, b, c may have. In the general case the ratios only of the coefficients are given when the representing point is given; but, if we suppose the coefficients to be integral numbers, and the form to be positive and primitive, the three coefficients themselves (and not merely their ratios) are given when the representing point is given.

If $\omega = \dfrac{\gamma + \delta\Omega}{\alpha + \beta\Omega}$, where $\begin{vmatrix} \alpha, \beta \\ \gamma, \delta \end{vmatrix}$ is an unit matrix, the two points ω and Ω are *equivalent*; and the equivalence is primary when the matrix $\begin{vmatrix} \alpha, \beta \\ \gamma, \delta \end{vmatrix}$ is primary. When two points are equivalent the corresponding quadratic forms are also equivalent; viz. if (a, b, c), (A, B, C) are the quadratic forms corresponding to ω and Ω respectively, (a, b, c) is transformed into (A, B, C) by $\begin{vmatrix} \alpha, \beta \\ \gamma, \delta \end{vmatrix}$.

36. *The Reduced Space.*

A positive quadratic form (a, b, c) of negative determinant is said to be *reduced* when the absolute values of its coefficients satisfy the conditions

$$\left.\begin{array}{ll} \text{(1)} & a \geqq [2b], \quad c \geqq [2b], \quad a \leqq c; \\ \text{of which the second is a consequence of the other two.} & \\ \text{(2)} & \text{If } a = [2b], \quad b > 0; \quad \text{if } a = c, \quad b > 0. \end{array}\right\} \quad . \; . \; \text{(ii)}$$

If $\omega = x + iy$ is a *reduced* point, *i. e.* a point corresponding to a reduced form, the conditions (ii) give

$$\left.\begin{array}{ll} (1) & [2x] \leqq x^2+y^2, \quad [2x] \leqq 1, \quad x^2+y^2 \leqq 1. \\ (2) & \text{If } x^2+y^2=[2x], \quad x<0\,; \quad \text{if } x^2+y^2=1, \quad x<0. \end{array}\right\} \quad \ldots \quad \text{(iii)}$$

In these formulæ the symbols $[2b]$ and $[2x]$ denote the absolute values of the quantities $2b$ and $2x$.

The geometrical interpretation (see figure 1) of these conditions is that the reduced point ω lies within the space AA' external to the semicircles $x^2+y^2=\pm 2x$, and internal to the semicircle $x^2+y^2=1$; we include in the space AA' that part of its contour which lies to the left of the axis of x, and we exclude from it the other part of its contour.

Since every binary quadratic form of a negative determinant is equivalent to one, and only one, reduced form, we have the corresponding theorem, "Every point in the plane (xy) is equivalent to one and only one point in the reduced space AA'." Again, if we regard two substitutions as identical which differ only in sign, we have the arithmetical theorem, "Every quadratic form is equivalent to its reduced form by one substitution only; except when the form appertains to the class (1, 0, 1) or (2, 1, 2); the number of reducing substitutions in these excepted cases being respectively two and three." We therefore infer that "any given point is equivalent to its reduced point by one substitution only, except when the reduced point is $+i$ or $\frac{1}{2}(-1+i\sqrt{3})$, in which cases there are respectively two or three reducing substitutions."

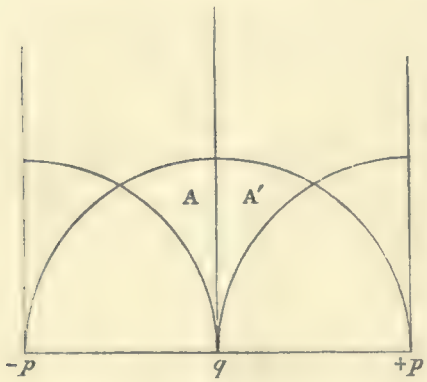


Fig. 1.

For our present purpose we have to modify the preceding enunciations by limiting ourselves to the consideration of *primary* equivalence. A form (a, b, c) is *primarily reduced*, when it satisfies the conditions

$$\begin{array}{ll} (1) & [b] \leqq a, \quad [b] \leqq c. \\ (2) & \text{If } [b]=a, \quad b>0\,; \quad \text{if } [b]=c, \quad b>0. \end{array}$$

Similarly, the point $\omega = x+iy$ is primarily reduced when it satisfies the conditions

$$\left.\begin{array}{ll} (1) & [x] \leqq x^2+y^2, \quad [x] \leqq 1. \\ (2) & \text{If } [x]=x^2+y^2, \quad x<0\,; \quad \text{if } [x]=1, \quad x<0. \end{array}\right\} \quad \ldots \quad \text{(iv)}$$

It may be proved, by a slight modification in the demonstration of the

theorem of Lagrange, (i) that "any quadratic form of negative determinant is primarily equivalent to one and only to one primarily reduced form;" (ii) that "the primary reducing substitution is also unique." If, therefore (see figure 2), we designate by Σ the space included between the parallel lines $x = \pm 1$, and external to the semicircles $x^2 + y^2 = \pm x$, we obtain the theorem, "Any given point in the plane xy is primarily equivalent to one, and only to one, point in the space Σ; and the two points are primarily equivalent by one substitution only" *.

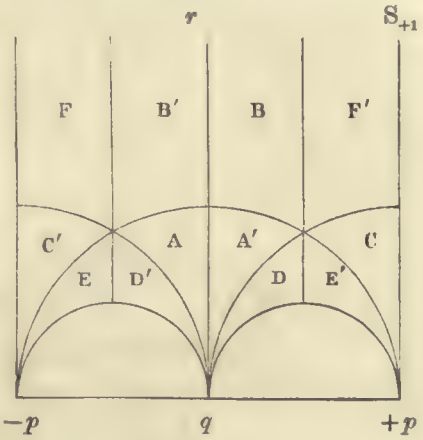


Fig. 2.

We reckon the line $x = -1$, and the semicircle $x^2 + y^2 = -x$ as appertaining to the space Σ, but we exclude the line $x = 1$, and the semicircle $x^2 + y^2 = x$. The lines $x = -1$ and $x = +1$ we denote by S_{-1} and S_{+1} respectively; the semicircles $x^2 + y^2 = -x$ and $x^2 + y^2 = x$ by T_{-1} and T_{+1}. The space Σ we shall henceforward term the reduced space. The *cornicular* points ∓ 1, 0, ∞, we denote by $\pm p$, q, r.

37. *The Circular Affinity of Moebius.*

If ω and Ω are two vectors connected by the relation $\omega = \dfrac{\gamma + \delta\Omega}{\alpha + \beta\Omega}$ where $\begin{vmatrix} \alpha, \beta \\ \gamma, \delta \end{vmatrix}$ are any quantities whatever, real or imaginary, but $\alpha\delta - \beta\gamma$ is not zero, any two corresponding loci described by the points ω and Ω are connected by a relation known as the *circular affinity of Moebius.* If $\beta = 0$, the relation between the two figures is one of similarity; in every other case the relation is one of inversion. For if β be different from zero, the equation connecting ω and Ω may be written in the form

$$\left(\omega - \frac{\delta}{\beta}\right)\left(\Omega + \frac{\alpha}{\beta}\right) = \frac{\beta\gamma - \alpha\delta}{\beta^2},$$

* See a Note on the primary periods of the elliptic functions in the 'Messenger of Mathematics,' new series, vol. xii. p. 73, Arts. 1–3 [vol. ii. p. 342]; where, however, the point corresponding to the form (a, b, c) is taken to be $-\dfrac{1}{\omega}$ instead of ω, or, what is the same thing, the point is taken to be $\dfrac{b + i\sqrt{\Delta}}{a}$ instead of $\dfrac{-b + i\sqrt{\Delta}}{c}$.

which shows that the vectors joining ω to $\frac{\delta}{\beta}$ and Ω to $-\frac{\alpha}{\beta}$ are inversely proportional. The fixed points $\frac{\delta}{\beta}$ and $-\frac{\alpha}{\beta}$, which, in each of the two figures, answer to the infinite in the other figure, are termed the centres of inversion. Figures related to one another by circular affinity possess the following properties:

(1) Circles in either figure which pass through the centre of inversion of that figure are transformed into straight lines, and *vice versâ*. All other circles are transformed into circles.

(2) The anharmonic ratio of four points on a circle or straight line in the one figure is equal to the anharmonic ratio of the corresponding four points in the other figure.

(3) The two figures are similar in their infinitesimal parts; except at the points $\frac{\delta}{\beta}$ and $-\frac{\alpha}{\beta}$ which (as has been said) correspond in the two figures respectively to the infinite of the other. The truth of this theorem may be inferred from the observation that to an infinitesimal circle (c) in one figure there corresponds an infinitesimal circle (C) in the other figure; and that to the centre of (c) there corresponds a point distant from the centre of (C) by an infinitesimal of the second order, the radii of the two circles being regarded as infinitesimals of the first order. But the theorem is only a particular case of the general proposition of Lagrange and Gauss, that if

$$X+iY=f(x+iy),$$

where X, Y, x, y are real quantities, the points $X+iY$ and $x+iy$ describe figures which are similar to one another in their infinitesimal parts, the only cases of exception presenting themselves at points where the derived function of f is zero, infinite, or indeterminate.

(4) To any simple contour in either figure, not passing through the centre of inversion, there answers in the other figure a simple contour, not passing through the centre of inversion; and the two contours are similarly described; *i. e.* positively or negatively in both figures alike*. It is sufficient, in order to

* By a simple contour we understand a closed and finite contour which does not intersect itself. Such a contour is said to be described positively or negatively according as the describing point has the enclosed space on the left or on the right; or, which is the same thing, according as the element of the contour and the element of the internal normal are situated with regard to one another as the vectors 1 and i, or as the vectors 1 and $-i$.

establish this assertion, to verify it in the three cases $\omega = \alpha + \Omega$, $\omega = \alpha\Omega$, $\omega = \frac{\alpha}{\Omega}$, α being any constant quantity.

38. *The Subdivisions of the Reduced Space.*

The twelve spaces into which Σ is divided by the lines and circles of figure 2 form, when taken in pairs, six regions which are in circular affinity with one another in the manner indicated by the equations

$$\left.\begin{aligned} \frac{A}{|1|} = \frac{B}{|\psi|} = \frac{C}{|\sigma|} = \frac{D}{|\tau|} = \frac{E}{|\rho|} = \frac{F}{|\rho^2|}, \\ \frac{A'}{|1|} = \frac{B'}{|\psi'|} = \frac{C'}{|\sigma'|} = \frac{D'}{|\tau'|} = \frac{E'}{|\rho'|} = \frac{F'}{|\rho'^2|}. \end{aligned}\right\} \quad \ldots\ldots \quad (\text{i})$$

The matrices $|1|, |\psi|, |\sigma|, |\tau|, |\rho|, |\rho^2|$ are those of which the values are given in Art. 21; the matrices denoted by the accented symbols are derived from these by changing the signs of the second and third constituents; thus, for example,

$$|\rho| = \begin{vmatrix} 1, & 1 \\ -1, & 0 \end{vmatrix}, \quad |\rho'| = \begin{vmatrix} 1, & -1 \\ 1, & 0 \end{vmatrix};$$

and we have

$$|\psi'| = -|\psi|, \quad |\sigma'| = |\sigma|^{-1}, \quad |\tau'| = |\tau|^{-1},$$
$$|\rho'| = -|\rho| \times |\sigma|^{-2}, \quad |\rho'^2| = -|\rho^2| \times |\tau|^{-2}.$$

The equations (I.) are to be interpreted as follows:—The space A is transformed into the space C by the equation

$$C = |\sigma| \times A, \quad \text{or} \quad C = 1 + A;$$

the space C is transformed into the space D by the equation

$$|\sigma|^{-1} \times C = |\tau|^{-1} \times D, \quad \text{or} \quad C = \sigma| \times |\tau|^{-1} \times D$$
$$= \begin{vmatrix} 1, & 0 \\ 1, & 1 \end{vmatrix} \times \begin{vmatrix} 1, & -1 \\ 0, & 1 \end{vmatrix} \times D = \begin{vmatrix} 1, & -1 \\ 1, & 0 \end{vmatrix} \times D = \frac{D}{1 - D}; \text{ \&c., \&c.}$$

39. *Quadratic Forms of a Positive Determinant.*

A form (a, b, c) of positive determinant D $(= b^2 - ac)$ is represented in the plane (xy) by the semicircle

$$a + 2bx + c(x^2 + y^2) = 0, \quad \ldots\ldots\ldots \quad (\text{i})$$

which we shall designate by the symbol $[a, b, c]$. The equation of the semicircle may be written in the form

$$-x+iy = \frac{a+b(x+iy)}{b+c(x+iy)} = \begin{vmatrix} b, c \\ a, b \end{vmatrix} \times (x+iy). \quad \ldots\ldots\ldots \quad \text{(ii)}$$

The point $x+iy$ (but not, except when $b=0$, or $x=0$, the point $-x+iy$) is a point on the semicircle. If $c=0$, a supposition which implies that D is a square, $[a, b, 0]$ is a straight line; for convenience, however, we shall regard this straight line as a semicircle of infinite radius.

Theorem I. "If (a, b, c) is a form of positive determinant, and (α, β, γ) a form of negative determinant $-\Delta$ (where $\Delta = a\gamma - \beta^2$) connected with (a, b, c) by the invariant relation $ac - 2b\beta + \gamma a = 0$, the point representing (α, β, γ) lies on the semicircle representing (a, b, c)."

For, if $x+iy$ be the point representing (α, β, γ), we have

$$x = -\frac{\beta}{\gamma}, \quad x^2+y^2 = \frac{a}{\gamma};$$

and the invariant relation becomes

$$a + 2bx + c(x^2+y^2) = 0.$$

Thus the semicircle $[a, b, c]$ may be regarded as representing the quadratic form (a, b, c), because the points of the semicircle represent quadratic forms of negative determinant which are harmonically associated to (a, b, c). If γ and c are of the same sign,

$$ac - 2b\beta + \gamma a \gtrless 0,$$

according as the point (α, β, γ) lies outside or inside the semicircle $[a, b, c]$.

By *equivalent* semicircles we understand semicircles which are related to one another by a circular affinity, the coefficients of the transformation being integral numbers, and the determinant a positive unit. We then have the following theorems:

Theorem II. "Equivalent points lie on equivalent semicircles; and equivalent semicircles correspond to equivalent quadratic forms."

Or, which is the same thing,

"If the form (a, b, c) of positive determinant is transformed into (A, B, C) by the linear substitution $\begin{vmatrix} \alpha, \beta \\ \gamma, \delta \end{vmatrix}$, the semicircle $[a, b, c]$ is transformed into the semicircle $[A, B, C]$ by the circular transformation $\omega = \frac{\gamma+\delta\Omega}{\alpha+\beta\Omega}$."

To establish this theorem, let $\omega = x+iy$, $\Omega = X+iY$, and let ω be a point on $[a, b, c]$, so that

$$-x+iy = \begin{vmatrix} b, & c \\ a, & b \end{vmatrix} \times (x+iy);$$

changing the sign of i in the equation

$$x+iy = \begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \times (X+iY),$$

we have also

$$-x+iy = \begin{vmatrix} \alpha, & -\beta \\ -\gamma, & \delta \end{vmatrix} \times (-X+iY);$$

whence

$$-X+iY = \begin{vmatrix} \delta, & \beta \\ \gamma, & \alpha \end{vmatrix} \times (-x+iy) = \begin{vmatrix} \delta, & \beta \\ \gamma, & \alpha \end{vmatrix} \times \begin{vmatrix} b, & c \\ a, & b \end{vmatrix} \times (x+iy)$$

$$= \begin{vmatrix} \delta, & \beta \\ \gamma, & \alpha \end{vmatrix} \times \begin{vmatrix} b, & c \\ a, & b \end{vmatrix} \times \begin{vmatrix} a, & \beta \\ \gamma, & \delta \end{vmatrix} \times (X+iY) = \begin{vmatrix} B, & C \\ A, & B \end{vmatrix} \times (X+iY);$$

or $X+iY$ lies on the semicircle $[A, B, C]$, and $[a, b, c]$ is transformed into $[A, B, C]$.

This Theorem is the geometrical expression of the invariantive character of the equation

$$ac - 2\beta b + \gamma a = 0.$$

Theorem III. "If

$$-x_1+iy_1 = \begin{vmatrix} b, & c \\ a, & b \end{vmatrix} \times (x_2+iy_2),$$

or, which is the same thing, if

$$-x_2+iy_2 = \begin{vmatrix} b, & c \\ a, & b \end{vmatrix} \times (x_1+iy_1),$$

the two points x_1+iy_1 and x_2+iy_2 are inverse points with regard to the circle $[a, b, c]$."

For, equating real and imaginary parts, we find

$$a+b(x_1+x_2)+c(x_1x_2+y_1y_2) = 0, \quad b(y_1-y_2)+c(y_1x_2-y_2x_1) = 0;$$

and of these equations the first expresses that x_1+iy_1, x_2+iy_2 are conjugate points with regard to $[a, b, c]$; the second expresses that the line joining them is a diameter.

40. *Automorphics of a Quadratic Form.*

The automorphics of the quadratic form (a, b, c), the coefficients of which we now suppose to be integral numbers, are included in the general formula

$$\frac{1}{m} \times \begin{vmatrix} T-bU, & -cU \\ aU\ , & T+bU \end{vmatrix}, \quad \ldots\ldots\ldots \quad \text{(iii)}$$

where T, U are any integers satisfying the equation $T^2 - DU^2 = m^2$, and $m = 1$, or 2, according as the form (a, b, c) is properly or improperly primitive. The meaning of course is that if

$$\begin{vmatrix} \alpha, \beta \\ \gamma, \delta \end{vmatrix} = \frac{1}{m} \times \begin{vmatrix} T-bU, & -cU \\ aU\ , & +bUT \end{vmatrix}$$

then $$(a, b, c)\,(x, y)^2 = (a, b, c)\,(\alpha n + \beta y, \gamma n + \delta y)^2.$$

By these automorphics the rational semicircle $[a, b, c]$ is transformed into itself, and the transformations are homographic (Art. 37, 1 and 2). Consequently the chords of the homography (*i. e.* the chords joining the pairs of corresponding points) all touch a conic of which the major axis is the diameter of the semicircle. This conic is an ellipse of which the equation is

$$a + 2bx + c\left(x^2 + \frac{T^2}{m^2} y^2\right) = 0,$$

and of which the eccentricity e is given by the equation

$$e = \frac{U\sqrt{D}}{T}, \quad \text{or} \quad \frac{1+e}{1-e} = \frac{1}{m^2}(T + U\sqrt{D})^2.$$

To establish these assertions we write the equation of transformation,

$$\omega = \frac{1}{m} \begin{vmatrix} T-bU, & -cU \\ aU\ , & T+bU \end{vmatrix} \times \Omega,$$

in the form

$$\left(m\omega + \frac{T+bU}{cU}\right)\left(\Omega - \frac{T-bU}{cU}\right) = -\frac{m^2}{U^2 c^2}; \quad \ldots\ldots \quad \text{(iv)}$$

and, denoting by Z, z the points $\dfrac{T-bU}{cU}$, $-\dfrac{T+bU}{cU}$ (which lie on the axis of x, outside the semicircle, and at a tangential distance $\dfrac{m}{cU}$ from it), we draw any two lines ZPQ, zpq, making equal angles with the axis, and meeting the semicircle

in the points PQ, pq respectively (see fig. 3). The equation (iv) shows that the pairs pP, qQ are pairs of corresponding points. Hence the tangent at the extremity of the minor axis of the ellipse passes through the points of contact of tangents drawn to the semicircle from z and Z. Thus the foci of the ellipse are the points inverse to Z and z with regard to the semicircle; from this conclusion the equation of the ellipse and the expression for its eccentricity are immediately derived.

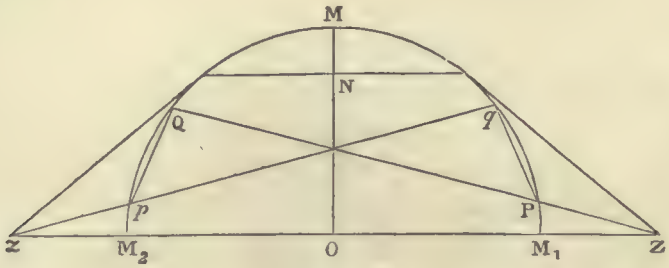


Fig. 3.

If pP are any two points on the semicircle connected by the automorphic (iii), the constant anharmonic ratio $\frac{M_1P}{M_1p} \div \frac{M_2P}{M_2p}$ is equal to $\frac{1 \pm e}{1 \mp e}$, M_1M_2 being the diameter of the semicircle. As all the automorphics of (a, b, c) are powers (with integral exponents positive or negative) of the fundamental automorphic (*i.e.* of that automorphic in which T and U are the least positive integral numbers satisfying the equation $T^2 - DU^2 = m^2$), so the homographic transformations into itself of the semicircle $[a, b, c]$ may be all obtained by repeating one of them continually in either direction. Let Σ be the fundamental automorphic, E the ellipse touched by the chords of the corresponding homography; also let PP_1, P_1P_2, P_2P_3,..., PP_{-1}, $P_{-1}P_{-2}$, $P_{-2}P_{-3}$,... be the two series of chords of the semicircle which, beginning with any given point P on the semicircle, can be drawn to touch E; we may suppose that PP_1 are connected by Σ, PP_{-1} by Σ^{-1}; then the homographic transformations (PP_2), $(PP_3), \ldots, (PP_{-2})$, $(PP_{-3}), \ldots$, which arise from the duplication, triplication, &c. of the fundamental transformation (PP_1) or (PP_{-1}), correspond to the automorphics of which the symbols are Σ^2, $\Sigma^3, \ldots . \Sigma^{-2}$, $\Sigma^{-3}, \ldots$ For all semicircles of the same determinant and of the same order (the semicircle appertains to the properly or improperly primitive order according as $m = 1$ or $m = 2$) the automorphic transformations are similar, since the eccentricities of the regulating ellipses depend only on T, U, D, and m.

Every point on the semicircle is equivalent by some one automorphic transformation to a point on the arc of the semicircle cut off by any tangent to the ellipse E; and no two points of such an arc can be equivalent by any automorphic transformation.

41.] *The Subclasses of Primarily Equivalent Quadratic Forms.*

The introduction of the conception of primary equivalence renders it necessary to divide each class of equivalent quadratic forms (as defined by Gauss) into subclasses of primarily equivalent forms. The number of subclasses in each class is six, three, or two, as will be seen from the following enumeration of the different cases:

We consider *first* under the headings I. and II. the case where the determinant is a positive number not a square; *secondly* the case where the determinant is a square or a negative number other than -1 or -3; and *thirdly* the case where the determinant is $= -1$ or -3.

First: I. *Properly Primitive Classes.*

Every form (a, b, c) of a properly primitive class belongs to one or other of the three sets:

$$\left.\begin{array}{ll} (A) & a \equiv c \equiv 1, \text{ mod } 2; \\ (B) & a \equiv 0, \quad c \equiv 1, \text{ mod } 2; \\ (C) & a \equiv 1, \quad c \equiv 0, \text{ mod } 2; \end{array}\right\} \quad \ldots\ldots\ldots \text{(i)}$$

and the result of applying unit transformations of the six types of Art. 21 to forms of these three sets is indicated in the following scheme:

	A	B	C
$A=$	$1, \psi$	ρ^2, σ	ρ, τ
$B=$	ρ, σ	$1, \tau$	ρ^2, ψ
$C=$	ρ^2, τ	ρ, ψ	$1, \sigma$

$\ldots\ldots$ (ii)

so that, for example, $(A) = (B) \times \rho^2 = (B) \times \sigma$, the symbols $(B) \times \rho^2$, $(B) \times \sigma$, repre-

senting the sets obtained by applying unit transformations of the type ρ^2, and of the type σ, to forms of the set (B).

We must now distinguish the cases in which the least number U_1, satisfying the equation $T^2 - DU^2 = 1$, is even and in which it is uneven.

(a) Let U_1 be even; then U_2, U_3, ... are all even, and all the automorphics of any given form f are primary. Hence the forms

$$f \times 1, \quad f \times \psi, \quad f \times \sigma, \quad f \times \tau, \quad f \times \rho, \quad f \times \rho^2. \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

are all primarily non-equivalent; since if any two of them were primarily equivalent, f would have a non-primary automorphic. Thus, in this case, each class contains six sub-classes of forms primarily non-equivalent.

(b) Let U_1 be uneven; then U_3, U_5, ... are uneven, U_2, U_4, ... are even; *i. e.* the automorphics of an even order are primary, and the automorphics of an uneven order are of the type ψ, τ, or σ according as the form is of the type (A), (B), or (C). In this case the six forms (iii) are primarily equivalent in pairs; for example, if f is of the type (A), f and $f \times \psi$ are primarily equivalent, because f has automorphics of the type ψ; *i. e.* f can be transformed into $f \times \psi$, not only by transformations of the type ψ, but also by transformations of the type $\psi \times \psi$ or 1. Each class therefore contains three subclasses of forms primarily non-equivalent.

II. *Improperly Primitive Classes.*

(1) Let $D \equiv 1$, mod 8; the values of T, U in the equation $T^2 - DU^2 = 1$ are all even, and the automorphics are all primary. The forms are of three types:

$$\left.\begin{array}{ll} (A') & a \equiv c \equiv 0, \text{ mod } 4, \\ (B') & a \equiv 2, \quad c \equiv 0, \text{ mod } 4, \\ (C') & a \equiv 0, \quad c \equiv 2, \text{ mod } 4, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

which are related to one another as the types (A), (B), (C) in the scheme (ii); and each class contains six subclasses of forms non-equivalent primarily.

(2) Let $D \equiv 5$, mod 8. The forms are of one or other of the two types

$$\left.\begin{array}{ll} (A'') & b \equiv +1, \text{ mod } 4, \\ (B'') & b \equiv -1, \text{ mod } 4, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

which are related to one another in the manner indicated in the scheme

	A''	B''
$A''=$	$1,\ \rho,\ \rho^2$	$\psi,\ \sigma,\ \tau$
$B''=$	$\psi,\ \sigma,\ \tau$	$1,\ \rho,\ \rho^2$

. . . . . (vi)

Here, again, we must distinguish between the two cases in which the equation $T^2 - DU^2 = 4$ admits, and in which it does not admit, of uneven solutions.

(a) Let $T^2 - DU^2 = 4$ admit of no uneven solutions. In this case all the automorphics are primary; and each class contains six different subclasses—viz. if f is any given form, $f, f\times\rho, f\times\rho^2$ are of the same type as f, and $f\times\psi, f\times\sigma$, $f\times\tau$ of the other type.

(b) Let $T^2 - DU^2 = 4$ admit of uneven solutions: then the least solution is uneven, and every third solution is even. There are thus automorphics of each of the types $1, \rho, \rho^2$, and each class contains but two subclasses, viz. one of each of the types (A'') and (B'').

There is no known criterion for distinguishing *à priori* between the two cases (a) and (b).

When $D \equiv 1$, mod 8, the criteria (v) as well as the criteria (iv) apply to the improperly primitive forms; and thus, if (a, b, c), (a', b', c') are two equivalent improperly primitive forms of such a determinant, they are or are not primarily equivalent according as the congruences

$$a \equiv a',\ b \equiv b',\ c \equiv c',\ \text{mod } 4 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{(vii)}$$

are or are not satisfied.

When $D \equiv 5$, mod 8, the conditions (vii) are necessary, but not sufficient, for the primary equivalence of two equivalent improperly primitive forms.

The conditions

$$a \equiv a',\ c \equiv c',\ \text{mod } 4 \quad . \ . \ . \ . \ . \ . \ . \ . \quad \text{(viii)}$$

are necessary (but, of course, not sufficient) for the primary equivalence of any two equivalent forms (whether properly or improperly primitive).

Secondly: If the determinant is a square, or a negative number other than -1 or -3, each class contains *six subclasses* of forms primarily non-equivalent.

Thirdly, for the determinants -1 and -3: there is but one class of forms of determinant -1; this class contains *three* subclasses, viz. one belonging to each of the sets (A), (B), (C); and each of these subclasses (besides the identical pair of automorphics ± 1) has a pair of automorphics which are respectively of the types $\pm\psi$, $\pm\tau$, $\pm\sigma$.

The class of improperly primitive forms of determinant -3 contains *two* subclasses, viz. one belonging to each of the sets (A'') and (B''), and each subclass has three pairs of automorphics, viz. a pair of each of the types ± 1, $\pm\rho$, $\pm\rho^2$; the properly primitive class of determinant -3 contains *six* subclasses.

Returning to quadratic forms of a positive determinant, two quadratic forms of a positive determinant, such as (a, b, c) and $(-a, -b, -c)$ correspond to one and the same semicircle: in certain cases these two forms are equivalent, but it follows from the conditions (vii) and (viii), that they can never be primarily equivalent. Hence the number of subclasses of semicircles of a given determinant primarily non-equivalent to one another is always half the number of the subclasses of forms of the same determinant. Let H and H' be the numbers of subclasses of properly and improperly primitive semicircles; h and h' the numbers of classes of properly and improperly primitive forms of the same determinant. Let $v = 1$ or 2, according as the equation $T^2 - DU^2 = 1$ admits or does not admit of solutions in which U is uneven; and let $v' = 1$, or $= 3$, according as the equation $T^2 - DU^2 = 4$ admits or does not admit of solutions in which T, U are uneven; we then have

$$H = \tfrac{3}{2} \times v \times h, \quad H' = v' \times h' \ldots\ldots\ldots \text{(ix)}$$

42.] *Reduction of Quadratic Forms of a Positive Determinant.*

A semicircle which enters the reduced space (figure 2, p. 507) is a reduced semicircle; its reduced arc is the arc lying within the reduced space. The boundaries of the reduced space, viz. the semicircles $x^2 + y^2 \pm x = 0$ and $x \pm 1 = 0$ are improperly primitive semicircles of the square determinant $+1$; of these, $x + x^2 + y^2 = 0$, and $1 + x = 0$ are reduced. The semicircle $x = 0$ is also an improperly primitive semicircle of determinant $+1$; and the three semicircles $x + x^2 + y^2 = 0$, $1 + x = 0$, $x = 0$ represent the three subclasses of improperly primitive semicircles of determinant 1, which exist in accordance with the general theory. The semicircles (α) $-1 + x^2 + y^2 = 0$, (β) $\pm 2x + x^2 + y^2 = 0$, (γ) $\pm 1 + 2x = 0$, represent the three subclasses of properly primitive semicircles of determinant 1. No other semicircle of determinant 1, besides those which we have named, can enter the reduced space; for the improperly primitive semicircles this is evident,

because the whole of each reduced semicircle lies within the reduced space, and any point on an improperly primitive circle of determinant 1 must be equivalent to a point on one of them, and cannot therefore be equivalent to a reduced point not lying on one of them; in the case of the properly primitive semicircles

$$\pm 2x + x^2 + y^2 = 0, \qquad \pm 1 + 2x = 0,$$

the reduced arc of either semicircle of each pair is equivalent to the unreduced arc of the other semicircle of the same pair; and thus the semicircle $-1 + x^2 + y^2 = 0$, and the reduced arcs of $\pm 2x + x^2 + y^2 = 0$, and of $\pm 1 + 2x$ are the only properly primitive reduced arcs of determinant 1. We may observe (i) that the reduced semicircles of determinant 1 form the boundaries of the twelve regions considered in Art. 36; (ii) that the only semicircles which lie wholly within the reduced space are the two boundaries $1 + x = 0$, $x + x^2 + y^2 = 0$, and the semicircles $x = 0$ and $-1 + x^2 + y^2 = 0$; (iii) that every rational semicircle which passes through one of the cornicular points p, q, r is a semicircle of square determinant.

Any given semicircle of determinant D is equivalent to a certain series of reduced arcs. For if ω be any point on the given semicircle, and Ω be the corresponding reduced point, as ω describes any arc of the given semicircle, Ω will describe the corresponding arc of the equivalent reduced semicircle: but when Ω reaches a boundary of the reduced space, the reducing substitution of ω changes, and Ω reenters the reduced space at the opposite boundary; viz. if before the arrival of Ω at the boundary, the reducing substitution is $\omega = |\alpha| \times \Omega$, we shall have immediately after the passage of the boundary $\omega = |\alpha| \times |\mathfrak{J}| \times \Omega$, where $|\mathfrak{J}|$ is $|\sigma|^{-2}$, $|\sigma|^2$, $|\tau|^{-2}$, $|\tau|^2$, according as Ω arrives from the interior of Σ upon the boundary S_{-1}, S_{+1}, T_{-1}, or T_{+1}. If Ω arrives at one of the cornicular points, ω arrives at the same time at the extremity of the given semicircle, and *vice versâ*.

The number of reduced rational semicircles of a given determinant is finite. For every reduced semicircle must intersect one at least of the lines $x = 0$, $\pm 1 + x = 0$; but the number of rational semicircles $[a, b, c]$ of a given determinant D, which intersect the line $x = 0$, is evidently finite, because, a and c being of opposite signs (except when one of them is zero, as may happen in the case of a square determinant), the Diophantine equation $b^2 - ac = D$ admits of only a finite number of solutions; and, if we shift the semicircles obtained by its solution to the right or to the left through the distance of a unit, we obtain the semicircles which intersect the lines $\pm 1 + x = 0$.

We have now to consider separately the cases in which D is, and in which it is not, a perfect square.

43.] *Case when the Determinant is not a square.*

No reduced arc can pass through one of the points p, q, r; thus the reduced arcs are of twelve different types, distinguished from one another by the boundaries which they traverse,

$$\left.\begin{array}{llll} [S_{-1}, S_{+1}], & [S_{+1}, S_{-1}], & [T_{-1}, T_{+1}], & [T_{+1}, T_{-1}], \\ [S_{-1}, T_{-1}], & [T_{-1}, S_{-1}], & [S_{-1}, T_{+1}], & [T_{+1}, S_{-1}], \\ [S_{+1}, T_{-1}], & [T_{-1}, S_{+1}], & [S_{+1}, T_{+1}], & [T_{+1}, S_{+1}]; \end{array}\right\} \quad . \quad . \quad . \quad \text{(i)}$$

so that, for example, $[S_{-1}, T_{+1}]$ is an arc entering the reduced space at S_{-1}, and quitting it at T_{+1}. Two types such as $[S_{-1}, S_{+1}]$ and $[S_{+1}, S_{-1}]$, or again such as $[S_{-1}, T_{+1}]$, $[T_{+1}, S_{-1}]$, are not essentially different; but it is convenient to distinguish between them in order to indicate the direction in which the reduced arc is described. If A_1 is a reduced arc having S_ϵ, or T_ϵ, for the second constituent of its type-symbol, and if A_2 is the reduced arc immediately succeeding A_1 (*i. e.* the reduced arc primarily equivalent to the continuation of A_1 immediately beyond the boundary in which A_1 terminates), the continuation of A_1 is transformed into A_2 by $|\sigma|^{2\epsilon}$, or $|\tau|^{2\epsilon}$; A_2 has for the first constituent of its type-symbol $S_{-\epsilon}$, or $T_{-\epsilon}$, and enters the reduced space at the point on $S_{-\epsilon}$, or $T_{-\epsilon}$, symmetrically situated (with regard to the axis of y) to the point on S_ϵ, or T_ϵ, at which A_1 quits the reduced space; lastly, the corresponding directions on A_1 and A_2 make equal angles with the corresponding directions on the boundaries S_ϵ, $S_{-\epsilon}$, or T_ϵ, $T_{-\epsilon}$.

The series of reduced arcs is periodic; for, if a point ω describe a given semicircle, the reduced point Ω will describe a series of reduced arcs. But we have just seen that a reduced arc quitting the reduced space at a point on S_{-1}, S_{+1}, T_{-1}, T_{+1}, is necessarily succeeded by an arc entering the reduced space at a point on S_{+1}, S_{-1}, T_{+1}, T_{-1}, symmetrically situated to the former point with regard to the axis of y. Thus the series of reduced arcs can never terminate, and as there is only a finite number of different reduced arcs, the same arcs must recur; further, each reduced arc determines the reduced arcs which immediately precede and follow it; hence the reduced arcs must arrange themselves in a periodic series in which they succeed one another in a definite order. All the reduced arcs must be included in this period, because as ω describes the given semicircle, the reduced point Ω either describes a reduced arc continuously, or passes discontinuously from the end of one reduced arc to the beginning of

another perfectly determinate arc; *i. e.* Ω can never leave the period of reduced arcs, and this point must therefore comprise every reduced arc equivalent to the given semicircle.

To determine the law of succession of the arcs of the period, we shall term arcs of any of the eight types (ST) or (TS) *transitive* arcs; and arcs of any of the four remaining types (SS) or (TT) *intransitive* arcs. A succession of intransitive arcs cannot form a period; and, as the transitive arcs must be alternately of the types (ST) and (TS), there must be an even number of them in any period. Let us take as the first arc of the period a transitive arc A_1, which we may suppose to be of the type $[T_{-\epsilon_0}, S_{\epsilon_1}]$, ϵ_0, ϵ_1 representing positive or negative units; let A_1 be followed by $\mu_1 - 1$ intransitive arcs of the type $(S_{-\epsilon_1}, S_{\epsilon_1})$; let these be followed by a transitive arc $(S_{-\epsilon_1}, T_{\epsilon_2})$, and this in its turn by $\mu_2 - 1$ intransitive arcs of the type $(T_{-\epsilon_2}, T_{\epsilon_3})$; let the next arc be a transitive arc of the type $(S_{-\epsilon_2}, S_{\epsilon_3})$, and let the series be continued until we arrive again at the transitive arc A_1 with which we began; the last transitive arc (that immediately preceding A_1) will be of the type $[S_{-\epsilon_{2s}-1}, T_{\epsilon_0}]$, the suffix $2s-1$ being uneven because arcs of the types $[ST]$ occur in even places in the series of transitive arcs. Writing ϵ_{2s} for ϵ_0, we find that the matrix

$$|A| = |\sigma|^{2\epsilon_1 \mu_1} \times |\tau|^{2\epsilon_2 \mu_2} \times |\sigma|^{2\epsilon_3 \mu_3} \times \ldots \times |\tau|^{2\epsilon_{2s} \mu_{2s}} \quad \ldots\ldots\ldots \quad \text{(ii)}$$

is an automorphic of the transitive semicircle A_1. For the transformations which change A_1 into the successive semicircles of the period are respectively

$$|\sigma|^{2\epsilon_1}, \ |\sigma|^{4\epsilon_1}, \ \ldots, \ |\sigma|^{2\epsilon_1 \mu_1},$$
$$|\sigma|^{2\epsilon_1 \mu_1} \times |\tau|^{2\epsilon_2}, \ |\sigma|^{2\epsilon_2 \mu_2} \times |\tau|^{4\epsilon_2}, \ \ldots,$$

so that (ii) is a transformation which changes A_1 into the arc immediately succeeding the last arc of the period, *i. e.* into itself.

Let $A_1 = [\alpha_0, \beta_0, \alpha_1]$, and let θ represent either root of the corresponding equation

$$\alpha_0 + 2\beta_0 \theta + \alpha_1 \theta^2 = 0; \quad \ldots\ldots\ldots \quad \text{(iii)}$$

the formula (ii) gives rise to the equation

$$\theta - |A_1| \times \theta = [2\mu_1 \epsilon_1, \ 2\mu_2 \epsilon_2, \ \ldots, \ 2\mu_{2s} \epsilon_{2s}, \ \theta]; \quad \ldots\ldots \quad \text{(iv)}$$

and hence if θ_2 be the greater of the two roots of the equation (iii), (one of the roots of that equation is, in absolute magnitude, greater and the other less than unity, since $[\alpha_0, \beta_0, \alpha_1]$ is a transitive semicircle) the quotients $2\mu_1 \epsilon_1, \ 2\mu_2 \epsilon_2 \ldots$ may

be obtained by developing θ_1 in a continued fraction in which the integral quotients are the even integers, positive or negative, which approach most nearly to the complete quotients. The roots of the equations corresponding to the successive reduced semicircles are

$$\theta_1,\ \theta_1-2\epsilon_1,\ \theta_1-4\epsilon_1,\ \ldots,\ \theta_1-2(\mu_1-1)\epsilon_1,$$

$$\frac{1}{\theta_2},\ \frac{1}{\theta_2-2\epsilon_2},\ \frac{1}{\theta_2-4\epsilon_2},\ldots,\ \frac{1}{\theta_2-2(\mu_2-1)\epsilon_2},$$

$$\theta_3,\ \theta_3-2\epsilon_3,\ \theta_3-4\epsilon_3,\ldots,\ \theta_3-2(\mu_3-1)\epsilon_3,$$

$$\ldots\quad\ldots$$

$$\frac{1}{\theta_{2s}},\ \frac{1}{\theta_{2s}-2\epsilon_{2s}},\ \frac{1}{\theta_{2s}-4\epsilon_{2s}},\ldots,\ \frac{1}{\theta_{2s}-2(\mu_{2s}-1)\epsilon_{2s}}$$

where

$$\theta_1=2\mu_1\epsilon_1+\frac{1}{\theta_2},\quad \theta_2=2\mu_2\epsilon_2+\frac{1}{\theta_3},\ \ldots,\ \theta_{2s}=2\mu_{2s}\epsilon_{2s}+\frac{1}{\theta_1},$$

$\theta_1, \theta_2, \theta_3, \ldots$, being all greater in absolute magnitude than unity, and the roots of the equations which correspond to the transitive semicircles being

$$\theta_1,\ \frac{1}{\theta_2},\ \theta_3,\ \frac{1}{\theta_4},\ldots.$$

Thus the succession of the reduced arcs in the period is regulated by the development in a continued fraction with even quotients of either root of the quadratic equation corresponding to any transitive semicircles of the period. According as we develope the greater or the lesser root of the quadratic equation, the transitive semicircle is of the type $[TS]$ or $[ST]$; we may thus describe the period either backwards or forwards, obtaining two continued fractions which consist of the same quotients in reverse order.

44.] *Comparison of the Geometrical and Arithmetical Reduction.*

(1) Given any reduced arc, we can by geometrical construction only, and without calculation, trace the complete period of reduced arcs; for the point of entrance of the arc succeeding any given arc is known, and also the tangent at that point. Again, if in the diagram exhibiting the complete period of reduced arcs we begin with the transitive arc A, or $[\alpha_0, \beta_0, \alpha_1]$, we can, as we have already seen, obtain the equation (iv), Art. 43, by counting the reduced arcs, and

attending to the order in which arcs of different types succeed each other. Lastly, if we write that equation in the form

$$\theta = \begin{vmatrix} A, & B \\ C, & D \end{vmatrix} \times \theta,$$

the integral numbers, α_0, β_0, α_1 are determined by the equalities

$$\frac{C}{\alpha_0} = \frac{D-A}{2\beta_0} = \frac{-B}{\alpha_1}.$$

Thus when a single reduced arc is given, we can not only complete the period geometrically, but can also ascertain the equations of the semicircles of which it is composed.

(2) On any given semicircle (see figure 3, p. 513) let PP_1 be an arc cut off by any tangent of the ellipse corresponding to the fundamental primary automorphic. While the point ω describes the arc PP_1, the point Ω describes the complete period of reduced arcs, and describes that period once only. For as soon as ω, after travelling from P to P_1, passes beyond P_1, it begins describing an arc primarily equivalent to the arc PP_1; therefore, as soon as ω passes P_1, Ω must begin the period over again; and, until ω has passed P_1, this repetition cannot begin, or we should obtain an automorphic of the given semicircle transforming points of the arc PP_1 into points of the same arc, which is impossible if PP_1 is an arc of the fundamental homography.

But, while ω describes the arc PP_1, Ω may occupy the same position more than once, because points on the arc PP_1 may be primarily equivalent, though not by an automorphic of the semicircle. Whenever this happens, the chain of reduced arcs intersects itself; and a certain number of such intersections does in general occur in every period.

(3) We have here assumed that all the primary automorphics of a quadratic form are powers of one of them. But this theorem may itself be established geometrically; for this purpose it is sufficient to consider reduced forms only. Let Γ be the reduced arc of the reduced semicircle A; and let there be an automorphic V of A by which Γ is transformed into γ, γ being an arc of A which lies wholly outside the reduced space. If a point ω setting out from any point Γ_1 of Γ travel along the circumference of A until it arrives at the equivalent point γ_1 of γ, the equivalent reduced point Ω will in the mean time describe the period of reduced arcs a certain number of times μ in the positive or negative

direction, returning at last to the point Γ_1. Thus the equation connecting γ_1 and Γ_1 will be

$$\gamma_1 = |A|^{\pm\mu} \times \Gamma_1;$$

i. e. the automorphic V is a power of the fundamental automorphic (A).

(4) If we fold the reduced space over itself so as to bring the boundary S_1 on S_{-1} and T_1 on T_{-1}, we may regard the chain of reduced arcs as forming an unbroken curve. We may, in fact, regard the reduced space as forming a quasi-cylindrical surface, having the singular point p at an infinite distance, and closed at its lower end by the semicircular curve qr joining the two singular points q and r; we may imagine the surface inflated so that qr is not a singular line upon it.

(5) To obtain the period of semicircles primarily equivalent to a semicircle of which the arithmetical symbol $[a_1, b_0, a_1]$ is given, we may employ an algorithm resembling that of Gauss for the reduction of a quadratic form of positive determinant. We compute the series of equivalent (but not primarily equivalent) forms,

$$(a_0, b_0, a_1), \quad (a_1, b_1, a_2), \quad (a_2, b_2, a_3), \quad (a_3, b_3, a_4), \ldots.$$

where a_{s+1} is determined by the equation

$$a_{s+1} = \frac{b_s^2 - D}{a_s},$$

and b_s, which lies as near as it can to $\sqrt{D}$, by the congruence

$$b + b_{s-1} \equiv 0, \text{ mod } 2a_s.$$

From this series we derive the series of primarily equivalent semicircles

$$[a_0, b_0, a_1], \quad [a_2, -b_1, a_1], \quad [a_2, b_2, a_3], \quad [a_4, -b_3, a_3], \ldots.$$

which eventually becomes a periodic series of transitive semicircles, the first period beginning with the first transitive semicircle. If $[a_0, \beta_0, a_1] = [a_{2h}, b_{2h}, a_{2h+1}]$ is a transitive semicircle occupying an uneven place in the series, we obtain the formulae of the last article by writing

$$\theta_1 = \frac{-\beta_0 - \sqrt{D}}{a_1}, \qquad \frac{b_{2h+j} - b_{2h+j-1}}{a_{2h+j}} = (-)^j \epsilon_j \mu.$$

In the period of transitive semicircles the coefficients β are alternately positive and negative; and two consecutive semicircles are of the types (TS),

(ST), or of the types (ST), (TS), according as they have the same last coefficient, or the same first coefficient.

If we use $-\sqrt{D}$ instead of $+\sqrt{D}$ throughout the process we obtain the same period of transitive semicircles, but in a reverse order.

If the semicircle which we have to reduce is given, not by its arithmetical symbol, but geometrically, the preceding algorithm indicates the following series of operations:—Let C_1 be the given semicircle, c_1 either of its terminal points. (i) We substitute for C_1 an equal equivalent semicircle C_2, such that the point c_2 equivalent to c_1 shall lie between -1 and $+1$. (ii) We substitute for C_2 an equivalent (but not a primarily equivalent) semicircle C_3, inverse to C_2 with regard to $x^2+y^2=1$, so that the point c_3, equivalent to c_2, lies outside of the segment $(-1, +1)$. The alternate repetition of these operations gives eventually the period of transitive semicircles; the alternate semicircles of the period not appearing themselves, but being represented by their inverses with regard to $x^2+y^2=1$.

To the algorithm for the reduction of a quadratic form of negative determinant there corresponds (in a similar manner) a geometrical process for the reduction of a point; and the substitution which reduces a form ϕ of negative determinant is also a reducing substitution for every quadratic form f of a positive determinant harmonically related to ϕ; and, *vice versâ*, every reducing substitution of f also reduces an infinite number of forms ϕ of negative determinant harmonically related to f.

45.] *Case when the Determinant is a Square.*

Let $D=\Delta^2$; we may suppose Δ positive; and, as we have already discussed the reduced semicircles of determinant 1, we may also suppose $\Delta>1$.

An *extreme* semicircle is a reduced semicircle passing through one of the points $\pm p, q, r$; two of these points cannot lie on the same semicircle, since $\Delta>1$; we may therefore regard the extreme semicircles as being of three different types, (p), (q), and (r).

Theorem. "The reduced semicircles, equivalent to a given semicircle Γ of a square determinant, comprise two, and only two, extreme semicircles; the remaining reduced semicircles form a chain of reduced arcs connecting the reduced arcs of the extreme semicircles."

"If Γ is properly primitive, the extreme semicircles are of the types (p, p), (q, q), (r, r), according as Γ is of the type (A), (B), (C)."

"If Γ is improperly primitive, the extreme semicircles are of the types (q, r), (r, p), (p, q), according as Γ is of the type (A'), (B'), (C')."

Let $\Gamma = [a, b, c]$; and let us consider the quadratic form f corresponding to Γ,

$$f = (a, b, c)(x, y)^2 = m(px + p'y)(qx + q'y),$$

where $m = 1$, or 2, according as Γ is properly or improperly primitive; p, p', and again q, q', are relatively prime; and $pq' - qp' = \frac{2\Delta}{m}$.

To demonstrate the theorem, we observe

(1) That if f is transformed into

$$F = (A, B, C)(x, y)^2 = m(Px + P'y)(Qx + Q'y)$$

by the matrix

$$\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix},$$

we have the equation

$$\begin{vmatrix} p, & p' \\ q, & q' \end{vmatrix} \times \begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} = \begin{vmatrix} P, & P' \\ Q, & Q' \end{vmatrix};$$

and hence, if $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$ be primary, the congruence

$$\begin{vmatrix} p, & p' \\ q, & q' \end{vmatrix} \equiv \begin{vmatrix} P, & P' \\ Q, & Q' \end{vmatrix}, \text{ mod } 2.$$

(2) That the form corresponding to an extreme semicircle is of one or other of the types

$$\left.\begin{aligned} &m(\eta x + \eta' y)(\lambda x + \lambda' y), \quad \eta\lambda' - \eta'\lambda = \frac{2\Delta}{m} \\ &m(\lambda x + \lambda' y)(\eta x + \eta' y), \quad \lambda\eta' - \lambda'\eta = \frac{2\Delta}{m} \end{aligned}\right\}; \quad \ldots\ldots \text{(i)}$$

where η and η' do not surpass unity in absolute magnitude, and λ and λ' are different from zero, and do not surpass $\frac{2\Delta}{m}$ in absolute magnitude.

(3) That if Γ is properly primitive, $\begin{vmatrix} p, & p' \\ q, & q' \end{vmatrix}$ is of the type

$$\begin{vmatrix} 1, & 1 \\ 1, & 1 \end{vmatrix}, \quad \begin{vmatrix} 0, & 1 \\ 0, & 1 \end{vmatrix}, \quad \begin{vmatrix} 1, & 0 \\ 1, & 0 \end{vmatrix}, \text{ mod } 2,$$

according as Γ is of the type (A), (B), (C); and that if Γ is improperly primitive, $\begin{vmatrix} p, p' \\ q, q' \end{vmatrix}$ is of the type

$$1 \text{ or } \psi\,;\ \rho \text{ or } \sigma\,;\ \rho^2 \text{ or } \tau\,;$$

according as Γ is of the type (A'), (B'), (C').

Hence the extreme semicircles, if they exist at all, are of the types assigned in the theorem. It only remains to show (1) that f can be transformed by a primary matrix into one, and only one, form of each of the two types (i); (2) that the two forms thus obtained give essentially different extreme semicircles.

Consider the equation

$$\begin{vmatrix} p, p' \\ q, q' \end{vmatrix} \times \begin{vmatrix} \alpha, \beta \\ \gamma, \delta \end{vmatrix} = \begin{vmatrix} \eta, \eta' \\ \lambda, \lambda' \end{vmatrix};$$

in which $\begin{vmatrix} \alpha, \beta \\ \gamma, \delta \end{vmatrix}$ is a primary unit matrix, and η, η', λ, λ' are subject to the in equalities stated above. We regard p, p', q, q' as given numbers; α, β, γ, δ, η, η', λ, λ' as numbers to be determined, and we have to show that the equation admits of one solution, and one only. For brevity, we attend only to the case in which η and η' are both units. We first determine α and γ by the conditions

$$p\alpha + p'\gamma = \eta, \quad \alpha \equiv 1, \text{ mod } 4, \quad \gamma \equiv 0, \text{ mod } 2,$$

$$[q\alpha + q'\gamma] < \frac{2\Delta}{m};$$

which are always satisfied by one, and only one, set of values for α, γ, and η; the equations

$$p\beta + p'\delta = \eta', \quad -\gamma\beta + \alpha\delta = 1,$$

then give

$$\eta\beta = \eta'\alpha - p', \quad \eta\delta = \eta'\gamma + p.$$

We have also

$$\lambda = q\alpha + q'\gamma, \quad \lambda' = q\beta + q'\delta = \eta\left(\eta'\lambda + \frac{2\Delta}{m}\right).$$

The last of these equations determines the unit η', for $\eta'\lambda$ must be negative in order that λ' may be less in absolute magnitude than $\frac{2\Delta}{m}$.

Thus the form

$$\phi = m(\eta x + \eta' y)(\lambda x + \lambda' y),$$

equivalent to f, is completely determined; similarly a form

$$\phi_1 = m(\eta_1 x + \eta_1 y)(\lambda_1 x + \lambda_1' y),$$

equivalent to f, may be found from the equations

$$\begin{vmatrix} p, p' \\ q, q' \end{vmatrix} \times \begin{vmatrix} \alpha_1, \beta_1 \\ \gamma_1, \delta_1 \end{vmatrix} = \begin{vmatrix} \lambda_1, \lambda_1 \\ \eta_1, \eta_1 \end{vmatrix}.$$

And these two forms correspond to essentially different semicircles; for the equation $\phi_1 = -\phi$ is impossible, and the equation $\phi_1 = \phi$, or

$$\left\| \begin{matrix} \eta, \eta' \\ \lambda, \lambda' \end{matrix} \right\| = \pm \left\| \begin{matrix} \lambda_1, \lambda_1' \\ \eta_1, \eta_1' \end{matrix} \right\|$$

implies the equation $\frac{2\Delta}{m} = 2$; or $\Delta = m = 1$, contrary to the hypothesis that $\Delta > 1$.

We may obtain the reduced arcs equivalent to a given semicircle Γ of a square determinant by the following geometrical construction :—Determine a reduced semicircle equivalent to Γ (Art. 44); and continue its reduced arc in both directions (or in one only if it is an extreme semicircle) by a chain of reduced arcs. The same considerations which we have already employed in the case of determinants which are not squares show that every reduced arc must be included in the chain. But when we arrive at the extreme semicircles (or at the extreme semicircle, if the semicircle from which we set out be itself extreme) the chain must terminate; for an extreme semicircle cannot be continued (in the direction of its extreme point) beyond the boundary of the reduced space; in fact, the extreme points of the two extreme semicircles answer to the extreme points of Γ; and while ω describes the circumference of Γ from one of its extreme points to the other, the reduced point Ω describes the complete chain of reduced arcs running from the extreme point of one of the two extreme semicircles to the extreme point of the other.

When Γ is given arithmetically by its symbol, we may obtain the symbols of the chain of reduced semicircles in the following manner :—We first determine, by the arithmetical process given above, one of the extreme semicircles primarily equivalent to Γ; and then by the same process we obtain the substitution which transforms this extreme semicircle into the other. Let this substitution be

$$\begin{vmatrix} \alpha, \beta \\ \gamma, \delta \end{vmatrix} = |\sigma|^{2\epsilon_1\mu_1} \times |\tau|^{2\epsilon_2\mu_2} \times |\sigma|^{2\epsilon_3\mu_3} \ldots,$$

where the first and last exponents may be zero; the substitutions

$$|\sigma|^{2\epsilon_1}, \ |\sigma|^{4\epsilon_1}, \ \ldots, \ |\sigma|^{2\epsilon_1\mu_1},$$

$$|\sigma|^{2\epsilon_1\mu_1} \times |\tau|^{2\epsilon_2}, \ \ldots, \ |\sigma|^{2\epsilon_1\mu_1} \times |\tau|^{2\epsilon_2\mu_2},$$

. . . . . .

applied successively to the first extreme semicircle, give the complete chain of reduced arcs taken in order.

When the chain of reduced arcs is given graphically, the matrix $\left|\begin{matrix}\alpha, & \beta\\ \gamma, & \delta\end{matrix}\right|$ is found by counting the reduced arcs of the different types; the equations of the extreme semicircles are then also known; viz. one of the factors of the form (a, b, c) answering to the first extreme semicircle is $x \pm y$, y, or x, according as that semicircle is of the type $\pm p$, q, or r; and the other factor is

$$(\gamma \pm \delta)\, x - (\alpha \pm \beta)\, y, \quad \gamma x - \alpha y, \text{ or } \delta x - \beta y,$$

according as the second extreme semicircle is of the type $\pm p$, q, or r.

Example.

Let the proposed semicircle be $[8, 9, 7]$, $D = 81 - 56, = 25$. We have

$$(8, 9, 7)\,(x, y)^2 = (2x + y)\,(4x + 7y);$$

and we find successively

$$\left|\begin{matrix}2, & 1\\ 4, & 7\end{matrix}\right| \times \left|\begin{matrix}1, & 0\\ -2, & 1\end{matrix}\right| = \left|\begin{matrix}0, & 1\\ -10, & 7\end{matrix}\right|,$$

$$\left|\begin{matrix}0, & 1\\ -10, & 7\end{matrix}\right| \times \left|\begin{matrix}-7, & -2\\ -10, & -3\end{matrix}\right| = \left|\begin{matrix}-10, & -3\\ 0, & -1\end{matrix}\right|,$$

$$\left|\begin{matrix}-7, & -2\\ -10, & -3\end{matrix}\right| = |\sigma|^2 \times |\tau|^{-2} \times |\sigma|^4.$$

Hence the chain of reduced circles equivalent to $[8, 9, 7]$ is

$$[0, -5, 7],\ [8, 9, 7],\ [8, -7, 3],\ [-8, -1, 3],\ [0, 5, 3].$$

Again, if we suppose the circle $[8, 9, 7]$ to be given geometrically, but its equation (or its symbol) not to be given, we have first to construct, as in Art. 44, the chain of semicircles; the aspect of the chain gives us the matrix

$$|\sigma|^2 \times |\tau|^{-2} \times |\sigma|^4 = \left|\begin{matrix}-7, & -2\\ -10, & -3\end{matrix}\right|,$$

which transforms the first of the extreme semicircles into the second; whence, finally, observing that the extreme semicircles are each of the type (q), we find that the symbol of the first of these semicircles is $[0, -5, 7]$.

Arts. 46–51. Geometrical Representation of the Modular Functions $\Phi(\omega)$ and $\Psi(\omega)$ $[=\phi^8(\omega)$ and $\psi^8(\omega)]$.

46. *Discussion of the Equation* $\Phi(\omega)=A$.

Let A be a given complex quantity having any value whatever except $0, +1, \infty$. The following theorems (of which the third results from the combination of the first and second) are of fundamental importance in the theory of the modular functions.

(A) 'A complex quantity Ω (having the coefficient of i different from zero and positive) can always be assigned satisfying the equation $\phi^8(\Omega)=A$.'

(B) 'If $\phi^8(\Omega)=\phi^8(\omega)$, the complex quantities Ω and ω (in each of which the coefficient of i is different from zero and positive) are primarily equivalent.'

(C) 'The equation $\phi^8(\omega)=A$ always admits of one, and only of one, reduced solution.'

These propositions are immediate consequences from a general theorem of Riemann ('Inaugural Dissertation,' Art. 21), relating to the representation of one plane surface upon another; but they may also be established by means of known properties of elliptic functions.

(A) The differential equation

$$\frac{du^2}{dx^2}=(1-u^2)(1-Au^2), \quad \ldots\ldots\ldots \quad \text{(i)}$$

taken with the initial conditions

$$x=0,\ u=0,\ \frac{du}{dx}=1, \quad \ldots\ldots\ldots \quad \text{(ii)}$$

defines $u=\lambda(x)$ as a uniform doubly periodic function of x, perfectly determined at every point of the plane upon which the complex variable x is represented. It is always possible (and, indeed, in an infinite number of different ways) to take as conjugate periods of $\lambda(x)$ a pair of *primary* periods, *i. e.* a pair of periods, $4L$ and $2iL'$, verifying the equations

$$(a)\ \ \lambda(L)=+1, \qquad (b)\ \ \lambda(L+iL')=\pm\frac{1}{\sqrt{A}},$$

and also satisfying the condition (c) that the coefficient of i in the quotient $\frac{iL'}{L}$ shall be positive; this coefficient cannot be zero, because the quotient of any two conjugate periods of a uniform doubly periodic function is always imaginary.

For if we represent by L_1 and iL_1' the *rectilinear* integrals

$$L_1 = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-Au^2)}}, \quad iL_1' = \int_1^{\pm\frac{1}{\sqrt{A}}} \frac{du}{\sqrt{(1-u^2)(1-Au^2)}},$$

the initial sign of the radical in the former integral being positive, and the sign of the radical in the second integral being so taken as to satisfy the condition (c), while the sign of $\frac{1}{\sqrt{A}}$ is the same as in the equation (b), it will be found that $4L_1$ and $2iL_1'$ are conjugate periods of $\lambda(x)$, and that the equations (a) and (b) are satisfied, *i. e.* that $[4L_1, 2iL_1']$ is a pair of elliptic periods of $\lambda(x)$*. There is an infinite number of pairs of primary periods, for if $\left|\begin{matrix}\alpha, \beta \\ \gamma, \delta\end{matrix}\right|$ be any primary unit matrix, the equations (a), (b), and the condition (c) are satisfied by the quantities

$$L = \alpha L_1 + \beta iL_1', \quad iL' = \gamma L_1 + \delta iL_1,$$

(the sign of $\sqrt{A}$ in the equation (b) changing when γ is unevenly even). If

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* For the method by which this result is obtained, see MM. Briot et Bouquet, 'Théorie des Fonctions Elliptiques' (Paris, 1875), pp. 351–368.

By a pair of *conjugate* periods we understand a pair of periods forming an elementary parallelogram of the doubly periodic function (*ibid.* pp. 231–234). The primary periods defined in the text differ from the *elliptic* periods of MM. Briot and Bouquet, only because we have left the sign of the radical $\sqrt{A}$ undetermined in the equation (b). This enables us to enunciate the theorem (D):

'If $[4L_1, 2iL_1']$ is any given pair of primary periods, all pairs of such periods are included in the formula $[4L, 2iL']$ where

$$(L, iL') = \left|\begin{matrix}\alpha, \beta \\ \gamma, \delta\end{matrix}\right| \times (L_1, iL_1')$$

and $\left|\begin{matrix}\alpha, \beta \\ \gamma, \delta\end{matrix}\right|$ is a primary matrix.'

If we adhere to the definition of MM. Briot and Bouquet, who suppose the sign of $\sqrt{A}$ to be fixed, we must add the condition that γ is to be evenly even. The theorem (D) is equivalent to the theorem (B) of the text; for both $[4K(\omega), 2iK'(\omega)]$ and $[4K(\Omega), 2iK'(\Omega)]$ are pairs of elliptic periods of one and the same function $u = \lambda(x)$.

In the Report on the Theory of Numbers (Reports of the British Association for 1865), Art. 125, pp. 330 *et seq.* [vol. i. p. 295], the theorems (A) and (B) were enunciated; and the principle of the demonstration here given was indicated. M. Hermite has recently called attention to the importance of the theorem (B) in a note on a Memoir by M. Fuchs ('Borchardt's Journal,' vol. lxxxiii. p. 29); and a demonstration of this theorem, depending on the theory of elliptic functions, has been given by M. Dedekind (*ibid.* pp. 266–269), who has also observed (p. 274) that the properties expressed by the theorems (A) and (B) follow from the principle of Riemann. See also the Note on the primary periods of the elliptic functions, cited in the footnote, p. 507.

$[4L, 2iL']$ is any pair of primary periods, the zero points of $\lambda(x)$ are

$$2mL+2niL',$$

and its infinite points are

$$2mL+(2n+1)iL',$$

m and n denoting any positive or negative integral numbers; and the zeros and infinities being all simple. Let $\frac{iL'}{L}=\omega$, and let K and K' be determined by the equations $K=K(\omega)$, $K'=\frac{\omega}{i}K(\omega)$; since $\frac{L'}{L}=\frac{K'}{K}$, we may write

$$L=\frac{K}{\mu}, \quad L'=\frac{K'}{\mu};$$

and we shall find that the doubly periodic function

$$\frac{1}{\phi^2(\omega)}\times\frac{\vartheta_1\left(\frac{\pi x}{2L},\omega\right)}{\vartheta\left(\frac{\pi x}{2L},\omega\right)}=\sin\text{am}\left[\mu x,\phi^8(\omega)\right]$$

has the same periods $4L$ and $2iL'$ as $\lambda(x)$; it has also the same zero points and the same infinite points; hence, by the principles of the Theory of Functions,

$$\lambda(x)=C\times\sin\text{am}(\mu x),$$

C denoting some constant multiplier; but $\lambda(L)=1$; and

$$\sin\text{am}(\mu L)=\sin\text{am}(K)=1;$$

i.e. $C=1$, and $\lambda(x)=\sin\text{am}(\mu x)$. Lastly,

$$\lambda(L+iL')=\sin\text{am}(\mu L+i\mu L')=\pm\frac{1}{\sqrt{A}};$$

but

$$\sin\text{am}(\mu L+i\mu L')=\sin\text{am}(K+iK')=\frac{1}{\phi^4(\omega)};$$

whence we infer

$$\phi^4(\omega)=\pm\sqrt{A}, \quad \phi^8(\omega)=A.$$

We may add (though this is not required to complete the preceding demonstration): (1) that $\mu=+1$; for, when $x=0$,

$$\lim\frac{\sin\text{am}(\mu x)}{\mu x}=1,$$

but also

$$\lim\frac{\sin\text{am}(\mu x)}{x}=\lim\frac{\lambda(x)}{x}=1.$$

(2) that the equation (b) is always satisfied when the equation (a) is satisfied; viz. if, of two conjugate periods $4L$ and $2iL'$, one satisfies the relation $\lambda(L)=1$, the equation $\lambda(L+iL')=\pm\frac{1}{\sqrt{A}}$ is always satisfied; for any two conjugate periods may be expressed by the formula

$$4L=4\alpha L_1+2\beta iL_1', \qquad 2iL'=4\gamma L_1+2\delta iL_1',$$

where $\alpha\delta-\beta\gamma=\pm1$. But if $\lambda(L)=1$, we must have $\alpha\equiv1$, mod 4, and $\beta\equiv0$, mod 4. Hence δ is uneven, and

$$\lambda(L+iL')=(-)^{\gamma}\lambda[L_1+iL_1']=\pm\frac{(-1)^{\gamma}}{\sqrt{A}}.$$

(B) If $\phi^8(\omega)=\phi^8(\Omega)=A$, we form the two elliptic functions

$$u_0=\frac{1}{\phi^2(\omega)}\times\frac{\vartheta_1\left(\frac{\pi x}{2K(\omega)},\omega\right)}{\vartheta_0\left(\frac{\pi x}{2K(\omega)},\omega\right)}, \qquad u_1=\frac{1}{\phi^2(\Omega)}\times\frac{\vartheta_1\left(\frac{\pi x}{2K(\Omega)},\Omega\right)}{\vartheta_0\left(\frac{\pi x}{2K(\Omega)},\Omega\right)}.$$

Each of these functions satisfies the differential equation (i) and the initial conditions (ii). Hence the two functions are identical, and their zero points and their infinite points are the same. Considering the zero point $2K(\omega)$, and the infinite point $iK'(\omega)$, of the former function, we must have

$$K(\omega)=\alpha K(\Omega)+i\beta K'(\Omega), \quad iK'(\omega)=\gamma K(\Omega)+i\delta K'(\Omega); \quad . \quad . \quad . \quad \text{(iii)}$$

where γ is even and δ uneven. Similarly, we should find

$$K(\Omega)=\alpha_1K(\omega)+i\beta_1K'(\omega), \quad iK'(\Omega)=\gamma_1K(\omega)+i\delta_1K'(\omega), \quad . \quad . \quad . \quad \text{(iv)}$$

where γ_1 is even and δ_1 uneven. But, if $\alpha\delta-\beta\gamma=m$, we may obtain from (iii)

$$K(\Omega)=\frac{\delta}{m}K(\omega)-i\frac{\beta}{m}K'(\omega), \quad iK'(\Omega)=-\frac{\gamma}{m}K(\omega)+i\frac{\alpha}{m}K'(\omega). \quad . \quad . \quad \text{(v)}$$

The systems (iv) and (v) must be identical; for otherwise the quotient $\frac{iK'(\omega)}{K(\omega)}=\omega$ would be real. Hence m divides $\alpha, \beta, \gamma, \delta$; *i.e.*, $m=\pm1$. The negative sign must be rejected, because the sign of i in Ω and in ω, $=\frac{\gamma+\delta\Omega}{\alpha+\beta\Omega}$, is the same; therefore $m=\alpha\delta-\beta\gamma=+1$. Since $\alpha=\delta_1$, α is uneven: and since

$$\sin\operatorname{am}[\alpha K(\Omega)+i\beta K'(\Omega)]=+1,$$

β must be even, and $\alpha \equiv 1$, mod 4. Thus the matrix $\left|\begin{matrix}\alpha, \beta \\ \gamma, \delta\end{matrix}\right|$ is primary, and ω, Ω are primarily equivalent.

47. *Limiting Values of* $\Phi(\omega)$.

We shall now suppose that the point ω is confined within the reduced space (see figure 2, p. 507), and we shall examine the values assumed by the function $\phi^8(\omega)$ in the vicinity of the cornicular points $\pm p$, q, r. Let τ be a constant quantity included between the limits -1 and $+1$, so that $-1 \leqq \tau < +1$; and let σ be a positive quantity increasing without limit.

(i.) Let $\omega = \tau + i\sigma$, so that ω moves along the straight line $x = \tau$, and approximates to the cornicular point ∞, or r. It is evident from the equation of definition of $\phi(\omega)$ that

$$\operatorname{Lim} \phi^8(\omega) = 16e^{-\pi\sigma}(\cos \tau\pi + i \sin \tau\pi);$$

i. e. $\phi^8(\omega)$ approximates without limit to zero, travelling in the direction indicated by the vector $-e^{i\pi\tau}$.

(ii.) Let $\omega = -\dfrac{1}{\tau + i\sigma}$, so that as σ increases without limit, ω moves along the semicircle $\tau(x^2+y^2)+x=0$ toward the cornicular point 0, or q. Then

$$\phi^8(\omega) = \psi^8\left(-\frac{1}{\omega}\right) = 1 - \phi^8\left(-\frac{1}{\omega}\right) = 1 - \phi^8(\sigma + i\tau)$$

$$= 1 - 16e^{-\pi\sigma}(\cos \pi\tau + i \sin \pi\tau) \text{ ultimately};$$

i. e. $\phi^8(\omega)$ converges to $+1$ in the direction indicated by the vector $e^{i\pi\tau}$; that direction depending on the curvature of the circular path pursued by ω.

(iii.) Let τ be positive, and let $\omega = 1 - \dfrac{1}{\tau + i\sigma}$, so that ω converges to the cornicular point $+1$ or $-p$, moving along the semicircle

$$\tau[(x-1)^2 + y^2] + x - 1 = 0.$$

Here $\phi^8(\omega) = 1 - \dfrac{1}{\phi^8(\tau + i\sigma)} = \frac{1}{2} - \frac{1}{16}e^{\pi\sigma}(\cos \pi\tau - i \sin \pi\tau)$ ultimately, omitting quantities of the order $e^{-\pi\sigma}$; so that $\Phi(\omega)$ becomes infinite, and travels, as σ increases, in the direction of the vector $-e^{-\pi i\tau}$. If τ is negative, we write $\omega = -1 - \dfrac{1}{\tau + i\sigma}$, so that ω converges to the cornicular point -1, or p, moving along the semicircle $\tau[(x+1)^2 + y^2] + x + 1 = 0$, and we obtain the sameresult as before.

The function $\phi^8(\omega)$ cannot attain the values 0, 1, ∞ at any point of the reduced space other than the cornicular points ∞, 0, ± 1, respectively. For if σ be finite and different from zero, the infinite products $\Pi\,(1+q^{2m-1})$ and $\Pi\,(1+q^{2m})$ are finite and different from zero; again, if σ converge to zero, so that ω approximates to the cornicular point 0 or ± 1, $\phi^8(\omega)$ converges (as we have just seen) either to 1 or to ∞; hence $\phi^8(\omega)$ cannot be zero, unless σ be infinite, *i. e.* unless ω approximate to the cornicular point ∞; and by using the transformations $\omega = -\frac{1}{\Omega}$, $\omega = \pm 1 - \frac{1}{\Omega}$, we may prove the corresponding assertions for the other cornicular points.

48. *Transformation of the Reduced Space by the Modular Functions* $\Phi(\omega)$ and $\Psi(\omega)$.

Denoting by X and Y real quantities, we shall now write

$$X + iY = \Phi(\omega) - \tfrac{1}{2} = \tfrac{1}{2} - \Psi(\omega);$$

and taking a pair of rectangular axes X, Y, in a new plane (XY), we shall regard the point $X+iY$ as answering to the point ω in the reduced space. We are thus in fact mapping the reduced space upon the plane XY; so that by a well-known theorem to which we have already referred, the transformation is *homœomeric*, *i. e.* the infinitesimal parts of the two figures are similar except at points at which the first derived function of the mapping function is infinite or zero. The derived function of $\Phi(\omega)$ is (Art. 9)

$$\Phi'(\omega) = \frac{4i}{\pi}\,\Phi(\omega) \times \Psi(\omega) \times K^2(\omega),$$

which is always finite and different from zero, so long as the real part of $i\omega$ is finite and different from zero; *i. e.* so long as ω does not approximate to one of the cornicular points. If $\omega = \tau + i\sigma$ approximate to the cornicular point ∞, we find

$$\operatorname{Lim} \Phi'(\omega) = i\pi \operatorname{Lim} \Phi(\omega) = 16\, i\pi e^{-\pi\sigma}(\cos \pi\tau + i \sin \pi\tau);$$

i. e. $\Phi'(\omega)$ converges to zero. Writing $\Omega = \tau + i\sigma$, and employing the substitutions

$$\omega = -\frac{1}{\Omega}, \quad \omega = \pm 1 - \frac{1}{\Omega},$$

which give

$$\Phi'(\omega) = -\Phi'(\Omega) \times \Omega^2, \quad \Phi'(\omega) = \frac{\Phi'(\Omega) \times \Omega^2}{\Phi^2(\Omega)},$$

we cause Ω to approximate to the cornicular point ∞, and we infer that if ω approximate to the cornicular point 0,

$$\text{Lim } \Phi'(\omega) = -16i\pi(\sigma+i\tau)^2 e^{-\pi\sigma}(\cos\pi\tau + i\sin\pi\tau) = 0;$$

and that if ω approximate to the cornicular point ± 1,

$$\text{Lim } \Phi'(\omega) = \tfrac{1}{16} i\pi e^{\pi\sigma}(\cos\pi\tau - i\sin\pi\tau)(\tau+i\sigma)^2 = \infty.$$

Thus the infinitesimal similarity of the two figures holds at all points of the reduced space except at the cornicular points ∞, 0, ± 1, to which there correspond in the plane (XY) the points $(-\frac{1}{2}, 0)$ or A_{-1}, $(+\frac{1}{2}, 0)$ or A_{+1}, and the point at an infinite distance; spaces in the vicinity of the cornicular points ∞ and 0 being infinitely contracted in the plane (XY) and spaces in the vicinity of the cornicular points ± 1 being infinitely expanded.

The correspondence between the reduced space and the plane (XY) is a correspondence *one to one*. For to every point in the reduced space there answers one, and only one, point in the plane (XY); and *vice versâ* to every point in the plane (XY) there answers one, and only one, point in the reduced space; this results either from the general theorem of Riemann, or from the proposition (C) of Art. 46, and from our examination of the values of $\Phi(\omega)$ in the vicinity of the cornicular points.

49. *Lines answering to the Semicircles of Determinant* $+1$.

We shall now determine the lines which in the plane (XY) answer to the reduced arcs bounding the twelve spaces into which the reduced space is divided (Art. 38). See figure 2 (p. 507) and figure 4 (p. 536).

(A) To the line $x=0$ considered as described from 0 to ∞, there answers the finite rectilinear segment $A_{-1}A_{+1}$, described from A_{-1} to A_{+1}. For if $\omega = \sigma i$, and the real quantity σ decrease from ∞ to 0, $\Phi(\omega) - \frac{1}{2}$ remains continually real, and the differential quotient $\dfrac{d\Phi(\omega)}{d\sigma}$ is real and negative; *i.e.* $\Phi(\omega) - \frac{1}{2}$ increases continually from $-\frac{1}{2}$ to $+\frac{1}{2}$.

Again, to the two equivalent lines $x = \pm 1$, considered as described from $+\infty$ to ± 1, there corresponds the line $[S]$, *i.e.* that part of the axis of X which lies between $-\frac{1}{2}$ and $-\infty$; for if $\omega = 1+\sigma i$, $\Phi(\omega)$ is real and negative, and $\dfrac{d\Phi(\omega)}{d\sigma}$ is positive, so that as σ decreases from ∞ to 0, $\Phi(\omega) - \frac{1}{2}$ decreases from $-\frac{1}{2}$ to $-\infty$. Lastly, to the equivalent circles $x^2+y^2 = \pm x$ considered as described from 0 to ± 1,

there corresponds the line $[T]$, *i. e.* that part of the axis of X which lies from $+\frac{1}{2}$ to $+\infty$; this result is immediately deduced from the last by employing the transformation

$$\omega = -\frac{1}{\Omega}, \quad \Omega = \sigma i, \quad \Phi(\omega) - \tfrac{1}{2} = \Psi(\Omega) - \tfrac{1}{2} = \tfrac{1}{2} - \Phi(\Omega).$$

In the equation $\Phi(\omega) = X + iY$, Y has the same sign as x in the equation $\omega = x + iy$. For very great values of y this is evident from the equation defining

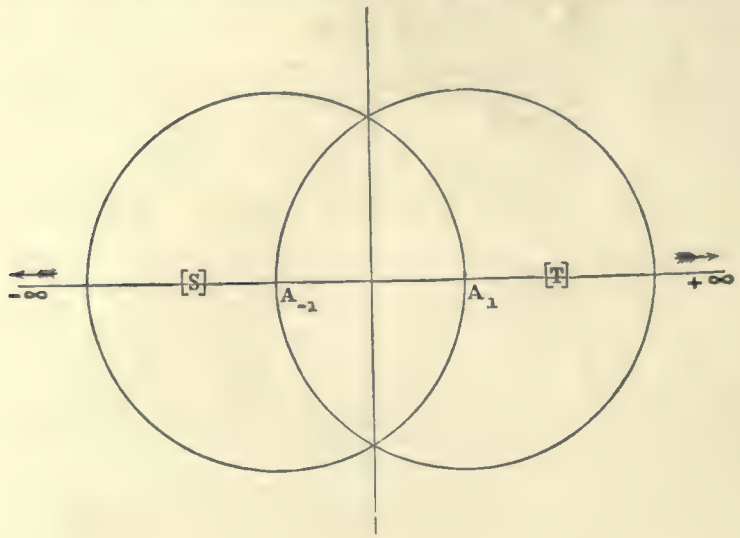


Fig. 4.

$\Phi(\omega)$; it is therefore true for all positive values of y, because, if x changes its sign, ω must either traverse the line $x = 1$, or must quit the reduced space at one of the boundaries to enter again at the equivalent boundary; in either case, by what has just been proved, $X + iY$ must traverse the axis of X; *i. e.* Y must change its sign with x.

(B) To the semicircle $x^2 + y^2 = 1$, considered as described from -1 to $+1$, there answers the axis of Y from $-\infty$ to $+\infty$. For if $\omega = x + iy$ lies on this semicircle, we have

$$\omega = x + iy = \frac{-1}{-x + iy},$$

or

$$\Phi(x + iy) = \Psi(-x + iy) = 1 - \Phi(-x + iy). \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

But by a principle which we shall frequently employ, $\Phi(x + iy)$ and

$\Phi(-x+iy)$ are conjugate imaginaries, so that

$$\text{if } \Phi(x+iy)=\tfrac{1}{2}+X+iY, \text{ then } \Phi(-x+iy)=-\tfrac{1}{2}+X-iY;$$

and the equation (i) becomes $X=0$. Hence to any point on the semicircle $x^2+y^2=1$, there answers a point on $X=0$. Conversely to any point on $X=0$, there answers a point on the semicircle. For if $\Phi(x+iy)=\frac{1}{2}+iY$, then

$$\Phi(-x+iy)=\tfrac{1}{2}-iY=1-\Phi(x+iy)=\Psi(x+iy)=\Phi\left(\frac{-1}{x+iy}\right).$$

But, because $x+iy$ is reduced, $-x+iy$ and $\dfrac{-1}{x+iy}$ are also reduced. Therefore the equation

$$\Phi(-x+iy)=\Phi\left(-\frac{1}{x+iy}\right)$$

implies the equation

$$-x+iy=-\frac{1}{x+iy}, \text{ or } x^2+y^2=1.$$

It is evident that to the spaces lying within and without the semicircle $x^2+y^2=1$ there answer in the plane (XY) the regions to the right and to the left respectively of the axis of Y.

(C) To the reduced parts of the lines $\pm 1+2x=0$ there answers in the plane (XY) the circle $(X-\frac{1}{2})^2+Y^2=1$; viz. to the line $-1+2x=0$, considered as drawn from ∞ to $\frac{1}{2}(1+i)$, there answers the semicircle drawn from $-\frac{1}{2}$ to $\frac{3}{2}$ through the upper region of the plane XY; and to the line $1+2x=0$, considered as drawn from $\frac{1}{2}(1-i)$ to ∞, there answers the semicircle drawn from $\frac{3}{2}$ to $-\frac{1}{2}$ through the lower region of the plane XY. For, if $x=\pm\frac{1}{2}$, we have

$$x+iy=\pm 1+(-x+iy); \text{ whence}$$

$$\Phi(x+iy)=-\frac{\Phi(-x+iy)}{\Psi(-x+iy)},$$

or

$$\tfrac{1}{2}+X+iY=\frac{\tfrac{1}{2}+X-iY}{-\tfrac{1}{2}+X-iY},$$

i. e.

$$(X-\tfrac{1}{2})^2+Y^2=1.$$

Conversely, it is readily seen that the equation $(X-\frac{1}{2})^2+Y^2=1$, implies the equation

$$\Phi(x+iy)=\Phi(\pm 1-x+iy);$$

but if $x+iy$ be reduced, $\pm 1-x+iy$ is also reduced, the ambiguous sign being that of x; hence

$$x+iy = \pm 1 - x + iy, \text{ or } 2x = \pm 1.$$

(D) Lastly, to the reduced quadrants of the semicircles $x^2+y^2 = \pm 2x$, considered as described from 0 to $1+i$, and from $-1+i$ to 0, there answers the circle $(X+\frac{1}{2})^2+Y^2=1$, considered as described from $\frac{1}{2}$ to $-\frac{3}{2}$, through the upper region, and from $-\frac{3}{2}$ to $\frac{1}{2}$ through the lower region of the plane (XY). We might establish this directly; but it is simpler to deduce it from the preceding result. As $-\frac{1}{\omega}$ describes the line $x=\frac{1}{2}$ from ∞ to $\frac{1}{2}(1+i)$, ω describes the quadrant $x^2+y^2+2x=0$ from 0 to $-1+i$. But

$$\Phi(\omega) - \tfrac{1}{2} = -\left[\Phi\left(-\frac{1}{\omega}\right) - \tfrac{1}{2}\right];$$

hence, while ω describes the quadrant, $\Phi(\omega)-\frac{1}{2}$ describes in the plane XY a locus centrally opposite to the locus corresponding to the reduced part of $x=\frac{1}{2}$; *i.e.* $\Phi(\omega)-\frac{1}{2}$ describes the semicircle $(X+\frac{1}{2})^2+Y^2=1$ from $\frac{1}{2}$ to $-\frac{3}{2}$ through the lower region of the plane XY.

The figure 4 shows the distribution of the spaces which in the plane XY answer to the twelve regions into which the reduced space is divided. The symmetry of the figure with regard to 0, the middle point of the segment $A_{-1}A_{+1}$, renders it convenient to take that point as the origin; *i.e.* to write, as we have done, $\Phi(\omega)=\frac{1}{2}+X+iY$, instead of writing $\Phi(\omega)=X+iY$. If P be any point in the plane XY, we have

$$A_{-1}P = \Phi(\omega), \quad A_{+1}P = -\Psi(\omega), \quad OP = \tfrac{1}{2}[\Phi(\omega) - \Psi(\omega)].$$

Jacobi ('Fundamenta Nova,' Art. 29) found that the modular equations of the 3rd and 5th orders assumed their simplest forms when expressed in terms of the quantities $1-2k^2$, $1-2\lambda^2$.

50. *Discussion of the Correspondence between the Reduced Space and the Plane XY.*

(A) The values of Φ at the points of intersection of the boundaries are as follows:

$$\left.\begin{array}{c} \Phi(0)=1, \quad \Phi(1)=\infty, \quad \Phi(\infty)=0; \\ \left.\begin{array}{c} \Phi(1+i)=\Phi(-1+i)=-1, \quad \Phi(i)=\frac{1}{2}, \\ \Phi(\frac{1}{2}+\frac{1}{2}i)=\Phi(-\frac{1}{2}+\frac{1}{2}i)=2; \end{array}\right\} \\ \Phi(\frac{1}{2}+\frac{1}{2}i\sqrt{3})=\frac{1}{2}+\frac{1}{2}i\sqrt{3}, \quad \Phi(-\frac{1}{2}+\frac{1}{2}i\sqrt{3})=\frac{1}{2}-\frac{1}{2}i\sqrt{3} \end{array}\right\} \quad \ldots\ldots \text{(i)}$$

These are the critical values of the anharmonic ratios of four points, corresponding to the cases in which (1) two of the points coincide, (2) the system is harmonic, (3) the system is equianharmonic.

(B) The broken line, made up of the straight lines qr, rp, and the semicircle pqr (which forms on the tricuspid surface a closed curve passing through the three singular points) is transformed in the plane (XY) into the straight line $A_{+1}A_{-1} \infty A_{+1}$, the evanescent angles at r and q (which are the halves of the cornicular angles at those points) being each changed into two right angles; a distortion which is not inconceivable because the infinitesimal similarity does not exist at the points p and q. In the same way the reduced parts of the straight lines $2x \pm 1 = 0$; and again the reduced arcs of the semicircles $x^2+y^2 \pm 2x = 0$ are represented by closed curves in the plane XY, not presenting any singularity at the points answering to r and q.

(C) The relations between the 12 spaces in the plane (XY), which answer to the 12 spaces of Art. 38, are as follows:

$$B = 1 - A, \quad C = \frac{A}{A-1}, \quad D = \frac{1}{A},$$

$$E = \frac{1}{1-A}, \quad F = \frac{A-1}{A},$$

A, B, C, ..., denoting vectors drawn from A_{-1}.

These equations, which hold for the accented as well as for the unaccented symbols, result from the formulæ (i) of Art. 38, compared with the formulæ of Art. 21. They show that the six pairs of spaces are related to one another as the six anharmonic ratios of four points, the points of intersection of the boundaries corresponding (as we have already seen) to the critical values.

(D) If (for brevity) we say that two points which are symmetrically situated with regard to a straight line are inverse to one another with regard to that straight line, we may enunciate the theorem:

'Two points which are inverse to one another with regard to any one of the boundaries in either figure, have corresponding points in the other figure which are inverse with regard to the corresponding boundary.'

This is a particular case of a general theorem to which we shall refer later, but it may also be verified independently. For example, in the reduced space, F is inverse to D' with regard to $x^2+y^2=1$, and C' is inverse to A with regard to $x = -\frac{1}{2}$; hence, in the plane XY, F and D' are inverse with regard to $X=0$, A and C' are inverse with regard to $(X-1)^2+Y^2=1$.

(E) The lines $[S]$ and $[T]$ are lines of discontinuity in the correspondence between the reduced quantity ω and the vector $X+iY=\Phi(\omega)-\frac{1}{2}$. Let the point $X+iY$ traverse the line $[S]$, passing (for example) from the upper to the lower part of the plane. Immediately before the passage, ω is close to the boundary S_{+1}; and immediately after the passage ω is close to the boundary S_{-1}; hence, when $X+iY$ traverses $[S]$, ω changes discontinuously from the value ω to the value $\omega-2=|\sigma|^{-2}\times\omega$. Similarly, if $X+iY$ traverses $[T]$, passing from the lower to the upper part of the plane, ω, which immediately before the passage is close to the boundary T_{-1}, is immediately afterwards close to the boundary T_{+1}; and thus changes abruptly from the value ω to the value $\dfrac{\omega}{1+2\omega}=|\tau|^{2}\times\omega$.

Let us now suppose the point $X+iY$, setting out from a given initial position P, to describe any closed contour whatever (C) in the plane (XY); and, still denoting by ω the reduced argument which satisfies the equation $\Phi(\omega)-\frac{1}{2}=X+iY$, let us represent by θ an argument, which satisfies the same equation, and which initially coincides with ω, but which, instead of remaining always reduced, varies continuously with $X+iY$; it is required to find the relation between θ and ω, when $X+iY$, after having described the given contour, returns again to P. If $X+iY$ traverses neither $[S]$ nor $[T]$, we have $\theta=\omega$ along the whole contour; if (C) be a *simple* contour including A_{-1}, but not including A_{+1}, we have, when $X+iY$ returns again to P, $\theta=|\sigma|^{\pm 2}\times\omega$, the lower or upper sign being taken according as the revolution round A_{-1} is positive or negative (*i.e.* left- or right-handed); if (C) be a simple contour, including A_{+1}, but not A_{-1}, we have, when $X+iY$ returns to P, $\theta=|\tau|^{\mp 2}\times\omega$, the ambiguous sign being determined as before; if $X+iY$ traverses $[S]$ or $[T]$ any even number of times, no passage across $[T]$ or $[S]$ intervening, the relation between θ and ω is the same after the last of these passages that it was before the first. Thus, without altering the ultimate value of θ, we may continuously deform the contour (C) in any manner we please, provided only that in so doing we never allow it to cross either of the points A_{-1} or A_{+1}. By such continuous deformation we may substitute for (C) a series of *elementary* contours, alternately surrounding A_{-1} and A_{+1}, and succeeding one another in a perfectly definite order. If (C) be thus reduced to a series of α_1 contours round A_{-1}, followed by β_1 contours round A_{+1}, followed by α_2 contours round A_{-1}, and so on, the numbers $\alpha_1, \alpha_2, \ldots$, $\beta_1, \beta_2, \ldots$, being positive or negative according as the elementary contours to which they refer are described in the positive or negative direction, we obtain

for the ultimate relation between θ and ω, when $X+iY$ finally returns to P, the equation

$$\theta = |\sigma|^{2\alpha_1} \times |\tau|^{-2\beta_1} \times |\sigma|^{2\alpha_2} \times |\tau|^{-2\beta_2} \times \ldots \times \omega;$$

where, however, the first exponent, α_1, is zero, if the set of elementary contours which come first in order surround A_{+1}, and not A_{-1}.

A particular case of the preceding result is occasionally useful. If (C) is a simple contour, including both A_{-1} and A_{+1}, and described in a negative direction round those points, (C) may also be regarded as a simple positive contour surrounding the point at an infinite distance in the plane (XY); and the relation between θ and ω is expressed by the equation

$$\theta = |\sigma|^{-2} \times |\tau|^{2} \times \omega = \begin{vmatrix} 1, & 2 \\ -2, & -3 \end{vmatrix} \times \omega,$$

or by the equation

$$\theta = |\tau|^{2} \times |\sigma|^{-2} \times \omega = \begin{vmatrix} -3, & 2 \\ -2, & 1 \end{vmatrix} \times \omega,$$

according as, setting out from P, and following the contour in the proper direction, we first traverse $[T]$ or $[S]$.

51. *Limiting Values of* $\Phi(\Omega)$.

Let n be any positive uneven number; let $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ be a primitive and primary matrix of determinant n, and let $\Omega = \dfrac{c+d\omega}{a+b\omega}$. We proceed to examine the values acquired by $\Phi(\Omega)$ when the reduced point ω lies in one of the cornicular angles r, q, p; *i. e.* when $\Phi(\omega)$ approaches one of the values $0, 1, \infty$.

(i.) As in Art. 47, let $\omega = \tau + i\sigma$, where $-1 \leqq \tau < +1$, and where σ is positive and increases without limit; also let $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ be equivalent, by primary premultiplication, to $\begin{vmatrix} g, & 0 \\ 2h, & g' \end{vmatrix}$, so that

$$\Phi(\Omega) = \Phi\left(\frac{2h+g'\omega}{g}\right);$$

it will be found that g' is the greatest common divisor of b, d, and that h is determined by the congruences

$$2h\frac{b}{g'} \equiv a, \quad 2h\frac{d}{g'} \equiv c, \bmod g.$$

Since $\Phi(\Omega) = \Phi\left(\frac{2h+g'\tau}{g} + \frac{g'}{g} i\sigma\right)$, $\Phi(\Omega)$ converges to zero at the same time with $\Phi(\omega)$, and we find (*cf.* Art. 47)

$$\operatorname{Lim} \Phi(\Omega) \div [\Phi(\omega)]^{\frac{g'}{g}} = 2^{4-4\frac{g'}{g}} \times e^{\frac{2h}{g} i\pi}.$$

The numbers g and g' may be negative, because $g \equiv 1$, mod 4; but $g \times g'$, $= n$, and $\frac{g'}{g}$, $= \frac{n}{g^2}$, are positive.

(ii.) Let $\omega = -\frac{1}{\tau + i\sigma}$, so that $\Phi(\omega) = 1 - \Phi(\tau + i\sigma)$; also let

$$\begin{vmatrix} a, b \\ c, d \end{vmatrix} \times \begin{vmatrix} 0, 1 \\ -1, 0 \end{vmatrix} = |v_1| \times \begin{vmatrix} g_1, 0 \\ 2h_1, g_1' \end{vmatrix},$$

where $|v_1|$ is a premultiplying unit of the type ψ; it will be found that g_1' is the greatest common divisor of a and c, and that h_1 is determined by the congruences

$$2h\frac{a}{g_1'} \equiv -b, \quad 2h\frac{c}{g_1'} \equiv -d, \text{ mod } g_1.$$

Since $\Phi(\Omega) = 1 - \Phi\left(\frac{2h_1 + g_1'\tau}{g_1} + \frac{g_1'}{g_1} i\sigma\right)$, $\Phi(\Omega)$ converges to $+1$ at the same time with $\Phi(\omega)$, and we have

$$\operatorname{Lim} [1 - \Phi(\Omega)] \div [1 - \Phi(\omega)]^{\frac{g_1'}{g}} = \operatorname{Lim} \Psi(\Omega) \div [\Psi(\omega)]^{\frac{g_1'}{g'}} = 2^{4-4\frac{g_1'}{g'}} \times e^{\frac{2h_1}{g} i\pi}.$$

(iii.) Let $\omega = \eta - \frac{1}{\tau + i\sigma}$, where $\eta = +1$ or -1 according as ω lies in the cornicular angle $+1$ or -1. We then have

$$\Phi(\omega) = 1 - \frac{1}{\Phi(\tau + i\sigma)},$$

and

$$\Omega = \begin{vmatrix} a, b \\ c, d \end{vmatrix} \times \begin{vmatrix} 0, 1 \\ -1, \eta \end{vmatrix} \times [\tau + i\sigma].$$

Let

$$\begin{vmatrix} a, b \\ c, d \end{vmatrix} \times \begin{vmatrix} 0, 1 \\ -1, \eta \end{vmatrix} = |v_2| \times \begin{vmatrix} g_2, 0 \\ 2h_2, g_2' \end{vmatrix},$$

$|v_2|$ being a premultiplying unit of the type ρ^2: g_2' is the greatest common divisor

of $\alpha+b\eta$ and $c+d\eta$, and h_2 is determined by the congruences

$$2h_2\frac{\alpha+b\eta}{g_2'}\equiv -b, \quad 2h_2\frac{c+d\eta}{g_2'}\equiv -d, \text{ mod } g_2.$$

We then have

$$\Phi(\Omega)=1-\frac{1}{\Phi\left(\frac{2h_2+g_2'\tau}{g_2}+\frac{g_2'}{g_2}i\sigma\right)};$$

so that $\Phi(\Omega)$ increases without limit at the same time with $\Phi(\omega)$, the ultimate values of $\Phi(\omega)$ and $\Phi(\Omega)$ being respectively

$$\tfrac{1}{2}-\tfrac{1}{16}e^{\pi\sigma}\times e^{-i\pi\tau}, \text{ and } \tfrac{1}{2}-\tfrac{1}{16}e^{\frac{g_2'}{g_2}\pi\sigma}\times e^{-i\pi\frac{2h_2+g_2'\tau}{g_2}};$$

whence we have $$\text{Lim } \Phi(\Omega)\div[\Phi(\omega)]^{\frac{g_2'}{g_2}}=2^{4\frac{g_2'}{g_2}-4}\times e^{-\pi i\frac{2h_2}{g_2}}.$$

The values of g_2, g_2', h_2 depend on the unit η; hence the relation between the limiting values of $\Phi(\Omega)$ and $\Phi(\omega)$ is different when $\Phi(\omega)$ is infinite in the upper, and when it is infinite in the lower region of the plane (XY).

If in the equation $\Omega=\begin{vmatrix}a, b\\ c, d\end{vmatrix}\times\omega$, ω is not reduced, let ω_1 be the reduced vector equivalent to ω, and let $\omega=\begin{vmatrix}\alpha, \beta\\ \gamma, \delta\end{vmatrix}\times\omega_1$; then

$$\Phi(\omega_1)=\Phi(\omega), \text{ and } \Omega=\begin{vmatrix}a, b\\ c, d\end{vmatrix}\times\begin{vmatrix}\alpha, \beta\\ \gamma, \delta\end{vmatrix}\times\omega_1.$$

Hence in this case also the ultimate relation of $\Phi(\Omega)$ to $\Phi(\omega)$, when $\Phi(\omega)$ approaches one of the values 0, 1, ∞, may be ascertained by the preceding formulæ, the numbers g, g', and h being determined from the matrix $\begin{vmatrix}a, b\\ c, d\end{vmatrix}\times\begin{vmatrix}\alpha, \beta\\ \gamma, \delta\end{vmatrix}$, and not from the matrix $\begin{vmatrix}a, b\\ c, d\end{vmatrix}$.

We may therefore enunciate the theorem:

'If Ω and ω are any two vectors connected by the relation

$$\Omega=\begin{vmatrix}a, b\\ c, d\end{vmatrix}\times\omega,$$

where $\begin{vmatrix}a, b\\ c, d\end{vmatrix}$ is a primitive and primary matrix of determinant n; and if one of

the two quantities $\Phi(\Omega)$ and $\Phi(\omega)$ approximates to one of the values 0, 1, ∞, the other approximates to the same value."

Again, if in the equation

$$\Omega = \left|\begin{matrix} a, b \\ c, d \end{matrix}\right| \times \omega,$$

the matrix $\left|\begin{matrix} a, b \\ c, d \end{matrix}\right|$ is not of the type (1), let $\omega = \left|\begin{matrix} \alpha, \beta \\ \gamma, \delta \end{matrix}\right| \times \omega_1$, ω_1 being reduced, and $\left|\begin{matrix} \alpha, \beta \\ \gamma, \delta \end{matrix}\right|$ being a unit matrix of the type inverse to $\left|\begin{matrix} a, b \\ c, d \end{matrix}\right|$; then

$$\Omega = \left|\begin{matrix} a, b \\ c, d \end{matrix}\right| \times \left|\begin{matrix} \alpha, \beta \\ \gamma, \delta \end{matrix}\right| \times \omega_1,$$

the matrix $\left|\begin{matrix} a, b \\ c, d \end{matrix}\right| \times \left|\begin{matrix} \alpha, \beta \\ \gamma, \delta \end{matrix}\right|$ being primary. The ultimate relation between $\Phi(\Omega)$ and $\Phi(\omega_1)$ may be found by the preceding formulæ; and thus the ultimate relation between $\Phi(\Omega)$ and $\Phi(\omega)$ is known.

Arts. 52–58. The Modular Equation.

52. *Definition of the Modular Equation.*

Theorem. If $|A|$ represent a system of primitive and primary matrices of an uneven determinant n, non-equivalent by post-multiplication, every rational and integral symmetric function H of the $\sigma'(n)$ quantities (see Art. 18)

$$\Phi(\Omega) = \Phi(|A|^{-1} \times \omega)$$

is a rational and integral function of $\Phi(\omega)$.

To establish this theorem we have to show (a) that H is a function of $\Phi(\omega) = \frac{1}{2} + X + iY$, one-valued throughout the whole plane (XY); (b) that H is finite for every finite value of $\Phi(\omega)$; (c) that, when $\Phi(\omega)$ is infinite, H is infinite, and comparable to a power of $\Phi(\omega)$ of a finite and integral exponent.

(a) If $\Phi(\omega) = \frac{1}{2} + X + iY$ describe any closed contour whatever in the plane XY, and if ω vary continuously with $\Phi(\omega)$, the value of ω, when $\Phi(\omega)$ returns to the point from which it set out, is $|\epsilon| \times \omega$, where $|\epsilon|$ is a primary unit matrix, depending on the form of the closed contour (Art. 50), and the values of Ω are at the same time $|A|^{-1} \times |\epsilon| \times \omega$. But the matrices $|A|^{-1} \times |\epsilon|$ are, in some definite order, equivalent by primary premultiplication to the

matrices $|A|^{-1}$. Hence the $\sigma'(n)$ quantities

$$\Phi(|A|^{-1} \times |\epsilon| \times \omega)$$

are in the aggregate the same as the quantities $\Phi(|A|^{-1} \times \omega)$; *i.e.* the symmetrical function H remains unchanged.

(*b*) If any one of the quantities $\Phi(\Omega)$ is infinite, $\Phi(\omega)$ is also infinite (Art. 51). Hence H is finite at every finite point of the plane.

(*c*) If $\Phi(\omega)$ is infinite, each of the quantities $\Phi(\Omega)$ is infinite, and comparable to a finite power of $\Psi(\omega)$. Hence H is also infinite, and comparable to a finite power of $\Phi(\omega)$. The exponent of this power must be integral, because the value of H does not change when $\Phi(\omega)$ describes a simple contour round the point ∞.

From this theorem it follows that the equation

$$F(\lambda^2, k^2) = \Pi(\lambda^2 - \Phi(\Omega)) = 0,$$

of which the roots are the squared modules resulting from the $\sigma'(n)$ primitive and primary transformations of order n, has for its coefficients rational and integral functions of $k^2 = \phi^8(\omega)$. This equation is termed the modular equation of order n: a discussion of its properties will occupy the following articles. The case of an even determinant is considered in Art. 64.

53. *Theorem* i. The modular equation is irreducible.

If possible, let

$$F(\lambda, k^2) = H_1(\lambda^2, k^2) \times H_2(\lambda^2, k^2), \quad \ldots\ldots\ldots \text{(i)}$$

each of the factors H_1, H_2 being rational and integral functions of λ^2 and k^2. When $k^2 = \Phi(\omega) = \frac{1}{2} + X + iY$ describes a closed contour in the plane XY, the roots of $H_1 = 0$, and of $H_2 = 0$, may change *inter se*, but each of these two sets of roots must, in the aggregate, remain unchanged, because the equations themselves remain unchanged. But by properly selecting the contour described by $\Phi(\omega)$, we can cause any root of $F = 0$ to change into any other. Let

$$\Omega_1 = |A_1|^{-1} \times \omega, \quad \Omega_2 = |A_2|^{-1} \times \omega,$$

so that $\Phi(\Omega_1)$ and $\Phi(\Omega_2)$ are two of the roots of $F = 0$. Let

$$|A_1|^{-1} = |\alpha| \times |A_2|^{-1} \times |\beta|,$$

the units $|\alpha|$ and $|\beta|$ being primary; if $\Phi(\omega)$ describe the contour (Art. 50) corresponding to $|\beta|$, the value of $\Phi(\Omega_2)$ after will be that of $\Phi(\Omega_1)$ before. Hence

the equation (i) is impossible, unless the roots of $H_1 = 0$ and $H_2 = 0$ are the same for all values of ω. And this supposition is inadmissible, because the equality of two values of $\Phi(\Omega)$ implies that ω is the root of a quadratic equation having integral coefficients.

Theorem ii. The equation $F(\lambda^2, k^2) = 0$ is symmetrical with respect to λ^2 and k^2.

If $k^2 = \Phi(|A_1|^{-1} \times \omega)$, one of the corresponding values of λ^2 is $\Phi(\omega)$. For one of the matrices $|A|^{-1}$ is equivalent by primary premultiplication to $\pm|A_1|$; if $|A_2|^{-1}$ be this matrix, we have

$$\Phi(|A_2|^{-1} \times |A_1|^{-1} \times \omega) = \Phi(|A_1| \times |A_1|^{-1} \times \omega)$$

$$= \Phi\left(\begin{vmatrix} n, & 0 \\ 0, & n \end{vmatrix} \times \omega\right) = \Phi(\omega),$$

i. e. $\Phi(\omega)$ is one of the values of λ^2. If therefore a, b is any pair of values satisfying the equation $F(\lambda^2, k^2) = 0$, b, a is a pair of values also satisfying it; *i. e.* the equation is symmetrical with regard to λ^2 and k^2.

Theorem iii. The equation $F = 0$ remains unchanged, if we write in it for k^2 and λ^2:

$$(1)\ 1 - k^2 \text{ and } 1 - \lambda^2, \quad (2)\ \frac{k^2}{k^2 - 1} \text{ and } \frac{\lambda^2}{\lambda^2 - 1}, \quad (3)\ \frac{1}{k^2} \text{ and } \frac{1}{\lambda^2},$$

$$(4)\ \frac{1}{1 - k^2} \text{ and } \frac{1}{1 - \lambda^2}, \quad (5)\ \frac{k^2 - 1}{k^2} \text{ and } \frac{\lambda^2 - 1}{\lambda^2}.$$

Let $|\epsilon|$ be any given unit matrix, if $k^2 = \Phi(|\epsilon| \times \omega)$, the values of λ^2 are comprised in the formula

$$\Phi(|\epsilon| \times \Omega) = \Phi(|\epsilon| \times |A| \times \omega).$$

For

$$|\epsilon| \times \Omega = |\epsilon| \times |A| \times |\epsilon|^{-1} \times |\epsilon| \times \omega,$$

and the matrices $|\epsilon| \times |A| \times |\epsilon|^{-1}$ are a complete system of primitive and primary matrices of determinant n (Art. 17). To prove the theorem we take in succession for $|\epsilon|$ the matrices $|\psi|$, $|\sigma|$, $|\tau|$, $|\rho|$, $|\rho|^2$, and employ the formulæ of Table *A*.

Theorem iv. If $k^2 = 0$, 1, ∞, the corresponding values of λ^2 are 0, 1, ∞ respectively. (See Art. 51.)

From these theorems it appears that the function $F(\lambda^2, k^2) = F(k^2, \lambda^2)$ is of the form

$$k^{2N} + \sum_{i=1}^{i=N-1} \sum_{j=1}^{j=N-1} a_{i,j} k^{2i} \lambda^{2j} + \lambda^{2N},$$

where $N=\sigma'(n)$ and the coefficients $\alpha_{i,j}$, if arranged in a square, are symmetrical with regard to both the diagonals of the square, so that

$$\alpha_{i,j}=\alpha_{j,i}=\alpha_{N-i,N-j};$$

the former of these equations resulting from the interchange of k^2 and λ^2, the latter from the simultaneous substitution of $\frac{1}{k^2}$ for k^2 and $\frac{1}{\lambda^2}$ for λ^2. Since $F(\lambda^2,1)=(\lambda^2-1)^N$, the sum of the coefficients in any row or column is equal to the corresponding coefficient in the expansion of $(\lambda^2-1)^N$. The identity

$$F(1-x,\ 1-y)=F(x,y),$$

which implies that $F(\frac{1}{2}+x,\ \frac{1}{2}+y)$ is an even function of x and y, gives another set of linear relations connecting the coefficients $\alpha_{i,j}$. And these are all the linear relations between the coefficients which are deducible from the theorems of this article.

54. *Theorem.* Let P and Q be two numbers relatively prime; and let

$$F(P,\lambda^2,k^2)=0,\quad F(Q,\lambda^2,k^2)=0,\quad F(PQ,\lambda^2,k^2)=0$$

be the modular equations of the orders P, Q, PQ respectively; the resultant of the elimination of z from the equations $F(P,\lambda^2,z)=0$ and $F(Q,z,k^2)=0$ is

$$F(PQ,\lambda^2,k^2)=0.$$

Let $|P|$ and $|Q|$ represent complete systems of primitive and primary matrices for the determinants P and Q respectively. If $k^2=\Phi(\omega)$, we have

$$F(Q,z,k^2)=\Pi_Q[z-\Phi(|Q|\times\omega)].$$

Again, if $z=\Phi(|Q|\times\omega)$, we have

$$F(P,\lambda^2,z)=\Pi_P[\lambda^2-\Phi(|P|\times|Q|\times\omega)].$$

The resultant of $F(Q,z,k^2)=0$ and $F(P,\lambda^2,z)=0$ is evidently

$$\Pi_Q\,\Pi_P[\lambda^2-\Phi(|P|\times|Q|\times\omega)]=0.$$

But $|P|\times|Q|$ is a complete system of primitive and primary matrices for the determinant PQ (Art. 16 *et seq.*); hence the resultant is $F(PQ,\lambda^2,k^2)=0$.

The theorem requires some modification when P and Q are not relatively prime. It suffices to consider the case in which Q is a prime p, and P is a power p^μ of p.

(*a*) Let $\mu > 1$. The system $|p^\mu| \times |p|$, resulting from the composition of the primitive and primary systems $|p^\mu|$ and $|p|$, consists of the primitive and primary matrices $|p^{\mu+1}|$, each taken once, and of matrices, each taken p times, derived from the primitive matrices $|p^{\mu-1}|$ by multiplying each constituent with p. Hence the resultant of $F(p^\mu, \lambda^2, z) = 0$ and $F(p, z, k^2) = 0$ is

$$F(p^{\mu+1}, \lambda^2, k^2) \times [F(p^{\mu-1}, \lambda^2, k^2)]^p = 0.$$

(*b*) Let $\mu = 1$. The matrices $|p| \times |p|$ consist of the primitive and primary matrices $|p^2|$, each taken once, and of $p+1$ matrices derived from primary unit matrices by multiplying each constituent with p. Hence the resultant of $F(p, \lambda^2, z) = 0$ and $F(p, z, k^2) = 0$ is

$$F(p^2, \lambda^2, k^2) \times (\lambda^2 - k^2)^{p+1} = 0.$$

It follows from the theorems of this article that if we have obtained the modular equations for primes, we can obtain the modular equations for composite numbers by elimination.

55. *Theorem* i. The coefficients $a_{i,j}$ are integral numbers.

It suffices to consider the case when n is a prime, since the theorem may be proved by elimination when n is composite.

As in Art. 5, let

$$f(q) = \phi(\omega) \div \sqrt[8]{2} q^{\frac{1}{8}}$$

so that $k^2 = 2^4 q f^8(q)$. The equation $F(n, \lambda^2, k^2) = 0$ is satisfied identically (*i. e.* whatever q may be, provided only that its analytical modulus is inferior to unity) by the values

$$k^2 = \Phi(\omega) = 2^4 q f^8(q), \quad \lambda^2 = \Phi(n\omega) = 2^4 q^n f^8(q^n).$$

We may therefore substitute these expressions for k^2 and λ^2 in the modular equation, and may determine the coefficients $a_{i,j}$ successively by equating to zero the coefficients of the powers of q; this is, in fact, the method employed by Sohncke in his memoir 'Aequationes modulares pro transformatione functionum ellipticarum' (Crelle, vol. vi. p. 97). Let the terms of F be arranged according to descending powers of λ^2 and k^2; *i. e.* let $a_{i,j} \lambda^{2i} k^{2j}$ precede $a_{i',j'} \lambda^{2i'} k^{2j'}$, if $i > i'$ or, if $i = i'$, $j > j'$. The lowest power of q arising from the term $a_{i,j} \lambda^{2i} k^{2j}$ is q^{in+j}, and this power is lower than the lowest power arising from any term which precedes $a_{i,j} \lambda^{2i} k^{2j}$. Writing for brevity $A_{i,j} = 2^{4(i+j-n-1)} . a_{i,j}$, then $A_{0,n+1} = 1$,

and denoting the coefficient of q^h in the expansion of $f^{8s}(q^n) \times f^{8s'}(q)$ by $C[h, s, s']$, so that (Art. 5)

$$C[h, s, s'] = \sum_{a=0}^{a=I\frac{h}{n}} [8s, a] \times [8s', h - n\alpha],$$

we find, on equating to zero the coefficient of q^{ni+j},

$$\Sigma C[(i-s)n + j - s', s, s'] \times A_{s,s'} = 0, \quad \ldots\ldots \text{(i)}$$

the sign of summation extending to all the coefficients $A_{s,s'}$ which do not precede $A_{i,j}$. This equation serves to calculate successively the coefficients $A_{i,j}$; and, since $C[0, i, j] = 1$, it shows that they are all integral numbers. Hence also, if $i + j \leqq n + 1$, the coefficients $a_{i,j}$, $= 2^{4(n+1-i-j)} A_{i,j}$ are integral; *i. e.* all these coefficients are integral, because the matrix $|a_{i,j}|$ is symmetric with regard to its diagonals.

We find from the formula (i)

$$a_{1,1} = -2^{4(n-1)}, \qquad a_{1,2} = 2^{4(n-2)} \times 8n,$$

$$a_{1,3} = -2^{4n-10} \times (8n^2 - 13n), \qquad a_{1,4} = 2^{4n-11} \times \tfrac{1}{3} \times n(n-2)(8n-23), \ldots;$$

and similarly, for any given index j, the value of $a_{1,j}$ may be determined. But a general expression of these coefficients cannot be obtained in this manner.

Theorem. When n is a prime, the coefficients $a_{i,j}$, except $a_{1,1}$, $= a_{n,n}$, $= -2^{4n-4}$, are divisible by n.

For, assuming that the theorem is true for the coefficients which follow $a_{i,j}$, and observing that $A_{1,1} = -1$, we have, from the equation (i)

$$A_{i,j} - C[(i-1)n + j - 1, 1, 1] + C[(i-1)n + j - 1, 0, n+1] \equiv 0, \text{ mod } n,$$

i. e. $A_{i,j} \equiv 0$, mod n; since by Fermat's theorem

$$f^8(q^n) \times f^8(q) \equiv f^{8n+8}(q), \text{ mod } n.$$

The coefficient $a_{i,j}$, where we may suppose $i + j \leqq n + 1$, is divisible by $2^{4(n+1-i-j)}$; but this is not always the highest power of 2 which divides $a_{i,j}$.

56. *The Sums of the Powers of the Roots.*

Let

$$\Sigma_s = \Phi^s(n\omega) + \Sigma \Phi^s(\Omega),$$

where n is still an uneven prime, and Ω is any one of the n quantities $\frac{\omega + 2h}{n}$.

The formula of Newton shows (1) that Σ_s is of the form

$$A_{s,1}\Phi(\omega)+A_{s,2}\Phi^2(\omega)+\ldots+A_{s,ns}\Phi^{ns}(\omega); \quad \ldots\ldots \text{(i)}$$

the coefficients $A_{s,h}$ being integral numbers; (2) that to obtain the coefficients $a_{i,j}$ of the modular equation, it is sufficient to know the n^2 coefficients comprised in the matrix

$$|A_{s,h}|, \begin{matrix} s=1, 2, \ldots n \\ h=1, 2, \ldots n, \end{matrix}$$

or, having regard to the relations which subsist among the coefficients $a_{i,j}$, the $\frac{1}{4}(n+1)^2$ coefficients which lie on or between two semidiameters of the matrix. Since

$$\Phi^s(n\omega)=\frac{1}{2^{4(n-1)}}\Phi^{ns}(\omega)$$

+terms containing higher powers of $\Phi(\omega)$, the series (i), as far as the term $A_{s,ns-1}\Phi^{ns-1}(\omega)$ inclusive, is the development of $\Sigma\Phi^s(\Omega)$ in a series proceeding by powers of $\Phi(\omega)$. Hence, if $h \leqq ns-1$, we have

$$A_{s,h}=\frac{1}{2\pi i}\int\frac{\Sigma\Phi^s(\Omega)\times\Phi'(\omega)d\omega}{\Phi^{h+1}(\omega)}, \quad \ldots\ldots \text{(ii)}$$

the track of the integration being such that $\Phi(\omega)$ describes a closed contour round 0; so that, for example, we may integrate in a straight line from $\omega=-1+i\tau$ to $\omega=1+i\tau$, τ being any constant greater than $\frac{1}{2}$, so as to bring the track of ω within the reduced space. But, employing the notation of Art. 5, and supposing $s \leqq n$,

$$\Sigma\Phi^s(\Omega)=2^{4s}\times n\times\sum_{x=1}^{x=\infty}[8s, nx-s]q^x$$

and

$$\frac{\Phi'(\omega)d\omega}{\Phi^{h+1}(\omega)}=-\frac{1}{h}d\left(\frac{1}{\Phi^h(\omega)}\right)=-\frac{1}{2^{4h}.h}d\left(\frac{1}{q^h f^{8h}(q)}\right)$$

$$=-\frac{1}{2^{4h}.h}\times\frac{dq}{q}\times\frac{1}{q^h}\times\Sigma(x-h)[-8h, x]q^x.$$

Hence, since $A_{s,h}$ is the absolute term in the coefficient of $\frac{dq}{q}$ within the sign of integration, we have finally

$$A_{s,h}=\frac{n}{h}\times 2^{4(s-h)}\times\sum_{a=1}^{a=h}a[8s, na-s]\times[-8h, h-a]. \quad \ldots \text{(iii)}$$

Since

$$\Sigma_{-s} = \frac{A_{s,1}}{\Phi(\omega)} + \frac{A_{s,2}}{\Phi^2(\omega)} + \ldots + \frac{A_{s,ns}}{\Phi^{ns}(\omega)},$$

we obtain by a similar process

$$A_{s,h} = \frac{n}{h} \times 2^{4(h-s)} \times \sum_{a=1}^{a=s} \alpha[8h, n\alpha - h] \times [-8s, s-\alpha], \quad \ldots \quad \text{(iv)}$$

so that $hA_{s,h}$ is symmetrical with regard to s and h.

The equations (iii) and (iv) serve to express the coefficients $A_{s,h}$, and consequently the coefficients $a_{i,j}$, in terms of the coefficients of the series $\Sigma \frac{\Delta(m)}{m} q^m$, and its powers up to $\left[\Sigma \frac{\Delta(m)}{m} q^m\right]^{\frac{1}{2}(n^2-1)}$ inclusively. (See Art. 5.)

57. *Development of the Roots of the Modular Equation.*

Employing the method of the last article we find

$$\left.\begin{aligned} \Phi(n\omega) &= \frac{n}{2^{4(n-1)}} \Phi^n(\omega) \times [a_0 + a_1\Phi(\omega) + \ldots + a_s\Phi^s(\omega) + \ldots], \\ a_s &= \frac{1}{2^{4s} \times (n+s)} \times \sum_{a=0}^{a=I\frac{s}{n}} (a+1)[8, a] \times [-8(n+s), s-n\alpha]; \end{aligned}\right\} \ldots \text{(i)}$$

$$\left.\begin{aligned} \Phi\left(\frac{\omega}{n}\right) &= \frac{2^{4\frac{n-1}{n}}}{n} \times \Phi^{\frac{1}{n}}(\omega) \times [\beta_0 + \beta_1 \Phi^{\frac{1}{n}}\omega) + \ldots + \beta_s \Phi^{\frac{s}{n}}(\omega) + \ldots], \\ \beta_s &= \frac{1}{2^{4\frac{s}{n}}(s+1)} \times \sum_{a=0}^{a=I\frac{s}{n}} (s+1-an)[8, s-an] \times \left[-8\frac{s+1}{n}, a\right]; \end{aligned}\right\} \ldots \text{(ii)}$$

the n values of the right-hand member of (ii) corresponding to the n quantities $\Phi\left(\frac{\omega+2h}{n}\right)$, $h = 0, 1, \ldots, n-1$. Writing $x = \Phi(\omega)$, we represent the right-hand member of (i) by Σ, and of (ii) by

$$\Sigma_0 + x^{\frac{1}{n}}\Sigma_1 + \ldots + x^{\frac{n-1}{n}}\Sigma_{n-1},$$

the infinite series $\Sigma_0, \Sigma_1, \ldots \Sigma_{n-1}$ being rational; if $\lambda^2 = y$, we have

$$F(x, y) = (\Sigma - y) \times \Pi[\Sigma_0 - y + x^{\frac{1}{n}}\Sigma_1 + \ldots + x^{\frac{n-1}{n}}\Sigma_{n-1}], \quad \ldots \quad \text{(iii)}$$

the sign of multiplication extending to the n values of the radical $x^{\frac{1}{n}}$ (see a paper on 'The Higher Singularities of Plane Curves,' Proceedings of the London Mathematical Society, vol. vi. p. 153*). The continued product II is equal to the determinant

$$\Delta(y, x) = \begin{vmatrix} \Sigma_0 - y, & x^{\frac{1}{n}}\Sigma_1, \ldots, & x^{\frac{n-1}{n}}\Sigma_{n-1} \\ x^{\frac{n-1}{n}}\Sigma_{n-1}, & \Sigma_0 - y, \ldots, & x^{\frac{n-2}{n}}\Sigma_{n-2} \\ \cdots & \cdots & \cdots \\ x^{\frac{1}{n}}\Sigma_1, & x^{\frac{2}{n}}\Sigma_2, \ldots, & \Sigma_0 - y \end{vmatrix}; \quad \ldots\ldots \text{(iv)}$$

thus, if $S_r(x)$ is the sum of the principal minors of order r in $\Delta(0, x)$, then the coefficient of y_i, $(i = 1, 2, \ldots n)$, in the modular equation, or

$$a_{i,1}x + a_{i,2}x^2 + \ldots + a_{i,n}x^n,$$

is equal to $(-1)^i S_{n-i+1}(x)$ taken as far as the term involving x^n. There is, however, one exception; the coefficient of x^n in $-S_1(x)$ exceeds $a_{n,n}$ by $-\frac{1}{2^{4(n-1)}}$.

Instead of the developments (i) and (ii) we may employ the series

$$\left.\begin{aligned} \frac{1}{\Phi(n\omega)} &= -2^{4(n-1)}n\Phi^{-n}(\omega) \times [\gamma_0 + \gamma_1\Phi(\omega) + \ldots + \gamma_s\Phi^s(\omega) + \ldots], \\ \gamma_s &= \frac{1}{2^{4s} \times (n-s)} \times \sum_{a=0}^{a=I\frac{s}{n}} (a-1) \times [-8, a] \times [8(n-s), s-na]. \end{aligned}\right\} \ldots \text{(v)}$$

$$\left.\begin{aligned} \frac{1}{\Phi\left(\frac{\omega}{n}\right)} &= 2^{-4\frac{n-1}{n}} \times \frac{1}{\Phi^{\frac{1}{n}}(\omega)} \times [\delta_0 + \delta_1\Phi^{\frac{1}{n}}(\omega) + \ldots + \delta_s\Phi^{\frac{s}{n}}(\omega) + \ldots], \\ \delta_s &= \frac{2^{4\frac{s}{n}}}{s-1} \times \sum_{a=0}^{a=I\frac{s}{n}} (s-1-an) \times [-8, s-an] \times \left[-8\frac{s-1}{n}, a\right]. \end{aligned}\right\} \ldots \text{(vi)}$$

58. *The Analytical Parallelogram.*

Returning to the case of any composite uneven number, let $1, \gamma_1, \gamma_2, \ldots n$ be the divisors of n in ascending order of magnitude, and let $n, \delta_1, \delta_2, \ldots 1$, be the conjugate divisors corresponding to them. If the terms of the equation be

* [Vol. ii. p. 101.]

arranged in an analytical parallelogram in the usual manner, the inferior limiting polygon (inside which the squares are empty) is obtained by joining in order the points

$$\begin{aligned}
&(0, N) \ldots\ldots\ldots 1\\
&(1, N-n) \ldots\ldots\ldots -2^{4(N-1)}\\
&(1+\gamma_1,\ N-n-\delta_1) \ldots\ldots 2^{4(N-1+\delta_1-\gamma_1)}\\
&(1+\gamma_1+\gamma_2,\ N-n-\delta_1-\delta_2) \ldots -2^{4(N-1+\delta_1+\delta_2-\gamma_1-\gamma_2)}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&(N-n, 1) \ldots\ldots\ldots -2^{4(N-1)}\\
&(N, 0) \ldots\ldots\ldots 1
\end{aligned}$$

The coefficients of the terms corresponding to the vertices of the polygon are given in a parallel column, and N is written for $\sigma'(n)$. The coefficients are obtained by causing k^2 to decrease without limit, and examining the initial terms of the expansions of the different values of λ^2 in series proceeding by powers (integral or fractional) of k^2.

The exterior limiting polygon, outside of which the squares are empty, is of course symmetrically situated on the other side of the diameter joining $(0, N)$ to $(N, 0)$.

Arts. 59–62. The Equation of the Multiplier.

59. *Theorem.* The reciprocals of the multipliers corresponding to the $\sigma'(n)$ primary and primitive transformations of an uneven determinant n are the roots of an irreducible equation of order $\sigma'(n)$: the first coefficient of the equation is unity, and the other coefficients are rational and integral functions of k^2 with integral coefficients.

Let $|A|$ represent a system of primary and primitive matrices of determinant n, and let the multiplier corresponding to the transformation $\omega = |A_i| \times \Omega$ (see Art. 26) be denoted by $M(\omega, |A_i|) = M_i$.

(i.) If $\Phi(\omega)$ describes any closed contour in the plane XY, and if ω at the close of this contour becomes $|\epsilon| \times \omega$, where $|\epsilon|$ is a primary unit matrix, the multiplier $M(\omega, |A_i|)$ becomes $M(|\epsilon| \times \omega, |A_i|)$, or, which is the same thing, $M(\omega, |\epsilon|^{-1} \times |A_i|)$.

Let
$$|\epsilon|^{-1} \times |A_i| = |A_j| \times |\eta_i|,$$
$|\eta_i|$ being a primary unit matrix; then
$$M(\omega, |\epsilon|^{-1} \times |A_i|) = M(\omega, |A_j| \times |\eta_i|) = M(\omega, |A_j|);$$

or M_i becomes M_j. Hence any symmetrical function of the multipliers remains unchanged, when $\Phi(\omega)$ describes any closed contour whatever in the plane XY; *i. e.* the $\sigma'(n)$ quantities $\frac{1}{M_i}$ are the roots of an equation

$$\mu\left(\frac{1}{M^2}, k^2\right) = 0, \quad \ldots \ldots \ldots \quad \text{(i)}$$

of which the coefficients are one-valued functions of k^2.

(ii.) To complete the demonstration of the theorem, it is convenient to examine the effect, upon the system of multipliers, of substituting for ω one of the expressions $|\psi| \times \omega$, $|\sigma| \times \omega$, $|\tau| \times \omega$, $|\rho| \times \omega$, $|\rho|^2 \times \omega$. The results are given in the following Table, in which $|\zeta|$ represents successively each of the five unit matrices, and the index j is that of the matrix $|A_j| = \begin{vmatrix} a_j, & b_j \\ c_j, & d_j \end{vmatrix}$ equivalent, by post-multiplication with a unit matrix of type (1), to the matrix $\pm |\zeta|^{-1} \times |A_i| \times |\zeta|$. The Table gives, for each of the five substitutions, the form of the equation of the multiplier, and the relation of its roots to the roots of the equation (i).

TABLE OF THE LINEAR TRANSFORMATIONS OF THE MULTIPLIER.

$\lvert\zeta\rvert =$	Equation.	Roots.
$\lvert\psi\rvert$	$\mu\left(\frac{1}{M}, 1-k^2\right) = 0$	$\frac{1}{M\left(-\frac{1}{\omega}, \lvert A_i\rvert\right)} = \frac{(-1)^{\frac{1}{2}(n-1)}}{M_j}$
$\lvert\sigma\rvert$	$\mu\left(\frac{1}{M}, \frac{k^2}{k^2-1}\right) = 0$	$\frac{1}{M(1+\omega, \lvert A_i\rvert)} = (-1)^{\frac{1}{2}b_j} \times \frac{\psi^4(\Omega_j)}{\psi^4(\omega)} \times \frac{1}{M_j}$
$\lvert\tau\rvert$	$\mu\left(\frac{1}{M}, \frac{1}{k^2}\right) = 0$	$\frac{1}{M\left(\frac{\omega}{1+\omega}, \lvert A_i\rvert\right)} = (-1)^{\frac{1}{2}c_j} \times \frac{\phi^4(\Omega_i)}{\phi^4(\omega)} \times \frac{1}{M_j}$
$\lvert\rho\rvert$	$\mu\left(\frac{1}{M}, \frac{1}{1-k^2}\right) = 0$	$\frac{1}{M\left(\frac{-1}{1+\omega}, \lvert A_i\rvert\right)} = (-1)^{\frac{1}{2}(n-1)+\frac{1}{2}b_j} \times \frac{\psi^4(\Omega_j)}{\psi^4(\omega)} \times \frac{1}{M_j}$
$\lvert\rho\rvert^2$	$\mu\left(\frac{1}{M}, \frac{k^2-1}{k^2}\right) = 0$	$\frac{1}{M\left(-\frac{1+\omega}{\omega}, \lvert A_i\rvert\right)} = (-1)^{\frac{1}{2}(n-1)+\frac{1}{2}c_j} \times \frac{\phi^4(\Omega_j)}{\phi^4(\omega)} \times \frac{1}{M_j}$

Of the five formulæ contained in the Table it will suffice to demonstrate one. Let

$$|\tau|^{-1}\times|A_i|\times|\tau| = (-1)^{\frac{1}{2}c_i}\times|A_j|\times|\eta|, \quad \ldots\ldots \text{(ii)}$$

where $|\eta| = \begin{vmatrix} \alpha, & \beta, \\ \gamma, & \delta, \end{vmatrix}$ is a primary unit matrix; so that

$$|A_i|^{-1}\times|\tau|\times\omega = |\tau|\times|\eta|^{-1}\times|A_j|^{-1}\times\omega = |\tau|\times|\eta|^{-1}\times\Omega_j;$$

then

$$(-1)^{\frac{1}{2}c_i}M_j = M(\omega, |\tau|^{-1}\times|A_i|\times|\tau|)$$

$$= \frac{1}{\phi^4(\omega)}\times M(|\tau|\times\omega, |A_i|)\times M(|A_i|^{-1}\times|\tau|\times\omega, |\tau|),$$

$$= \frac{1}{\phi^4(\omega)}\times M\left(\frac{\omega}{1+\omega}, |A_i|\right)\times M(|\tau|\times|\eta|^{-1}\times\Omega_j, |\tau|)$$

$$= \frac{1}{\phi^4(\pi)}\times M\left(\frac{\omega}{1+\omega}, |A_i|\right)\times\frac{1}{\phi^4(|\tau|\times|\eta|^{-1}\times\Omega_j)},$$

$$= \frac{1}{\phi^4(\omega)}\times M\left(\frac{\omega}{1+\omega}, |A_i|\right)\times(-1)^{\frac{1}{2}\gamma}\phi^4(\Omega_j).$$

But, from equation (ii),

$$c_i = (-1)^{\frac{1}{2}c_i}\times(c_j\alpha + d_j\gamma), \quad \text{or } c_i \equiv c_j+\gamma, \text{ mod } 4.$$

We have therefore, finally,

$$\frac{1}{M\left(\frac{\omega}{1+\omega}, |A_i|\right)} = (-1)^{\frac{1}{2}cj}\frac{\phi^4(\Omega_j)}{\phi^4(\omega)}\times\frac{1}{M_j};$$

and these $\sigma'(n)$ quantities are the roots of the equation

$$\mu\left(\frac{1}{M}, \phi^8\left(\frac{\omega}{1+\omega}\right)\right) = 0, \quad \text{or } \mu\left(\frac{1}{M}, \frac{1}{k^2}\right) = 0.$$

(iii.) We proceed to determine the values of the multiplier when ω is at one of the cornicular points, or when $k^2 = 0,\ 1,\ \infty$.

(1) If $k^2 = 0$, we have

$$\mu\left(\frac{1}{M}, 0\right) = \Pi\left(\frac{1}{M} - (\alpha-1)^{\frac{1}{2}(g-1)}g\right)^{\delta(g')},$$

g and g' being conjugate divisors of n. For, taking for $|A|$ the system

$$(-1)^{\frac{1}{2}(g-1)}\times\begin{vmatrix} g, & 0 \\ -2h, & g' \end{vmatrix},$$

we have

$$\frac{1}{M} = (-1)^{\frac{1}{2}(g-1)} g \, \frac{K\left(\frac{2h+g\omega}{g'}\right)}{K(\omega)}, \quad \ldots\ldots\ldots \quad \text{(iii)}$$

and, when ω approximates to the cornicular point,

$$\lim K(\omega) = \tfrac{1}{2}\pi, \quad \lim K\left(\frac{2h+g\omega}{g'}\right) = \tfrac{1}{2}\pi, \quad \lim \frac{1}{M} = (-1)^{\frac{1}{2}(g-1)} g;$$

and there are $\delta(g')$ roots of which this is the limiting value, because there are $\delta(g')$ values of h corresponding to any given value of g.

(2) If $k^2 = 1$, we have

$$\mu\left(\frac{1}{M}, 1\right) = \Pi\left(\frac{1}{M} - (-1)^{\frac{1}{2}(g'-1)} g\right)^{\delta(g')},$$

since

$$\mu\left(\frac{1}{M}, 1-k^2\right) = \mu\left(\frac{(-1)^{\frac{1}{2}(n-1)}}{M}, k^2\right).$$

(3) To find the limiting values of the roots when k^2 is infinite, we have

$$\lim_{k^2=0} \mu\left(\frac{1}{M}, \frac{1}{k^2}\right) = \lim_{\sigma=\infty} \Pi\left(\frac{1}{M} - (-1)^{\frac{1}{2}(g-1)} g \times e^{\frac{i\pi h}{g'}} \times e^{-\frac{1}{2}\pi\sigma\frac{g-g'}{g'}}\right)^{\delta(g')}.$$

In this equation, which is derived from (iii) and from the formula (τ) of the Table, we have supposed that h is even, and that $k^2 = \phi^8(\sigma i)$, σ being real, and increasing without limit. Thus, when k^2 increases without limit, the roots are partly zero and partly infinite, except when n is a square, in which case there are as many finite roots as there are numbers prime to $\sqrt{n}$ and not surpassing it. The product of the roots is finite, and equal to $\pm \Pi g^{\delta(g')}$, the sign being $(-1)^{\frac{1}{2}(n-1)}$ when n is a prime, and $+$ in every other case.

(iv.) Since the multiplier cannot become infinite except when ω approximates to one of the cornicular points, we infer from the preceding discussion that the coefficients of the powers of $\frac{1}{M}$ in the equation (i) never become infinite except when k^2 is infinite; and that, when k^2 is infinite, they either remain finite, or become infinite as a finite power of k^2. These coefficients are therefore rational and integral functions of k^2.

60. *Case when n is a Prime.—Determination of the Coefficients.*

When n is a prime, let $A_j(k^2) = A_j$ be the coefficient of $\frac{1}{M}$, in the equation of the multiplier, and let z_∞, z_h be the reciprocals of the multipliers appertaining to the transformations

$$(-1)^{\frac{1}{2}(n-1)} \times \begin{vmatrix} n, & 0 \\ 0, & 1 \end{vmatrix}, \quad \begin{vmatrix} 1, & 0 \\ -4h, & n \end{vmatrix} \text{ respectively.}$$

If in the equation $\mu\left(\frac{1}{M}, \frac{1}{k^2}\right) = 0$, we put $\omega = \sigma i$, and, allowing σ to increase without limit, regard $\Phi(\omega)$ as infinitesimal of order 1, the root $z_\infty \times \frac{\phi^4(\Omega_\infty)}{\phi^4(\omega)}$ is infinitesimal of the order $\frac{1}{2}(n-1)$, and each of the roots $z_h \times \frac{\phi^4(\Omega_h)}{\phi^4(\omega)}$ is infinite of the order $\frac{1}{2}\left(1-\frac{1}{n}\right)$. Thus the coefficient $A_j\left(\frac{1}{k^2}\right)$ cannot contain k^2 to any power higher than $(n+1-j) \times \frac{1}{2}\left(1-\frac{1}{n}\right)$; in particular, A_n and A_{n-1} are of the order zero, and A_1 is of the highest order possible, viz. $\frac{1}{2}(n-1)$.

Let $A_j(k^2) = \Sigma_s a_{j,s} k^{2s}$; if in the equation $\mu\left(\frac{1}{M}, \frac{1}{k^2}\right) = 0$ we put

$$\frac{1}{M} = z_\infty \times \frac{\phi(\Omega_\infty)}{\phi(\omega)}, \quad \omega = \sigma,$$

then the term

$$a_{j,s} \frac{1}{M^j k^{2s}}, \quad \ldots\ldots\ldots\ldots \text{(iv)}$$

is infinitesimal of the order $\frac{1}{2}(n-1)j - s$; and no two terms (except the absolute term $(-1)^{\frac{1}{2}(n-1)}n$ and the penultimate term $a_{1,\frac{1}{2}(n-1)}\frac{1}{Mk^{n-1}}$) are of the same order. If therefore we substitute for z and $\frac{1}{k^2}$ their developments in series proceeding by powers of q, we can successively determine the coefficients $a_{j,s}$ by equating to zero the coefficients of like powers of q. Or, again, we may put $\frac{1}{M} = z_h \times \frac{\phi^4(\Omega_h)}{\phi^4(\omega)}$: the term (iv) is now infinite of the order $\frac{1}{2}\left(1-\frac{1}{n}\right)j + s$; and no two terms (except $\frac{1}{M^{n+1}}$ and the penultimate term) are of the same order. Since the coefficient of $q^{-\frac{1}{2}\left(1-\frac{1}{n}\right)j-s}$ in (iv) is $\frac{a_{j,s}}{2^{4s}}$, we infer that $a_{j,s}$ is integral and divisible

by 2^{4n}. All the coefficients $a_{j,s}$ are divisible by n, except when $j=1$; because the coefficients of $\frac{1}{M^{n-1}}, \frac{1}{M^{n-2}}, \ldots, \frac{1}{M}$ in the expression

$$\frac{\mu\left(\frac{1}{M}, k^2\right)}{\frac{1}{M} - (-1)^{\frac{1}{2}(n-1)} n \frac{K(n\omega)}{K(\omega)}} = \Pi\left(\frac{1}{M} - \frac{K\left(\frac{\omega+2h}{n}\right)}{K(\omega)}\right),$$

when developed in series proceeding by powers of q, contain n as a factor in every term; similarly it may be shown that

$$a_{1,0} + a_{1,1}k^2 + \ldots + a_{1,\frac{1}{2}(n-1)}k^{n-1}$$
$$\equiv -\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \ldots + \left(\frac{1.3\ldots n-2}{2.4\ldots n-1}\right)^2 k^{n-1}\right\}, \text{ mod } n^2,$$

the series within the brackets on the right-hand side consisting of the first $\frac{1}{2}(n-1)$ terms of the development of $\frac{2K}{\pi}$.

61. *Theorem of Jacobi.*

Let n be a prime number as before, and let b denote one of the $\frac{1}{2}(n-1)$ numbers 1, 2, 3, . . . , $\frac{1}{2}(n-1)$. Since

$$\sqrt{\frac{2}{\pi}K\left(\frac{\omega}{n}\right)} = \sum_{m=-\infty}^{m=+\infty} q^{\frac{m^2}{n}} = \Sigma q^{nm^2} + \sum_{b=1}^{b=\frac{1}{2}(n-1)} 2 \sum_{m=-\infty}^{m=+\infty} q^{\frac{1}{n}(mn+b)^2},$$

and, writing for shortness

$$\frac{\Sigma q^{nm^2}}{\Sigma q^{m^2}} = A, \qquad \frac{\Sigma q^{\frac{1}{n}(mn+b)^2}}{\Sigma q^{m^2}} = A_b,$$

we have

$$\sqrt{M}_\infty = \sqrt{n} \times A, \qquad \sqrt{M}_h = A + \sum_{b=1}^{b=\frac{1}{2}(n-1)} 2\epsilon^{2i\pi\frac{hb^2}{n}} A_b;$$

so that the square roots of the $n+1$ multipliers can be expressed linearly in terms of the $\frac{1}{2}(n+1)$ quantities A.

62. *Expression of the Multiplier as a Rational Function of k^2 and λ^2.*

We have (Art. 11)

$$\frac{d\Phi(\omega)}{d\omega} = \frac{4i}{\pi}\Phi(\omega)\Psi(\omega)K^2(\omega),$$

$$\frac{d\Phi(\Omega)}{d\omega} = \frac{d\Phi(\Omega)}{d\Omega} \times \frac{d\Omega}{d\omega} = \frac{4i}{\pi}\Phi(\Omega)\Psi(\Omega)K^2(\Omega) \times \frac{(a+b\Omega)^2}{n}.$$

Hence, by division,

$$\frac{1}{M^2} = (a+b\Omega)^2 \times \frac{K^2(\Omega)}{K^2(\omega)}$$

$$= n\frac{\Phi(\omega)\,\Psi(\omega)\,d\Phi(\Omega)}{\Phi(\Omega)\,\Psi(\Omega)\,d\Phi(\omega)}$$

$$= n\frac{k(1-k^2)d\lambda}{\lambda(1-\lambda^2)dk} = -n\frac{k^2(1-k^2)}{\lambda^2(1-\lambda^2)}\frac{\left(\frac{dF}{d.k^2}\right)}{\left(\frac{dF}{d.\lambda^2}\right)}. \quad \ldots\ldots \quad \text{(i)}$$

This theorem holds for every transformation of any order whatever, even or uneven ; $F=0$ representing the modular equation between k^2 and λ^2.

But $\frac{1}{M}$ can itself be expressed as a rational function of k^2 and λ^2. Attending only to the case when n is an uneven number, we write the equation of the multiplier in the form

$$\mu_1\left(\frac{1}{M^2}\right) + \frac{1}{M}\mu_2\left(\frac{1}{M^2}\right) = 0\,;$$

and we substitute for $\frac{1}{M^2}$ in μ_1 and μ_2 the value given by equation (i). We thus obtain for $\frac{1}{M}$ an expression of the form

$$\frac{1}{M} = -\mu_1\left(\frac{1}{M^2}\right) \div \mu_2\left(\frac{1}{M^2}\right),$$

the numerator and denominator being rational functions of k^2 and λ^2, which may simultaneously vanish for particular pairs of values of k^2 and λ^2, but which do not vanish by virtue of the modular equation ; since, if they did so vanish, $-\frac{1}{M}$ would be a root of the equation of the multiplier as well as $\frac{1}{M}$, which is not in general the case. The expression of $\frac{1}{M}$ thus obtained may be put into an infinite number of different forms, because λ^2 and k^2 are not independent, but are connected by the modular equation. In particular the equation

$$\frac{1}{M} = \frac{f_n(\lambda^2, k^2)}{\Delta(k^2)}$$

(when f_n is a rational and integral function of λ^2 and k^2 of the order n in λ^2

and having integral coefficients, and $\Delta(k^2)$, which contains only k^2, is the discriminant of the modular equation) may be deduced by the following method, which is due to Professor Cayley. It may be shown as before that the $\sigma'(n)$ functions B_s defined by the equations

$$\Sigma_j \frac{\lambda_j^{2s}}{M_j} = B_s, \qquad \begin{matrix} s = 0, & 1, \ldots, \sigma'(n)-1, \\ j = 0, & 1, \ldots, \sigma'(n)-1, \end{matrix}$$

are rational and integral functions of k^2 with integral coefficients. Solving these equations for the $\sigma'(n)$ multipliers which enter linearly into them, and denoting by f_1 and f_n rational and integral functions of λ^2 and k^2 in which we may suppose that the highest exponent of λ^2 is $\sigma'(n)-1$, we find

$$\frac{1}{M_j} = \frac{f_1(\lambda^2, k^2)}{F'(\lambda_j^2, k^2)} = \frac{f_n(\lambda_j^2, k^2)}{\Delta(k^2)}.$$

The theory of the equation of the multiplier in the case of an even determinant presents no special difficulty. The results are of the same general character as in the case of an uneven determinant; but the discussion involves many points of detail, and cannot be attempted within our present limits.

Arts. 63–73. The Modular Curves.

63. *The Modular Curves of an Uneven Order.*

We have seen (Art. 51) that, if we represent by $\epsilon(x)$ any one of the six anharmonic functions

$$x, \quad 1-x, \quad \frac{1}{x}, \quad \frac{1}{1-x}, \quad \frac{x}{x-1}, \quad \frac{x-1}{x}, \quad \ldots\ldots \quad \text{(i)}$$

the modular equation is unchanged by the simultaneous substitution of $\epsilon(k^2)$ for k^2, and $\epsilon(\lambda^2)$ for λ^2. Hence if $\epsilon_1(x)$, $\epsilon_2(x)$ denote any two, the same or different, of the six functions (i), the thirty-six substitutions

$$F[\epsilon_1(k^2), \quad \epsilon_2(\lambda^2)]$$

give only six different equations, viz.:

$$\left.\begin{array}{ll} (1)\quad F(k^2, 1-\lambda^2) = 0, & (4)\quad F(k^2, \lambda^2) = 0, \\ (2)\quad F\left(k^2, \frac{1}{\lambda^2}\right) = 0, & (5)\quad F\left(k^2, \frac{\lambda^2-1}{\lambda^2}\right) = 0, \\ (3)\quad F\left(k^2, \frac{\lambda^2}{\lambda^2-1}\right) = 0, & (6)\quad F\left(k^2, \frac{1}{1-\lambda^2}\right) = 0; \end{array}\right\} \quad \ldots\ldots \quad \text{(ii)}$$

for these we shall employ the notation

$$F_s(k^2, \lambda^2) = 0, \qquad s = 1, 2, 3, 4, 5, 6,$$

the left-hand members being supposed integral; so that *e. g.*

$$F_2(k^2, \lambda^2) = \lambda^{2N} \times F\left(k^2, \frac{1}{\lambda^2}\right), \qquad F_4(k^2, \lambda^2) = F(k^2, \lambda^2).$$

If $k^2 = \Phi(\omega)$, $\omega = |A| \times \Omega$, $|A|$ representing a system of primary matrices of determinant Δ, and of the type ψ, τ, σ, 1, ρ, ρ^2 in the six cases respectively, the roots of the six equations are comprised in one and the same formula

$$\lambda^2 = \Phi(\Omega).$$

The first three equations (as well as the fourth) are symmetric with regard to k^2 and λ^2; the fifth and sixth are changed each into the other by the interchange of k^2 and λ^2. If then we denote by X and Y rectangular Cartesian coordinates, the equations

$$F_s(\tfrac{1}{2} + X + iY, \ \tfrac{1}{2} + X - iY) = 0,$$

$$s = 1, 2, 3, 4$$

are real, and, as we shall presently see, represent real curves, which we shall term the first, second, third, and fourth modular curves; we observe, however, that when $\Delta \equiv 3 \bmod 4$, the fourth modular curve reduces itself to the two conjugate points $[\pm \frac{1}{2}, 0]$; the fifth and sixth equations represent a pair of conjugate imaginary curves.

The first and fourth curves are each of them symmetric with regard to both axes; the fourth curve is its own inverse with regard to each of the two real circles

$$(X \pm \tfrac{1}{2})^2 + X^2 = 1,$$

and the first curve with regard to each of the two imaginary circles

$$X^2 + (Y \pm \tfrac{1}{2}i)^2 = -1.$$

The second and third curves are symmetric with regard to the axis of X, and symmetric to one another with regard to the axis of Y; the fifth and sixth (imaginary) curves are symmetric with regard to the axis of Y, and symmetric to one another with regard to the axis of X. The second and third curves are

respectively the inverses of the first curve with regard to one of the circles

$$(X-\tfrac{1}{2})^2+Y^2=1, \quad (X+\tfrac{1}{2})^2+Y^2=1;$$

each of these curves is also its own inverse with regard to the other of these circles. Similarly the two imaginary curves are the inverses of the fourth curve with regard to the two imaginary circles

$$X^2+(Y\pm\tfrac{1}{2}i)^2=-1.$$

Lastly, the substitution $X=iY'$, $Y=-iX'$, changes the first curve into the fourth, the second into the fifth, the third into the sixth, and *vice versâ*. These assertions are the geometrical equivalents of the Theorems ii. and iii. of Art. 53.

Let g and g' be any two conjugate divisors of Δ, and let η be the greatest common divisor of g and g'. Resolve g into the product of two factors γ_1 and γ_2, of which γ_1 contains only those prime divisors of g which do not occur in η, and γ_2 contains only those prime divisors of g which do occur in η. Representing by $f(z)$ the number of numbers prime to any given number and not surpassing it, we write

$$f'(g)=\gamma_1 f(\gamma_2),$$

and we define the numbers ν, A, B, by the equations

$$2\nu=\Sigma f(\eta), \quad \text{(iii)}$$

$$A=\Sigma f'(g), \ g\geqq\sqrt{\Delta}, \quad \text{(iv)}$$

$$B=\Sigma f'(g), \ g\leqq\sqrt{\Delta}. \quad \text{(v)}$$

In (iii) the summation extends to every divisor g of Δ: and in general each term $f(\eta)$ occurs twice in $\Sigma f(\eta)$, because η is the same for each of any two conjugate divisors; but if $\Delta=\theta^2$ is a perfect square, the term $f'(\theta)=f(\theta)$ occurs only once in $\Sigma f(\eta)$; in the same case we divide the term $f'(\theta)=f(\theta)$ equally between A and B, so that in every case

$$A+B=\Sigma f'(g)=\sigma'(\Delta).$$

The first and fourth modular curves are of the order $2A$; when Δ is not a square they are each of them completely and parabolically cyclic, having at each cyclic point ν branches of the aggregate order $A-B$ and class B, all touching the line at an infinite distance. When $\Delta=\theta^2$ is a perfect square, there are at each cyclic point only $\nu-\frac{1}{2}\theta'$ branches, of the aggregate order $A+B$ and class $B-\frac{1}{2}\theta'$:

in this case each of the curves has θ' real points at an infinite distance, the asymptotes being

$$\left.\begin{aligned} Y\cos\tfrac{1}{2}u - X\sin\tfrac{1}{2}u = 0 \\ Y\sin\tfrac{1}{2}u + X\cos\tfrac{1}{2}u = 0 \end{aligned}\right\}, \quad u = \frac{2s+1}{\theta}\pi, \quad \ldots\ldots \quad \text{(vi)}$$

for the two curves respectively. The points $(\pm\frac{1}{2}, 0)$ always belong to the fourth modular curve, but only as isolated points when N is not a square; when N is a square this curve has θ' real branches at each of these points, the tangents being

$$Y\sin\tfrac{1}{2}u + (X \pm \tfrac{1}{2})\cos\tfrac{1}{2}u = 0.$$

The points $(\pm\frac{1}{2}, 0)$ and $(0, \pm\frac{1}{2})$ are foci, and indeed the only foci, of both curves; of these the points $(0, \pm i)$ lie on the first curve, and the two real points $(\pm\frac{1}{2}, 0)$ are its two foci (properly so called), the axis of Y being the only corresponding cyclic axis, or directrix. The fourth modular curve has, in the strictest sense of the term, no foci at all, as the cyclic lines passing through the points $[\pm\frac{1}{2}\ 0]$ touch the curve at those points only.

For a demonstration and fuller discussion of the preceding results, the reader is referred to a paper 'On the Singularities of the Modular Equations and Curves,' in vol. ix. of the Proceedings of the London Mathematical Society*.

64. *The Modular Curves of an Even Order.*

Let $|C|$ represent a system of primitive matrices of an even determinant $2^\mu \times \Delta$, non-equivalent by primary premultiplication, but all of one and the same of the nine types $C_{i,j}$ of Art. 23; let also $\Omega = |C| \times \omega$. If $|a|$ is a given primary unit matrix the matrices $|C| \times |a|$ are equivalent by primary premultiplication to the matrices $|C|$ taken in a certain order. It may therefore be proved, as in Art. 52, that any symmetrical function of the $2^\mu \times \sigma'(\Delta)$ quantities

$$\Phi(\Omega) = \Phi(|C| \times \omega)$$

is a rational function of $\Phi(\omega)$. There are thus nine modular equations of any even order; which we may represent by the formula

$$F_{i,j}(2^\mu\Delta, k^2, \lambda^2) = 0.$$

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* [Vol. ii. p. 242.]

Each of these equations is changed into each of the other eight by four of the thirty-six anharmonic substitutions, and is transformed into itself by the remaining four; viz. if α and β are any two of the six typical units, it may be inferred from the Tables in Art. 25 that $\alpha^{-1} \times |\, C_{i,j} \,| \times \beta$ is of the type $C_{i',j'}$ for four of the combinations α, β, whence it follows that $F_{i,j}$ is changed into $F_{i',j'}$ when we write simultaneously $\Phi(|\, \alpha \,| \times \Omega)$ for $\Phi(\Omega)$, and $\Phi(|\, \beta \,| \times \omega)$ for $\Phi(\omega)$. In all the equations alike the numerical coefficients are rational, as may be shown by an application of Sohncke's method; for brevity we omit the discussion of the relations subsisting among the coefficients, and of the methods of determining them.

Of the nine modular equations three, and only three, are symmetrical with regard to k^2 and λ^2, because the matrices $C_{1,1}$, $C_{2,2}$, $C_{3,3}$, and these only, have reciprocals which are of the same type as the matrices themselves. Thus there are three modular curves

$$F_{i,i}(\tfrac{1}{2}+X+iY,\ \tfrac{1}{2}+X-iY)=0,$$
$$i=1, 2, 3,$$

which we shall designate as the first, second, and third respectively, and which represent (exactly as when the determinant is uneven) the three sets of properly primitive forms of the determinant $2^{\mu}\Delta$. The real relations subsisting between these curves are the same as in the case of an uneven determinant, viz. the first curve is symmetric with regard to the axis of Y, the first and second curves are inverse to one another, and the third inverse to itself with regard to the circle

$$(X-\tfrac{1}{2})^2+Y^2=0\,;$$

the first and third curves, and the second curve, have the same relation with regard to the circle

$$(X+\tfrac{1}{2})^2+Y^2=1.$$

The imaginary properties are different; for example, the first curve is transformed into itself by the substitutions $X=\pm iY'$, $Y=\mp iX'$; *i.e.* it is symmetric with regard to each of the imaginary cyclic lines intersecting at its centre.

65. *Representation of the Rational Semicircles by the Modular Curves.*

The following theorem establishes a remarkable connexion between the modular equations of order Δ and the quadratic forms of the positive determinant Δ.

Theorem. If the reduced space Σ be mapped on the plane XY by the equation

$$\Phi(\omega) = \tfrac{1}{2} + X + iY,$$

the reduced arcs of the properly primitive semicircles of determinant Δ are represented by the first, second, or third modular curve of order Δ, according as these reduced arcs appertain to the subclass (A), (B), or (C); and when $\Delta \equiv 1$, the reduced arcs of the improperly primitive semicircles are represented by the fourth modular curve.

Let $[a, b, c]$ be any rational semicircle of determinant Δ (even or uneven); and let the equation of $[a, b, c]$ be written in the form

$$-x + iy = \frac{a + b(x+iy)}{b + c(x+iy)}.$$

If

$$k^2 = \Phi(x+iy), \qquad \lambda^2 = \Phi(-x+iy),$$

we may write simultaneously

$$k^2 = \tfrac{1}{2} + X + iY, \quad \lambda^2 = \tfrac{1}{2} + X - iY,$$

because k^2, λ^2 are conjugate complex quantities, $\Phi(\omega)$ being a real function of $i\omega$.

(i) Let $[a, b, c]$ be properly primitive; when Δ is uneven the matrix $\left|\begin{matrix} b, & c \\ a, & b \end{matrix}\right|$ is of the type ψ, σ, or τ; and when Δ is even that matrix is of the type $C_{1,1}$, $C_{2,2}$, or $C_{3,3}$, according as $[a, b, c]$ appertains to the set (A), (B), or (C) of Art. 41. Hence k^2 and λ^2 satisfy the equation $F_i(k^2, \lambda^2) = 0$; or, which is the same thing, the point X, Y lies on the curve

$$F^i\left(\tfrac{1}{2} + X + iY,\ \tfrac{1}{2} + X - iY\right) = 0,$$

where i is 1, 2, or 3, according as $[a, b, c]$ appertains to the set (A), (B), or (C).

(ii) Let $[a, b, c]$ be improperly primitive; Δ is uneven, the matrix $\left|\begin{matrix} b, & c \\ a, & b \end{matrix}\right|$ is of the type 1, and the point X, Y lies on the fourth modular curve.

Thus every point which in the plane (xy) lies on a rational semicircle of determinant Δ is represented in the plane (XY) by a point lying on one of the modular curves of order Δ. Conversely, the points ω, which in the plane xy answer to any given point of one of the modular curves, lie on rational semicircles of determinant Δ. For if X and Y are real quantities satisfying the last-mentioned equation, $\frac{1}{2} + X - iY$ is one of the modules derived from

$\frac{1}{2}+X+iY$ by a transformation of order Δ. Hence, if as before

$$\tfrac{1}{2}+X+iY=\Phi(x+iy), \qquad \tfrac{1}{2}+X-iY=\Phi(-x+iy),$$

$-x+iy$ and $x+iy$ are connected by an equation of the form

$$-x+iy=\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \times (x+iy),$$

the quantities x and y being real, and $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$ being a matrix of determinant Δ.

Separating in this equation the real and imaginary parts, we find

$$\alpha=\delta, \qquad \gamma+2\alpha x+\beta(x^2+y^2)=0,$$

or $x+iy$ lies on the semicircle $[\gamma, \alpha, \beta]$ of determinant $\alpha^2-\beta\gamma=\Delta$. When two modules, either of which can be transformed into the other, are conjugate complex quantities, the transformation is said to be symmetrical. It is evident that all modules represented by points appertaining to modular curves of order Δ, and these modules only, admit of symmetrical transformation by a matrix of determinant Δ.

66. *Case when the Determinant is not a Square.*

Let A be not a square; and let a point ω describe in the reduced space Σ the period of reduced arcs appertaining to any given subclass of semicircles of determinant Δ; the corresponding point X, Y will describe a continuous closed contour in the plane (XY) forming one of the *ovals* of which the corresponding modular curve is composed *. Since no semicircle of determinant Δ passes through any one of the points p, q, r, no oval of the modular curve can pass through either of the points A_{-1}, A_{+1}, or can have a point at an infinite distance. The portion of an oval terminated at each extremity by one of the boundaries $(-\infty, A_{-1})$, $(A_{+1}, +\infty)$ we term a *filament*. The filaments are of six types, four transitive and two intransitive, corresponding to the six essentially different types of reduced arcs (Art. 43).

(1) An intransitive filament $[S_{-1}, S_{+1}]$ lies wholly on the left-hand side of the axis of Y; it begins at a point on the boundary $[-\infty, A_{-1}]$, and enters

* By an *oval* of a modular curve we understand a distinct and separate portion of the curve forming a continuous and closed series of real points. The term oval is suggested by the form of the portions of the curve in the simplest instances; but an oval may intersect itself any number of times, and (when the determinant is a square) may have points at an infinite distance.

the lower region of the plane XY (*i. e.* the region in which Y is negative); it afterwards bends upwards and traverses the segment $A_{-1}0$, curving round so as to terminate in a point (in general different from that at which it began) upon the boundary $[-\infty, A_{-1}]$.

The intransitive filaments $[T_{-1}, T_{+1}]$ are symmetrical with regard to the axis of Y to filaments of the type $[S_{-1}, S_{+1}]$.

(2) A transitive filament $[S_{-1}, T_{-1}]$ begins at a point on the boundary $[-\infty, A_{-1}]$ and enters the lower region (which it never leaves) of the plane XY; it traverses the axis of Y, and terminates in a point on the boundary $[A_{+1}, +\infty]$.

A transitive filament $[S_{-1}, T_{+1}]$ begins at a point on the boundary $[-\infty, A_{-1}]$ and enters the lower region of the plane; it afterwards traverses the segment A_{-1}, A_{+1}, and the axis of Y, and terminates in the upper region of the plane at a point on the boundary $[A_{+1}, +\infty]$. Such a filament always has a point of inflexion; it may traverse first the segment A_{-1}, A_{+1} and afterwards the axis of Y, or *vice versâ*; or it may cut them both simultaneously, passing through the point 0.

The transitive filaments $[S_{+1}, T_{+1}]$, $[S_{+1}, T_{-1}]$ are symmetrical with regard to the axis of X to the transitive filaments $[S_{-1}, T_{-1},]$, $[S_{-1}, T_{+1}]$ respectively.

As in the case of reduced arcs, if we change the order of the letters in any type symbol we do not change the type of the filament, but only the direction in which it is drawn.

Every oval contains at least two transitive filaments, and its general form is as follows:—Beginning, as we may suppose, with a transitive filament of the type TS, it winds a certain number of times round the focus A_{-1}; it then crosses the axis of Y by a second transitive filament, and winds a certain number of times round the focus A_{+1}: it thus keeps crossing from one focus to the other, and winding round each of them alternately until it finally closes.

Combining this discussion with preceding results, we see that given any oval, we can, from a consideration of the filaments of which it is composed, construct the complete system of reduced quadratic forms appertaining to the subclass represented by the oval. Beginning with a transitive filament (TS), let $\mu_1 - 1$ be the number of non-transitive filaments immediately following it; let the next transitive filament be followed by μ_2 non-transitive filaments, and so on continually. Attribute to $\mu_1, \mu_3, \ldots$ a positive or negative sign according as the revolution round A_{-1} is positive or negative; and to $\mu_2, \mu_4, \ldots$ a positive or negative sign according as the revolution round A_{+1} is negative or positive. The fundamental automorphic of the reduced form, represented by the first

transitive filament, is

$$|\sigma|^{2\mu_1} \times |\tau|^{2\mu_2} \times |\sigma|^{2\mu_3} \times |\tau|^{2\mu_4} \times \ldots ;$$

and the determination of this automorphic is equivalent to a determination of the period of reduced forms represented by the oval.

We may therefore say that the modular curves of order Δ present us with a species of chart of the reduced forms of determinant Δ, each oval representing a subclass, each filament a reduced form, and the periodic continued fraction corresponding to each subclass exhibited by the aspect of the oval.

The number of ovals contained in each of the first three modular curves is evidently (Art. 41) $\frac{1}{3}H$, and H' in the fourth.

67. *Case where the Determinant is a Square.*

Let $\Delta = \theta^2$ be a perfect square ; and let δ represent the number of solutions of the congruence

$$\alpha^2 + 1 \equiv 0, \mod 2\theta,$$

so that $\delta = 0$, if θ is divisible by 2 or by any prime of the form $4m+3$. As in the former case each modular curve consists of a certain number of ovals; but, in certain cases, an oval corresponds to more than one chain of reduced semicircles ; viz. the oval may be divisible into two, three, or six sections, each section answering to a single chain of reduced arcs. Each oval is subject to the condition that it must pass either through the point A_{-1}, or through the point A_{-2}, or else have a point P_∞ at an infinite distance. If the oval satisfies this condition only once, it is a simple oval, and represents but one chain of reduced semicircles ; if it satisfies the condition more than once, it is a multiple oval, and corresponds to as many different chains of reduced arcs as it has branches passing through A_{-1}, A_{+1}, or points at an infinite distance ; each section of the oval beginning and ending at one of the points A_{-1}, A_{+1}, A_∞. Each of the first three modular curves has δ simple ovals and $\frac{1}{6}(H - 3\delta)$ double ovals ; the ovals of the first curve having one or two infinite branches, and those of the second and third curves having one or two branches passing the points A_{-1} and A_{-2} respectively. Lastly, the fourth modular curve (which exists only when θ is uneven) contains δ triple ovals and $\frac{1}{6}(H' - 3\delta)$ sextuple ovals ; each oval having, according as it is triple or sextuple, one or two branches of each of the types A_{-1}, A_{+1}, A_∞. The two, three, or six subclasses represented by different sections of the same multiple oval always appertain to the same

Gaussian class; and it is to be observed that if the extremities of a section lie at the same point A_{-1} or A_{+1}, they have different tangents, and if they both lie at an infinite distance they belong to branches having different asymptotes.

It will suffice to establish the preceding results in the case of the fourth modular curve. Consider the pairs of extreme semicircles

$$[2a, b, 0] \quad \text{and} \quad [2a, b, 0]\times|\sigma|^{-1} = [2a-2b, b, 0],$$

$$[0, b, 2a] \quad \text{and} \quad [0, b, 2a]\times|\tau|^{-1} = [0, b, 2a-2b],$$

$$[2a-2b, b-2a, 2a] \text{ and } [2a-2b, b-2a, 2a]\times|\psi| = [2a, 2a-b, 2a-2b],$$

which are all improperly primitive, if b is uneven, and reduced, if a is less in absolute magnitude than b. It appears from a preceding investigation, that if ω describes either semicircle of one of these pairs up to its extreme point, and then passing through the extreme point describes the other semicircle of the same pair, the corresponding point (XY), determined by the equation $\frac{1}{2}+X+iY=\Phi(\omega)$, will describe a line of continuous curvature passing through A_{-1}, A_{+1}, or a point at an infinite distance, as the case may be. Thus a semicircle P_1 of the type (r), ending a chain P of the type (qr), is continued by a semicircle Q_0 of the same type (r), which begins a chain Q of the type (rp); the chain Q ends in a semicircle Q_1 of type (p), which in its turn is continued by a semicircle R_0 of the same type (p) beginning a chain R of the type (pq). Lastly, the semicircle R_1, of type (q), which terminates the chain R, is continued by a semicircle P_0' of the same type, beginning a chain P' of the original type (qr). The chain P' is equivalent to the chain P by a transformation of the type $\sigma\times\psi\times\tau=\psi$; but the two chains may also be primarily equivalent to one another. If they are primarily equivalent the oval closes, and is triple; if they are not primarily equivalent we begin with the chain P', and perform the preceding process over again; we thus arrive at a chain P'', which is certainly equivalent to P, because $\psi^2=-1$: in this case the oval is sextuple. To determine whether P, P' are primarily equivalent or not, we have only to examine whether P_0 is or is not equivalent to itself by a transformation of type ψ. Now P_0 is of the type $[4\xi, \theta, 0]$; and this is transformed into

$$-[4\xi, \theta, 0] \quad \text{by} \quad \left|\begin{matrix} -2\xi, & \theta \\ \dfrac{1+4\xi^2}{\theta}, & 2\xi \end{matrix}\right|;$$

i. e. there are as many triple ovals as there are solutions of the congruence

$4\xi^2+1\equiv 0$, mod θ, or, which is the same thing, of the congruence

$$\xi^2+1\equiv 0, \text{ mod } 2\theta.$$

It will be observed that two sections of an oval which meet at either of the points A_{-1} or A_{+1} are analytically inverse with regard to the other of those two points; and that two sections which have the same asymptote have the origin for a common centre. In fact, when Δ is an uneven square, each oval of the modular curve taken separately has the properties of symmetry and anallagmatism which in the general case characterize the curve as a whole.

68. *Geometrical Construction of the Transformed Modulus by means of the Modular Curve.*

Let P be a given point in the plane of a geometric curve C of order n; each of the imaginary cyclic lines passing through P meets the curve in $n-\epsilon$ imaginary points at a finite distance, if the curve has branches of the aggregate order ϵ passing through each cyclic point. The $n-\epsilon$ real lines on which these $n-\epsilon$ pairs of conjugate imaginary points are situated, are the cyclic axes of P with regard to the curve; and the $n-\epsilon$ points inverse to P with regard to these lines are termed the points inverse to P with regard to the curve; or, which is the same thing, the points inverse to P with regard to the curve are the points which, taken one by one with P, are the antipoints of pairs of imaginary points lying on the curve. If P is a focus of the curve, or, again, if P is one of the antipoints of a pair of conjugate imaginary multiple points of the curve, some of the cyclic axes of P, and some of the points inverse to P, may become coincident.

Theorem. Let $f(\xi, \eta)=0$ be an algebraic equation between ξ and η of the order μ in η: and let $\eta=x_s+iy_s$, $s=1, 2, 3, \ldots, \mu$, be the μ values of η corresponding to the value $x+iy$ of ξ; if the equation $f(x+iy, x-iy)=C=0$ is real, the points x_s-iy_s are the inverses of $x+iy$ with regard to the curve C.

The condition that $C=0$ is real implies that $f(\xi, \eta)$ is of the same order μ in ξ that it is in η; and that $\mu=n-\epsilon$. Again, the point T, of which the coordinates are

$$X=\tfrac{1}{2}(x+x_s)+\tfrac{1}{2}i(y+y_s),$$

$$Y=\tfrac{1}{2}(y-y_s)-\tfrac{1}{2}i(x-x_s),$$

is an imaginary point on C; the conjugate point T' is therefore also a point on C: but the antipoints of T, T' are the points $x+iy$, x_s-iy_s.

The condition that $C=0$ should be real is that $f(\xi, \eta)$ should be of the form $f_1+i(\xi-\eta)f_2$, f_1 and f_2 denoting symmetric functions of ξ and η. If this condition be not satisfied, the theorem still subsists if we replace the curve C by two conjugate imaginary curves, each of which has to be regarded as cut by one of the cyclic lines passing through the point (x, y).

We obtain from the theorem the following construction for the transformed modules corresponding to any given modulus P. Let $Q_1, Q_2, \ldots$ be the points inverse to P with regard to one of the modular curves; and $Q_{-1}, Q_{-2}, \ldots$ the points inverse to $Q_1, Q_2, \ldots$ with regard to the axis of X; the transformed modules determined by the modular equation corresponding to the curve considered, are represented by $Q_{-1}, Q_{-2}, \ldots$.

The following general theorem, which, however, must be taken as subject to certain limitations, may serve to show the importance of inverse points in the theory of functions.

'If the plane (xy) be mapped upon the plane XY by means of the equation

$$X+iY=f(x+iy),$$

points inverse with regard to any curve are transformed into points inverse with regard to the corresponding curve.'

Let $x=\alpha(\theta)$, $y=\beta(\theta)$, where θ is a variable parameter, be the equations of a curve in the plane (xy); let also

$$X+iY=f[\alpha(\theta)+i\beta(\theta)]=A(\theta)+iB(\theta), \quad \ldots\ldots\ldots \text{(i)}$$

so that $X=A(\theta)$, $Y=B(\theta)$ are the equations of the transformed curve. If θ_1, θ_2 are a pair of conjugate imaginary quantities,

$$\alpha(\theta_1), \beta(\theta_1); \quad \alpha(\theta_2), \beta(\theta_2); \quad \text{or } (\alpha_1, \beta_1), (\alpha_2, \beta_2),$$

are a pair of conjugate imaginary points on γ; the corresponding points on Γ we may represent by $(A_1, B_1), (A_2, B_2)$. The coordinates of the antipoints of these two pairs of imaginary points are obtained by equating real and imaginary parts in the equations

$$x_1+iy_1=\alpha_1+i\beta_1; \quad x_2+iy_2=\alpha_2+i\beta_2;$$
$$X_1+iY_1=A_1+iB_1; \quad X_2+iY_2=A_2+iB_2.$$

For each of these pairs the equation (i) gives

$$X+iY=A+iB,$$

which establishes the theorem.

In the demonstration the functions $\alpha(\theta)$, $\beta(\theta)$ are supposed to be real when θ is real; the three functions $\alpha(\theta)$, $\beta(\theta)$, $f(x+iy)$ are also regarded as monodromic throughout the portion of the plane xy considered. The theorem may be regarded as defining for any Gaussian transformation $X+iY=f(x+iy)$ the relation between the imaginary points appertaining (in the sense of coordinate or projective geometry) to any figure in the given plane, and the corresponding imaginary points in the plane (XY): viz. this relation may be expressed by saying 'corresponding pairs of imaginary points have corresponding antipoints,' or, again, by saying 'the imaginary cyclic lines correspond to one another in the two planes.'

By virtue of the theorem, the construction for the transformed modules given above is an immediate consequence from Art. 39, Theorem III.

69. *Theorems relating to the Multiplier.*

Theorem I. The multiplier corresponding to a symmetrical transformation of determinant n, has $\sqrt{n}$ for its analytical modulus.

Let $k^2=\Phi(\omega)=\frac{1}{2}+X+iY$, $\lambda^2=\Phi(\Omega)=\frac{1}{2}+X-iY$, $\omega=|A|\times\Omega$, where $|A|$ is a matrix of determinant n; if $\omega=-x+iy$, then Ω is equivalent to $x+iy$, and the equation between ω and Ω assumes the form

$$-x+iy=\frac{a+b(x+iy)}{b+c(x+iy)},$$

a, b, c being integral numbers satisfying the equation $b^2-ac=n$. Let

$$\frac{1}{M}=p+iq, \quad K=A+iB, \quad iK'=(-x+iy)(A+iB)=A'+iB',$$

the quantities p, q, A, B, A', B' being real; then $\Lambda=K(x+iy)=A-iB$, $i\Lambda'=-A'+iB'$. Equating real and imaginary parts in the equations (Art. 26, ii)

$$(p+qi)K=b\Lambda+ci\Lambda',$$
$$(p+qi)K'=a\Lambda+bi\Lambda',$$

we obtain the system

$$\begin{aligned}
(p-b)A+ \quad qB \quad -cA'=0,\\
-qA-(p+b)B \quad -cB'=0,\\
-aA+ \quad (p+b)A'-qB'=0,\\
aB+ \quad qA'-bB'=0,
\end{aligned}$$

of which the determinant is $(p^2+q^2-n)^2$. Since A, B, A', B' cannot be all equal to zero, the determinant must vanish, and we have $\sqrt{p^2+q^2}=\sqrt{n}$.

The same result might be obtained from the equation of Jacobi (Art. 61) which, in the case here considered, assumes the form

$$\frac{1}{M^2}=n\frac{\Phi(-x+iy)\,\Psi(-x+iy)\,\Phi'(\quad x+iy)\,d(\quad x+iy)}{\Phi(\quad x+iy)\,\Psi(\quad x+iy)\,\Phi'(-x+iy)\,d(-x+iy)},$$

$$=n\frac{P-Qi}{P+Qi},$$

when P and Q are real quantities.

Thus, while the point representing $-\frac{1}{2}+k^2$ describes an oval of the modular curve, the point representing the corresponding multiplier describes an arc of the circle $X^2+Y^2=n$; this point, however, does not always move in one direction on the circumference of the circle, and the stationary points, at which the direction of its motion is reversed, answer to values of $\frac{1}{M}=\sqrt{n}e^{i\theta}$, for which two roots of the equation $\mu\left(\frac{1}{M},\,k^2\right)=0$ become equal.

Theorem II. The square of the multiplier appertaining to the transformation of any given modulus into its conjugate is

$$n\times e^{2i(\alpha+\alpha'-\theta)}$$

when n is the order of the transformation and (the given modulus being represented by the point P on the modular curve) α, α', and θ are the angles made with the positive axis of X by the focal radii vectores of P, and by the tangent at P respectively.

This is the geometrical interpretation of the equation of Jacobi in the case in which the transformation is symmetrical.

70. *Symmetry of the Modular Curves.*

The first and fourth modular curves are symmetric, each to itself, with regard to both the axes; the third and fourth modular curves are symmetric, each to itself, with regard to the axis of X, and each to the other with regard to the axis of Y.

In certain cases, which we proceed to enumerate, ovals appertaining to the modular curves are symmetric with regard to one or both axes, or have the origin for a centre.

I. Ovals symmetric with regard to the axis of X.

If $[a, b, c]$ is a semicircle appertaining to the period of an oval symmetric with regard to the axis of X, $[a, -b, c]$ is also a semicircle appertaining to the period; *i.e.* either (1) the forms (a, b, c) and $(a, -b, c)$, or else (2) the forms (a, b, c) and $(-a, b, -c)$ are primarily equivalent. The former hypothesis is inadmissible in the case of improperly primitive forms, the latter in the case of properly primitive forms.

Theorem. If any one subclass of forms contained in a given class is primarily ambiguous, every subclass contained in that class is also primarily ambiguous.

For if (a, b, c) is transformed into $(a, -b, c)$ by the primary matrix $|\theta|$ and into (A, B, C) by any unit matrix $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$, then (A, B, C) is transformed into $(A, -B, C)$ by the primary matrix

$$\begin{vmatrix} \delta, & -\beta \\ -\gamma, & \alpha \end{vmatrix} \times |\theta| \times \begin{vmatrix} \alpha, & -\beta \\ -\gamma, & \delta \end{vmatrix}.$$

Hence each of the first three modular curves contains the same number of ovals symmetric with regard to the axis of X.

(A) Properly Primitive Classes.

(1) Let U_1, the least number which satisfies the equation

$$T^2 - DU^2 = 1,$$

be uneven. Then every ambiguous class contains three subclasses, each of which is primarily ambiguous. For every ambiguous class contains forms of one or other of the types $(a, 0, c)$ or $(2a, a, c)$. Of these $(a, 0, c)$ is identical with its opposite, and therefore primarily equivalent to it; and $(2a, a, c)$ is transformed into $(2a, -a, c)$ by

$$\begin{vmatrix} T - aU_1, & -cU_1 \\ 2aU_1, & T + aU_1 \end{vmatrix} \times \begin{vmatrix} 1, & -1 \\ 0, & 1 \end{vmatrix}, \quad \ldots\ldots\ldots \quad \text{(i)}$$

which is primary.

(2) Let U_1 be even. In this case no ambiguous class can contain forms of each of the types $(a, 0, c)$ and $(2a, a, c)$. Ambiguous classes containing forms of the first of these types are, as before, primarily ambiguous; but the matrices (i) are no longer primary, and hence ambiguous classes containing forms of the type $(2a, a, c)$ are not primarily ambiguous. When $D \equiv 1, 2, 4, 5, 6$, mod 8, there are no forms of this type; when $D \equiv 0, 3, 7$, mod 8, one half of the ambiguous classes contain forms of each of the two types.

Combining the two results, we find that each of the first three modular curves contains as many ovals symmetric with regard to the axis of X, or half as many, according as $D \equiv 1, 2, 4, 5, 6$, mod 8, or $D \equiv 0, 3, 7$, mod 8.

The symmetry is of the first species*; for if $[a, b, c]$ is a transitive semicircle of the period, so is also $[a, -b, c]$, which is symmetrical to it. Then two semicircles occur, one in an even, the other in an uneven place in the period; *i. e.* one is described in the direction $[TS]$, the other in the direction $[ST]$.

(B) Improperly Primitive Classes.

If (a, b, c), any form of determinant D, is equivalent to $(-a, b, -c)$, we have $T^2 - Du^2 = -1$; and conversely, if $T^2 - Du^2 = -1$, every form (a, b, c) of determinant D is equivalent to $(-a, b, -c)$; and, if the forms are improperly primitive, the equivalence is primary, for

$$\begin{vmatrix} T - bu, & cu \\ au, & -T - bu \end{vmatrix}, \quad \ldots\ldots\ldots \quad \text{(ii)}$$

which transforms (a, b, c) into $(-a, b, -c)$, is primary. Hence when the equation $T^2 - Du^2 = -1$ is resoluble, every oval of the fourth modular curve is symmetric with respect to the axis of X; but, except in that case, no oval of the curve possesses this symmetry. The symmetry is of the second species; for the semicircles $[a, b, c]$, $[-a, b, -c]$ occur either both in even places, or else both in uneven places.

II. Ovals symmetric with respect to the axis of Y.

If $[a, b, c]$ is any semicircle of a period, the oval corresponding to the period is symmetric with respect to the axis of Y, (1) if (a, b, c) is primarily equivalent to (c, b, a), (2) if (a, b, c) is primarily equivalent to $(-c, -b, -a)$. In either case the extreme coefficients must be either both even or both uneven; so that ovals symmetric with regard to the axis of Y exist only in the first and fourth modular curves.

(A) Properly Primitive Classes.

(1) If (a, b, c) is primarily equivalent to (c, b, a), the class containing (a, b, c) must be ambiguous, and (a, b, c) must be equivalent to $(a, -b, c)$ by a trans-

* When any closed line is symmetric with regard to an axis, a point describing the line continuously may either describe corresponding portions in corresponding directions, or it may describe them in opposite directions. In the latter case the symmetry is said to be of the first species, in the former of the second. The symmetry of an ellipse with regard to either axis is of the first species; the symmetry of a hyperbola, or of a lemniscate, with regard to one axis is of the first species, and with regard to the other of the second.

formation of the type ψ. It may be inferred that (a, b, c) is equivalent in this manner to $(a, -b, c)$, (α) when U_1 is uneven (see I. (A) (1) *supra*, and observe that (a, b, c) has automorphics of the type ψ); (β) when U_1 is even, and the class (a, b, c) contains a form of the type $(2a, a, c)$; viz. in this case (a, b, c) is not primarily equivalent to $(a, -b, c)$ (see I. (A) (2) *supra*), and must therefore be equivalent to it by a transformation of the type ψ.

(2) If (a, b, c) is primarily equivalent to $(-c, -b, -a)$, then (a, b, c) is equivalent to $(-a, b, -c)$ by a transformation of type ψ, and the equation $T^2 - Du^2 = -1$ is resoluble, the values of U being necessarily uneven. Conversely, if this equation be resoluble, every form (a, b, c) of determinant D, with its extreme coefficients uneven, is primarily equivalent to $(-c, -b, -a)$.

Combining these results we find (1) that if $D \equiv 0, 3, 7$, mod 8, the first modular curve has one half as many ovals symmetric with respect to the axis of Y as there are properly primitive ambiguous classes of determinant D; and (2) that if the equation $T^2 - Du^2 = -1$ is resoluble, all the ovals of the first modular curve are symmetric with respect to the axis of Y. The symmetry is of the first species in the first case, and of the second species in the second case; the two cases cannot coexist, because $D \equiv 0, 3, 7$, mod 8 in the first case, and $D \equiv 1, 2, 5$, mod 8 in the second.

(B) Improperly Primitive Classes.

The forms (a, b, c) and $(-a, -b, -c)$ cannot be primarily equivalent. If (a, b, c) is primarily equivalent to (c, b, a), the class containing (a, b, c) is ambiguous, and therefore contains a form of the type $(2a, \beta, 2a)$. If U_1 is uneven, the class contains only two subclasses; these we may represent by $(2a, \beta, 2a)$ and $(2a, -\beta, 2a)$, since these two forms are certainly not primarily equivalent. If U_1 is even, the class contains six subclasses; these we may represent by the forms

$$(2a, \pm\beta, 2a), \quad (2a, \pm\beta, 2a) \times \rho, \quad (2a, \pm\beta, 2a) \times \rho^2.$$

If (a, b, c) is one of the four last of these forms, (a, b, c) is not primarily equivalent to (c, b, a); for example, if $(a, b, c) = (2a, \beta, 2a) \times \rho$, then (a, b, c) is transformed into (c, b, a) by transformations of the type ρ, and of no other type. Hence every ambiguous class (whether U_1 is even or uneven) contains two and only two subclasses (a, b, c), such that (a, b, c) is primarily equivalent to (c, b, a).

We infer therefore that the fourth modular curve contains as many ovals

symmetric with respect to the axis of Y as there are ambiguous classes. The symmetry is of the first species.

III. Ovals symmetric with respect to both axes.

Such ovals exist only in the first and fourth modular curves, and in them only in the following cases :

(1) When the equation $T^2 - Du^2 = -1$ is resoluble, every oval in the first or fourth curve respectively is symmetric with regard to the axis of Y or of X. Hence all ovals of the first curve which are symmetric with regard to the axis of X, and all ovals of the fourth curve which are symmetric with regard to the axis of Y, are symmetric with regard to both axes; the symmetry with respect to the two axes being dissimilar.

(2) If in the equation $T^2 - Du^2 = 1$, U_1 is uneven, every oval in the first modular curve which is symmetric with regard to either axis is also symmetric with regard to the other axis. The symmetry is of the first species with regard to both axes. It will be observed that, when U_1 is uneven, the equation $T^2 - Du^2 = -1$ cannot be resoluble.

IV. Central ovals.

The ovals considered in III. are of course central; we need therefore only consider here those ovals which are central, but symmetrical with regard to neither axis: central ovals can, of course, only exist in the first and second modular curves; and the oval is central, if any form (a, b, c) belonging to it is primarily equivalent either to $(c, -b, a)$ or to $(-c, b, -a)$.

(A) Properly primitive Classes.

(1) If in the equation $T^2 - Du^2 = 1$, U_1 is uneven, every form (a, b, c), of which the extreme coefficients are uneven, is primarily equivalent to $(c, -b, a)$; for every such form (a, b, c) has automorphics of the type ψ.

(2) When D is the sum of two squares prime to one another, a certain number of classes of determinant D are such that (a, b, c) and $(-c, b, -a)$ are equivalent; viz. the number of such classes is equal to the number of ambiguous classes: the two sets of classes being entirely distinct when the equation $T^2 - Du^2 = -1$ is irresoluble, and coinciding when that equation is resoluble. It will be observed that, in the case here considered, $D \equiv 1, 2, 5$, mod 8, and that U_1 is therefore even.

Thus, if U_1 is uneven every oval of the first modular curve is central, and of the first species; if D is the sum of two squares relatively prime, there

are half as many central ovals as there are ambiguous classes. The symmetry is of the second species: if the equation $T^2 - Du^2 = -1$ is not resoluble, the central ovals are entirely distinct from the ovals representing ambiguous classes; but if this equation is resoluble, the two sets of ovals coincide.

(B) Improperly Primitive Classes.

Here the forms (a, b, c) and $(c, -b, a)$ cannot be primarily equivalent; hence the only case in which the fourth modular curve contains central ovals is that in which the determinant is uneven and is the sum of two squares relatively prime. In this case the number of central ovals is the same as in the first modular curve (viz. half as many as the improperly primitive classes); and (as in the case of the first modular curve) the central ovals are simply central, or are symmetric with respect to both axes, according as the equation $T^2 - Du^2 = -1$ is not or is resoluble.

For a fuller explanation of the arithmetical principles upon which the preceding discussion of the symmetry of the ovals is based, the reader is referred to a memoir, 'Note on the Theory of the Pellian Equation and of Binary Quadratic Forms of a Positive Determinant,' in the Proceedings of the London Mathematical Society, vol. vii. pp. 196–208 *, where, however, the equivalence considered is general, and not as here primary.

71. *Transformation of a Modular Curve by Inversion with regard to another Modular Curve.*

Let $|H|$ be a given matrix of determinant h, and let $\omega = |H| \times \Omega$; if ω describes a semicircle C of determinant Δ, then Ω describes the semicircle $C \times |H|$, derived from C by applying to it the transformation $|H|$. Thus any transformation of a quadratic form by a linear substitution of which the determinant is greater than unity, gives rise to a transformation of the modular curve answering to the quadratic form. We have in fact the general theorem, 'The inverse of any modular curve with regard to itself or any other modular curve is a modular curve, or is composed of modular curves.' For the arithmetical theory itself we refer to the 'Disquisitiones Arithmeticæ,' section 5; to a memoir by Professor Lipschitz (Crelle, vol. liii. p. 238); and to the Report on the Theory of Numbers (Reports of the British Association for 1862, p. 503, Art. 106 *et seq.*) †; confining ourselves in this place to a few examples in illustration of the geometrical theorem.

* [Vol. ii. p. 148.] † [Vol. i. p. 231.]

To the symbol $\left(\frac{\Delta}{p}\right)$ we attribute the value $+1$, -1, or 0, according as Δ is a quadratic or a non-quadratic residue of p, or is divisible by p.

Let the modular curves of any given determinant n be denoted by $F_s(n)$, $s=1, 2, 3, 4$; let Δ and h be both uneven, and $h \equiv 1$, mod 4; in the equation $T^2 - \Delta U^2 = 1$, let U_μ be the first of the numbers $U_1, U_2, \ldots$ which is divisible by h; and let p be any prime divisor of h. Let P be a point on $F_1(\Delta)$, and let the points inverse to P with regard to the curve $F_4(h)=0$ be denoted by $Q_1, Q_2 \ldots$. There are in all $h\Pi\left(1+\frac{1}{p}\right)$ points Q; of these $m = h\Pi\left[1-\left(\frac{\Delta}{p}\right)\frac{1}{p}\right]$ describe the curve $F_1(\Delta h^2)$ while P describes $F_1(\Delta)$; viz. $F_1(\Delta h^2)$ has $\frac{m}{\mu}$ times as many ovals as $F_1(\Delta)$, and while P describes any oval of $F_1(\Delta)$, the points Q describe $\frac{m}{\sigma}$ different ovals of $F_1(\Delta h^2)$, σ points describing each oval simultaneously.

When Δ is a non-quadratic residue of every prime p, $F_1(\Delta h^2)$ is the complete inverse of $F_1(\Delta)$ with regard to $F_4(h)$. In every other case the complete inverse includes certain modular curves $F_1\left(\frac{\Delta h^2}{\delta^2}\right)$, δ_2 being a divisor of Δh^2 and containing only primes which divide h. For brevity we confine ourselves to the particular case in which $h=p$ is a prime and $\left(\frac{\Delta}{p}\right) = +1$. In this case there are $p+1$ points Q, of which $p-1$ describe the curve $F_1(\Delta p^2)$, and two, Q_1, Q_2, the curve $F_1(\Delta)$ itself. Thus the inversion of $F_1(\Delta)$ with regard to $F_4(p)$ gives a 'one to two' transformation of $F_1(\Delta)$ into itself. The prime p is always represented either by an ambiguous form of determinant Δ, or by each of two opposite and non-equivalent forms. (1) If p is represented by an ambiguous form of determinant Δ, p^2 is represented by the principal form, and while P describes any oval S of $F_1(\Delta)$, the two points Q_1 and Q_2 both describe another oval S_1, and both describe it in the same direction, never meeting one another, although they may cross at a double point. In particular, if p is itself represented by the principal form of determinant Δ, S_1 is the same as S. Whether S_1 coincides with S or not, the transformation may be repeated *ad infinitum*; viz. if the points inverse to P are P_{-1} and P_1, then the points inverse to P_1 are P and P_2, the points inverse to P_2 are P_1 and P_3, and so on continually, the points $P, P_1, P_2 \ldots$ forming a series which never closes, and lying alternately on S and S_1 when those two ovals are different. (2) If p is not represented by an ambiguous form, the ovals described by Q_1 and Q_2

are different, except when the equation $T^2 - \Delta U^2 = -1$ is resoluble; in which case Q_1 and Q_2 describe the same oval, but describe it in opposite directions. Let p be represented by a fundamental form f of the principal genus; *i.e.* by a form which compounded with itself gives all the forms of the principal genus (we suppose the determinant Δ to be regular); and let ν be the number of forms in the principal genus; the points $P, P_1, \ldots, P_{\nu-1}$ will all lie on the ν different ovals appertaining to one and the same genus, and the point P_ν will lie on the same oval as P. In fact if that oval represent the form Γ, the points $P_1 \ldots P_{\nu-1}$ lie on the ovals answering to the forms $\Gamma \times f$, $\Gamma \times f^2, \ldots, \Gamma \times f^{n-1}$. We may observe (3) that the number ν of ovals which are thus cyclically changed into one another is always the number of forms in the composition period of the form f; *i.e.* the index ν for which f^ν becomes the principal class; (4) that if instead of $F_4(p)$ we use $F_4(p^2)$, or any modulus curve of an uneven square determinant prime to Δ, as the transforming curve, only ovals of the same genus can be transformed into one another.

The preceding discussion may suffice to show that the arithmetical theories of the division of the classes into genera, and of the composition of classes, have a certain geometrical signification in the theory of the modular curves. But, so far as is yet known, there is nothing in the aspect of the ovals to indicate the order in which they are related to one another by composition, or their distribution into genera.

72. *Multiple Points and Points of Intersection.*

A point ω, defined by a quadratic equation, of which the determinant is negative, and the coefficients integral, may be termed a quadratic point; and the same designation may, for brevity, be applied to the corresponding point $-\frac{1}{2} + \Phi(\omega)$ in the plane (XY). We do not regard the cornicular points in the plane (xy), or the points corresponding to them in the plane XY, as quadratic points. The determinant of a quadratic point ω, or $-\frac{1}{2} + \Phi(\omega)$, is the determinant $\alpha\gamma - \beta^2$ of the primitive quadratic equation $\alpha + 2\beta\omega + \gamma\omega^2 = 0$ by which ω is defined.

The theta and elliptic functions of which the modules are quadratic admit of complex multiplication; and, conversely, if the functions admit of complex multiplication, the modulus is quadratic. Let $\alpha\gamma - \beta^2 = \Delta$, $\tau^2 + \upsilon^2\Delta = n$, the transformation of order n,

$$\begin{vmatrix} \tau - \upsilon\beta, & -\upsilon\gamma \\ \upsilon\alpha, & \tau + \upsilon\beta \end{vmatrix},$$

by which (α, β, γ) is changed into $n \times (\alpha, \beta, \gamma)$, does not alter the value of ω, and consequently leaves $K(\omega)$, $iK'(\omega)$, $\Phi(\omega)$, $\Phi'(\omega)$ unchanged: the multiplier is the complex number defined by the equation

$$\frac{1}{M} = \frac{1}{\mu}[\tau + v\beta + v\gamma\omega] = \frac{1}{\mu}[\tau + iv\sqrt{\Delta}];$$

i. e. the transformation is equivalent to a multiplication of the argument by a complex number.

For an account of the researches to which the theory of complex multiplication has given rise we must refer to a note of Abel ('Œuvres,' vol. i. p. 272), to the Memoirs of MM. Kronecker and Hermite, and to the Report on the Theory of Numbers (Arts. 126 and 130–138*).

Theorem. 'Every real point of intersection of a modular curve with itself, or with another modular curve, is a quadratic point; and an infinite number of modular curves can be drawn through any quadratic point in the plane (XY).'

For if Z be a point at which two branches of a modular curve, or of two different modular curves, intersect, the corresponding reduced semicircles C_1 and C_2 intersect in a quadratic point z lying in the reduced space; through this point there pass an infinite number of rational semicircles $\lambda_1 C_1 + \lambda_2 C_2$; *i. e.* there pass through Z an infinite number of modular curves.

Theorem. 'The inverse of any quadratic point in the plane XY with regard to any modular curve is a quadratic point; and the determinants of the two points are to one another as two squares.'

If $-\frac{1}{2} + \Phi(x_2 + iy_2)$, $-\frac{1}{2} + \Phi(x_1 + iy_1)$ of the determinants Δ_2, Δ_1, are the two points, we have

$$-x_2 + iy_2 = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} + (x_1 + iy_1);$$

whence, if either of the two points is quadratic, the other is so too; and if $ad - bc = D$, $\Delta_1 = \frac{D^2}{\delta^2}\Delta_2$, δ^2 being a square divisor of $D^2\Delta_2$, containing only primes which divide D.

Theorem. 'If two points are inverse to one another with regard to two different modular curves, or are doubly inverse with regard to the same curves, they are quadratic points.'

For if $-x_2 + iy_2 = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times (x_1 + iy_1) = \begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix} \times (x_1 + iy_1)$, the two matrices

[* Vol. i. pp. 303 and 321.]

being different, we have

$$x_1+iy_1=\begin{vmatrix} d, & -b \\ -c, & a \end{vmatrix}\times\begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix}\times(x_1+iy_1);$$

i. e. x_1+iy_1 is a quadratic point, and therefore also x_2+iy_2.

Theorem. 'If the determinants of two quadratic points are to one another as two square numbers, the two points are inverse with regard to an infinite number of modular curves.'

For if the determinants of $\Phi(x_1+iy_1)$ and $\Phi(x_2+iy_2)$ are as two square numbers, we can always, and in an infinite number of ways, establish between $-x_2+iy_2$ and x_1+iy_1 an equation of the form

$$x_2+iy_2=\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}\times(x_1+iy_1);$$

viz. if $|A_1|, |A_2|$ are two matrices satisfying the equation, it is also satisfied by the matrices $|\lambda_1 A_1+\lambda_2 A_2|$. For brevity we omit the demonstration, which depends on the theory of the composition of quadratic forms.

A complete discussion of the intersections and multiple points of the modular curves, involving a systematic examination of the properties of the quadratic modules, is not exempt from difficulty, and would surpass our present limits. We confine ourselves to a few observations which may serve to show the general character of the theory.

(i.) The modular curves have no superlinear branches, except those at the cyclic points, and at the points $(\pm\frac{1}{2}, 0)$ or $(0, \pm\frac{1}{2}i)$, which arise from the cornicular values of ω. (See the 'Memoir on the Singularities of the Modular Equations and Curves,' Proceedings of the London Mathematical Society, vol. ix. pp. 242–272*.)

Again, except at the points $(\pm\frac{1}{2}, 0)$ no modular curve can have a conjugate point (*i. e.* a real point) with pairs of conjugate imaginary branches passing through it.

For if the point ω corresponding to the real point $-\frac{1}{2}+\Phi(\omega)$ be not a cornicular point, it may be shown that as many real semicircles (appertaining to the modular curve) pass through ω as there are branches of the modular curve passing through $-\frac{1}{2}+\Phi(\omega)$.

In speaking of the intersections and multiple points of the modular curves we shall henceforward exclude the cyclic points and the two pairs of points $(\pm\frac{1}{2}, 0), (0, \pm\frac{1}{2}i)$.

[* Vol. ii. p. 242.]

(ii.) Two branches of a modular curve, or of two different modular curves, cannot touch.

For real points this is evident, since two rational semicircles cannot touch, except when the determinant is a square ; even in this case, semicircles passing through one of the cornicular points correspond to curves in the plane XY which do not touch.

But the same thing is true for imaginary points. At a point of intersection, whether real or imaginary, we must have

$$\Omega = \begin{vmatrix} a_1, & b_1 \\ c_1, & d_1 \end{vmatrix} \times \omega = \begin{vmatrix} a_2, & b_2 \\ c_2, & d_2 \end{vmatrix} \times \omega ;$$

the two matrices not being equivalent by permultiplication, or else the two branches would be the same. If there be contact, $\frac{\Phi'(\Omega)}{\Phi'(\omega)} \times \frac{d\Omega}{d\omega}$, and consequently $\frac{d\Omega}{d\omega}$, must have the same value for each of the two determinants of Ω. We thus obtain

$$\frac{D_1}{(a_1 + b_1\omega)^2} = \frac{D_2}{(a_2 + b_2\omega)^2},$$

D_1, D_2 being the determinants of the two matrices. But this equation implies that ω is real.

The same conclusion might be obtained by considering the multipliers appertaining to the two transformations.

(iii.) Let D_1, D_2 be two uneven numbers relatively prime, and let us consider for simplicity the modular curves of the fourth species only. The equation

$$F(\tfrac{1}{2} + X, \tfrac{1}{2} + X, D_1 \times D_2) = 0$$

determines the intersections of $F(D_1 \times D_2)$ with the axis of X ; viz. the real values of X give the real intersections of $F(D_1 \times D_2)$ with the axis, and the imaginary pairs of values give the real antipoints of the pairs of imaginary intersections. The system of points (X) thus obtained, consisting of points on the axis of X, and of pairs of points symmetrically situated with regard to the axis of X, is the same as the system (Y) consisting of the real intersections of the curves $F(D_1)$, $F(D_2)$, and of the antipoints of their imaginary intersections ; *i.e.* if P_1 is any point of (X), P_1 has, with regard to $F(D_1)$ and $F(D_2)$, a common inverse Q_1 which is itself a point of (X) ; and P_1, Q_1 are the antipoints of an imaginary pair of intersections of $F(D_1)$ and $F(D_2)$.

If P_2, Q_2 are the points symmetrical to P_1, Q_1 with regard to the axis of X, then P_2, Q_2 belong to the system (X), and are the antipoints of another pair of imaginary intersections; if P_1 coincide with Q_1, and P_2 with Q_2, we have two real intersections instead of two pairs of imaginary intersections: if P_1 coincide with Q_2, and P_2 with Q_1, we have a single pair of imaginary intersections on the axis of X.

To establish this theorem we observe that $F(\lambda^2, k^2, D_1 \times D_2) = 0$ is the resultant of the elimination of z from $F(D_2, \lambda^2, z) = 0$ and $F(D_1, z, k^2) = 0$. Hence $F(D_1 \times D_2, k^2, k^2) = 0$ is the resultant of the elimination of λ^2 from $F(D_2, \lambda^2, k^2) = 0$ and $F(D_1, \lambda^2, k^2) = 0$. And, in accordance with this result, it can be shown by a method due to M. Kronecker, that, if $|D_1|$, $|D_2|$, and $|D_1 \times D_2|$ denote complete systems of primitive and primary matrices for the determinants D_1, D_2 and $D_1 \times D_2$, the two equations

$$\Pi[\Phi(\omega) - \Phi(|D_1 \times D_2| \times \omega)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$\Pi[(|D_1| \times \omega) - \Phi(|D_2| \times \omega)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

(of which the first is $F(D_1 \times D_2, k^2, k^2) = 0$, and the second is the resultant of $F(D_1)$ and $F(D_2)$) give precisely the same values for $\Phi(\omega)$; and give each value with the same multiplicity, the cornicular values being always excepted. It can also be shown that if θ is a value of ω satisfying the two equations, so that one of the quantities $|D_1| \times \theta$ is equivalent to one of the quantities $|D_2| \times \theta$, either of these equivalent quantities is itself a value of ω satisfying the two equations. These considerations suffice to establish the identity of the system (X) with the system (Y). But the way in which the points of the system (X) are paired so as to form the system (Y) depends on the theory of the composition of quadratic forms; viz. if the system (X) contains a group of points (Δ), having the determinant $-\Delta$, then D_1 and D_2 can be represented by one and the same form $-\gamma$ of that determinant, and the pairing of the points (Δ) in the system (Y) depends on the composition of the forms corresponding to the points (Δ) with successive powers of γ.

The same principles are applicable to the case in which D_1 and D_2 are not relatively prime. But in this case the system (Y) includes, besides the system (X), the systems (X'), (X''), ... representing the intersections of the curves, $F\left(\frac{D_1 D_2}{\delta'^2}\right)$, $F\left(\frac{D_1 D_2}{\delta''^2}\right)$, ...; δ', δ'', ... being common divisors of D_1 and D_2, and each of the systems (X'), (X''), ... having in (Y) a certain multiplicity.

To find the multiple points of the modular curve $F(D)=0$, we have to compare the discriminantal equation $\Pi[\Phi(|D|\times\omega)-\Phi(|D'|\times\omega)]^2=0$, where $|D|, |D'|$ are any two different matrices of the complete system, with the system of equations

$$\Pi[\Phi(\omega)-\Phi(|d^2|\times\omega)]=0,$$

where d is any divisor of D; and the system of the discriminantal points and antipoints is the same as the aggregate of the systems representing the intersections of the curves $F(d^2)=0$ by the axis of X, each such system being taken with a certain multiplicity.

(iv.) When two modular curves intersect at a real point, the angle of intersection is the same as for the corresponding semicircles. Let $[A, B, C]$, $[A', B', C']$ be two semicircles of determinants D and D' intersecting at the point $[\alpha, \beta, \gamma]$; let $\tau=AC'-2BB'+A'C$; and let υ be the greatest common divisor of the matrix $\left|\begin{matrix}A, & B, & C\\ A', & B', & C'\end{matrix}\right|$, so that

$$\upsilon\times|\gamma, -2\beta, \alpha|=\left|\begin{matrix}A, & B, & C\\ A', & B', & C'\end{matrix}\right|;$$

lastly, let ζ be the external angle of intersection of the semicircles; we then have

$$\cos\zeta=\frac{\tau}{\sqrt{DD'}}, \qquad \sin\zeta=\frac{\upsilon\sqrt{\Delta}}{\sqrt{DD'}}.$$

If therefore the determinant Δ of a point of intersection of two modular curves of order D and D' is known, a measurement of the angle of intersection gives the decomposition $DD'=\tau^2+\upsilon\Delta$, the values of the indeterminates τ and υ being relatively prime when D, D' are relatively prime.

73. *Intersections of the Modular Curves with the Axis.*

As an example of this theory let $D'=1$, and let us consider the real intersections of the first modular curve of an uneven order D by the axis of X. The number of these intersections is

$$\Sigma 2^\mu\times[D=\tau^2+\upsilon^2\Delta],$$

where μ is the number of primes dividing Δ, and $[D=\tau^2+\upsilon^2\Delta]$ is the number of solutions of the equation $D=\tau^2+\upsilon^2\Delta$, in which τ and υ are relatively prime, and υ is positive and uneven; the sign of summation Σ extending to every value

of Δ from 0 to D. This result may be obtained by counting the number of the reduced semicircles, each of which traverses twice the boundaries of the reduced space, while some of them in addition traverse the line $x=0$. For any uneven value of Δ, let $\Delta = p \times q$, p and q being relatively prime; then $F_1(D)$ cuts the axis of X at the two points $-\frac{1}{2}+\Phi\left(i\sqrt{\frac{p}{q}}\right)$ and $-\frac{1}{2}+\Phi\left(i\sqrt{\frac{q}{p}}\right)$, lying upon the line $A_{-1}A_{+1}$ at equal distances on either side of the centre. For any even value of Δ, let $\Delta = 2^\sigma \times p \times q$, p and q being uneven and relatively prime; then $F_1(D)$ cuts the axis of X between $-D$ and A_{-1} in the points

$$\frac{1}{2}-\frac{1}{\Psi\left(i\sqrt{\frac{2^\sigma p}{q}}\right)}, \qquad \frac{1}{2}-\frac{1}{\Psi\left(i\sqrt{\frac{2^\sigma q}{p}}\right)},$$

and in the symmetrically situated points between A_{+1} and $A_{+\infty}$.

When $\tau = 0$, and consequently $\upsilon = 1$, the point of intersection is single, and the curve cuts the line $A_{-1}A_{+1}$ at right angles 2^d times, if D has d prime factors. When τ is not zero, there is always a double intersection of the axis of X by two symmetrical branches, the angles of intersection being given by the equations

$$\cos\zeta = \frac{\tau}{\sqrt{D}}, \qquad \sin\zeta = \pm\frac{\upsilon\sqrt{\Delta}}{\sqrt{D}}.$$

Thus for each equation $D = \tau^2 + \upsilon^2\Delta$ in which τ is not zero (as there are two solutions, since τ may be either positive or negative) we have 2^μ double points with parallel tangents; and if for the same value of Δ, there are h sets of values of τ^2, υ^2 satisfying the equation, there are h pairs of branches at each of the 2^μ points, the tangents at corresponding pairs being parallel. It thus appears that if the modular curve be traced, we have only to measure the angles at which it cuts the axis of X to obtain all the representations of which D is susceptible by the principal class of any negative determinant, the value of the second indeterminate being uneven. The following are special cases.

(α) The modular curve $F_1(D)$ passes through the origin as many pairs of times as D is the sum of an even square τ^2 and an uneven square υ^2: each pair of symmetrical branches cutting the axis of X at angles of which the trigonometrical tangents are $\pm\frac{\upsilon}{\tau}$.

This theorem may be expressed without any reference to the geometrical

considerations by which it has been obtained; viz. if we consider the modular equation (as Jacobi has done) between $q = 2k^2 - 1$, $l = 2\lambda^2 - 1$, the lowest terms in q and l are

$$A \times \Pi[D(q^2 + l^2) + 2(\tau^2 - v^2)ql],$$

A being an integral number; we have thus a direct method for decomposing a number into the sum of two squares relatively prime, all the decompositions being obtained simultaneously.

(*b*) The curve $F_1(D)$ passes through each of the points $X = \pm\frac{1}{2}\sqrt{2}$ as many pairs of times as D can be decomposed into an uneven square and the double of an uneven square, the tangents of the angles corresponding to the decomposition $D = \tau^2 + 2v^2$ being $\pm\frac{v\sqrt{2}}{\tau}$.

(*c*) The curve $F_1(D)$ passes through each of the points $\pm\frac{1}{4}\sqrt{3}$ as many pairs of times as D can be decomposed into the sum of an even square and the triple of an uneven square; the tangents of the angles being $\pm\frac{v\sqrt{3}}{\tau}$, if $D = \tau^2 + 3v^2$.

Arts. 74–82. Theory of the Modular Functions $\phi(\omega)$ and $\psi(\omega)$.

74. *Theory of* $\phi(\omega)$. *Primary Matrices.*

The unit matrices $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ which satisfy the equation

$$\phi\left(\frac{c + d\omega}{a + b\omega}\right) = \phi(\omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

form a group, characterized by the congruences

$$b \equiv 0, \text{ mod } 2; \quad c \equiv d^2 - 1, \text{ mod } 16. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this article we shall designate as primary those matrices of uneven determinant, and those only, which satisfy the congruences (2) and also the congruence $a \equiv 1$, mod 4; by equivalent matrices we shall understand matrices which can be reduced to one another by (pre- or post-) multiplication with a primary unit matrix. We then have the theorems:

(i.) Every unit matrix can be expressed in one way, and in one only, in each of the forms

$$\pm|A| \times |\sigma|^{2\mu} \times |\zeta|; \quad \pm|\zeta| \times |\sigma|^{2\mu'} \times |A'|;$$

where $|A|$, $|A'|$ are primary units; μ, μ' have values included in the series $0, \pm 1, \pm 2, \pm 3, 4$; and $|\zeta|$ is one of the six unit matrices of Art. 21.

In particular every unit matrix of type (1) is equivalent, by premultiplication, and also by postmultiplication, to one of the matrices $\pm|\sigma|^{2\mu}$, the exponents μ, μ' being in this case the same.

(ii.) If a unit matrix of type (1) be expressed in the form

$$|\sigma|^{2\alpha_1} \times |\tau|^{2\beta_1} \times |\sigma|^{2\beta_2} \times |\tau|^{2\beta_2} \times \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

it is primary or not according as the congruence $\Sigma\alpha \equiv 0$, mod 8, is or is not satisfied. And two unit matrices such as (3) are or are not equivalent according as the congruence $\Sigma\alpha \equiv \Sigma\alpha'$, mod 8, is or is not satisfied.

(iii.) If $|X|$ is any unit matrix, the least exponent for which $|X|^{\mu}$ is primary is always a divisor of 48 (other than 48 itself).

(iv.) Let $$U_\mu = |\sigma|^{2\mu} \times |\tau|^{-2} \times |\sigma|^{-2\mu}, \ (\mu = 0, \pm 1, \pm 2, \pm 3);$$

$$U_4 = |\sigma|^{8} \times |\tau|^{-2} \times |\sigma|^{8}; \qquad U_0 = |\sigma|^{16};$$

every primary unit matrix can be expressed in one way, and in one way only, as a product of powers of the nine matrices U_μ, U_4, U_0.

(v.) Every prime matrix of an uneven determinant Δ is equivalent to one of the $48 \times \sigma'(\Delta)$ pairs of matrices included in the formula

$$\pm |\sigma|^{2\mu} \times |\zeta| \times \begin{vmatrix} g', & 0 \\ 8h, & g \end{vmatrix},$$

where $gg' = \Delta$; $g' \equiv 1$, mod 4; $h = \frac{1}{8}(g^2 - 1)$, mod 2.

(vi.) If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ is a matrix of determinant Δ and of type (1), the exponents 2μ and 2λ, for which the matrices

$$|\sigma|^{2\mu} \times \begin{vmatrix} a, & b \\ c, & d \end{vmatrix}, \qquad \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times |\sigma|^{2\lambda}$$

are primary, are respectively determined by the congruences

$$c + 2\mu d\Delta \equiv d^2 - 1, \mod 16,$$

$$c + 2\lambda d \equiv d^2 - 1, \mod 16.$$

If $|A|$ represents a complete set of primary and primitive matrices of

determinant Δ, and if $|\alpha|$, $|\beta|$ are given primary unit matrices, the formulæ

$$|\alpha| \quad \times |A| \times |\beta|,$$
$$|\sigma|^{2\mu\Delta} \times |A| \times |\sigma|^{-2\mu},$$
$$|\tau|^{\mu} \quad \times |A| \times |\tau|^{-\mu},$$

also represent complete sets of primary and primitive matrices of determinant Δ.

75. *Theory of* $\phi(\omega)$. *The Reduced Space.*

Every point ω in the plane xy is equivalent to one point, and only one, in a certain reduced space lying between the lines $x = \pm 8$, and outside the sixteen semicircles

$$\mu^2 + \mu - (2\mu+1)x + x^2 + y^2 = 0 \ (\mu = -8, -7, \ldots, 6, 7) \quad . \quad . \quad . \quad . \quad \text{(i)}$$

For in the reduced space of Art. 36 there is always one point Ω, and only one equivalent, in the sense defined in that article, to any given point ω.

Let $\Omega_1 = |\alpha| \times \omega$, so that $|\alpha|$ is of type (1), and let $|\sigma|^{2\mu} \times |\alpha|$ be positive, μ being one of the numbers $0, \pm 1, \pm 2, \pm 3, 4\epsilon$, where ϵ is a positive or negative unit according as the real part of Ω_1 is negative or positive; then the point

$$\Omega = |\sigma|^{2\mu} \times \Omega_1 = |\sigma|^{2\mu} \times |\alpha| \times \omega$$

lies in the reduced space and is the only point in that space equivalent to ω.

The sixteen semicircles (ii) are equivalent in pairs; viz. the semicircle

$$U_\nu^{-1} = -[8\nu^2 - 4\nu, \ 1 - 4\nu, \ 2], \ \mu = 2\nu - 1$$

is transformed into the semicircle

$$U_\nu = [8\nu^2 + 4\nu, \ -1 - 4\nu, \ 2], \ \mu = 2\nu$$

by the transformation

$$|U_\nu| = |\sigma|^{2\nu} \times |\tau|^{-2} \times |\sigma|^{-2\nu} = \begin{vmatrix} 1-4\nu, & -2 \\ 8\nu^2, & 1-4\nu \end{vmatrix}$$

for the seven values of ν from -3 to $+3$ inclusive; and the extreme circle

$$U_4^{-1} = -(112, \ -15, \ 2), \ \mu = 7,$$

is changed into the extreme semicircle

$$U_4 = (112, \ 15, \ 2), \ \mu = -8,$$

by the transformation

$$|U_4| = |\sigma|^8 \times |\tau|^{-2} \times |\sigma|^8 = \begin{vmatrix} -15, & -2 \\ -112, & -15 \end{vmatrix}.$$

Similarly the semicircle $U^{-1} = -[16, -1, 0]$ is changed into $[16, 1, 0]$ by the transformation $|U| = |\sigma|^{16} = \begin{vmatrix} 1, & 0 \\ 16, & 1 \end{vmatrix}$.

We may form the reduced space into a closed surface by folding it on itself so that the two boundaries $x = \pm 8$ coincide, and also the two semicircles of each equivalent pair; the eight points $x = \pm 1, \pm 3, \pm 5, \pm 7$ will then coincide and form a singular point on the surface from which the eight pairs of coincident semicircles diverge; the surface has also nine other singular points; viz. one at an infinite distance, and eight at the points $x = 0, \pm 2, \pm 4, \pm 6, 8$, which are the extremities of the semicircles remote from the extremity common to them all.

76. *Transformation of the Reduced Space by the Modular Function* $\phi(\omega)$.

We now map the reduced space on the plane XY by the equation $\phi(\omega) = X + iY$. The seven pairs of equivalent semicircles $[8\nu^2 \pm 4\nu, \mp 1 - 4\nu, 2]$, $\nu = 0, \pm 1, \pm 2, \pm 3$, are represented by the lines $Re^{\frac{1}{4}\nu i\pi}$, R being real and varying from 1 to ∞ as ω passes from $x = 2\nu$ to $x = 2\nu \pm 1$ along one or other of the equivalent semicircles; the equivalent pair of semicircles $[112, \pm 15, 2]$ from $x = -8$ to $x = -7$, or from $x = 8$ to $x = 7$, are represented by the axis of X from -1 to ∞; lastly, the equivalent boundaries $x = \pm 8$ from $y = \infty$ to $y = 0$ are represented by the axis of X from 0 to -1.

If ω and Ω are the reduced values of ω immediately before and immediately after the passage by $\phi(\omega) = X + iY$ of any boundary U_ν, the equation of passage is

$$\omega = |U_\mu| \times \Omega, \text{ or } \omega = |U_\mu|^{-1} \times \Omega$$

according as $X + iY$ traverses the boundary from right to left or from left to right as seen from 0.

The seven lines $x = 2\nu$ (none of which are boundaries) from $y = \infty$ to $y = 0$ are represented by the lines $Re^{\frac{1}{4}\nu i\pi}$, from $R = 0$ to $R = 1$; the eight lines $x = 2\nu - 1$, $\nu = 0, \pm 1, \pm 2, \pm 3, 4$ from $y = \infty$ to $y = 0$ are represented by the lines $Re^{\frac{1}{8}(2\nu-1)i\pi}$ from 0 to ∞; the eight semicircles $[4\nu^2 - 4\nu, 1 - 2\nu, 1]$ from $x = 2\nu$ to $x = 2\nu - 2$ are represented by the eight arcs of the circle $X^2 + Y^2 = 1$

from $e^{\frac{1}{4}\nu i\pi}$ to $e^{\frac{1}{4}(\nu-1)i\pi}$ respectively. The seven semicircles

$$[4\nu^2-1,\ -2\nu,\ 1],\quad \nu=0,\ \pm1,\ \pm2,\ \pm3,$$

from $2\nu-1$ to $2\nu+1$ are represented by seven of the infinite branches of the curve $R^8\cos 8\theta=\frac{1}{2}$, lying respectively between the asymptotes $\theta=\frac{1}{16}(4\nu-1)\pi$, $\theta=\frac{1}{16}(4\nu+1)\pi$, and drawn from the former of these lines to the latter: the two quadrants $[4\nu^2-1,\ -2\nu,\ 1]$, $\nu=\pm4$, from $x=7$ to the boundary $x=8$, and then from the boundary $x=-8$ to $x=-7$, form the remaining loop from $\theta=\frac{15}{16}\pi$ to $\theta=\frac{17}{16}\pi$. Lastly, the pairs of lines $[2\nu\pm1,\ -1,\ 0]$, $\nu=0,\ \pm1,\ \pm2,\ \pm3$, and the pair $[\pm15,\ 1,\ 0]$ are represented by the eight loops of the inverse curve $R^8=2\cos 8\theta$; these loops lie in the same angles as the loops of the curve $R^8\cos 8\theta=\frac{1}{2}$; each of them represents the reduced parts of the pair of straight lines answering to it; viz. if the point ω describe the line $x=2\nu-1$ from 0 to $2\nu-1+\frac{1}{2}i$, $X+iY$ describes the loop from 0 on the tangent $\theta=\frac{1}{16}(4\nu-1)\pi$ to the point $R=2^{\frac{1}{4}}$, $\theta=\frac{1}{4}\nu\pi$, and then as θ describes the line $x=2\nu+1$ from $2\nu+1+\frac{1}{2}i$ to ∞, θ continues the loop to the point 0 on the tangent $\theta=\frac{1}{16}(4\nu+1)\pi$. All the semicircles, properly and improperly primitive, of determinant $+1$ are included in the preceding enumeration.

77. *Theory of $\phi(\omega)$. The Rational Semicircles.*

Given any semicircle we can always find a reduced semicircle C_1 equivalent to it. Consider this semicircle as described by a point ω in a definite direction. For brevity, we attend only to the case in which the determinant is uneven and not a square. Let U' be the boundary at which C_1 quits the reduced space, and let $|U'|$ be the transformation by which U' is transformed into the equivalent boundary; then if C_2 is the semicircle by which the continuation of C_1 is represented in the reduced space, C_1 is transformed into C_2 by $|U'|$. Continuing this process we shall certainly arrive at C_1 again, because the number of reduced semicircles equivalent to a given rational semicircle is finite, and because each reduced semicircle determines the reduced semicircles which precede and follow it. The reduced semicircles equivalent to C_1 therefore form a period including C_1 itself; and $T=|U'|\times|U''|\times|U'''|\times\ldots$ is an automorphic of C_1. It can be shown that T is one of the two *primary* fundamental automorphics of C_1; *i. e.* that every primary automorphic is a power (with a positive or negative exponent) of T. Hence, conversely, if the quadratic form C_1 be given, and also T the fundamental automorphic of C_1, we have only to

express T as a product of powers of the unit matrices U, in order to obtain analytically the series of forms (or of semicircles) equivalent to C_1.

78. *Theory of* $\psi(\omega)$.

We shall state very briefly some of the corresponding results relating to the functions $\psi(\omega) = \phi\left(-\frac{1}{\omega}\right)$ and $\frac{\phi(\omega)}{\psi(\omega)} = e^{\frac{1}{8}i\pi} \times \phi(-1+\omega)$.

If $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ is a unit matrix satisfying the conditions $b \equiv c \equiv 0$, mod 2, $b \equiv a^2 - 1$, mod 16, we have $\psi(\omega) = \psi\left(\frac{c+d\omega}{a+b\omega}\right)$; and conversely, if

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = |\sigma|^{2\mu_1} \times |\tau|^{2\nu_1} \times |\sigma|^{2\mu_2} \times |\tau|^{2\nu_2} \times \ \dots,$$

the condition $b \equiv a^2 - 1$, mod 16, is equivalent to the condition $\Sigma\nu \equiv 0$, mod 8.

If $|A|$ is a unit matrix appertaining to the group of $\phi(\omega)$, $|\psi| \times |A| \times |\psi|^{-1}$ is a unit matrix appertaining to the group of $\psi(\omega)$: we have in fact

$$|\psi| \times |\sigma|^{2\mu_1} \times |\tau|^{2\nu_1} \times \ \dots \ \times |\psi|^{-1} = |\tau|^{-2\mu_1} \times |\sigma|^{-2\nu_1} \times \ \dots .$$

For the reduced space of $\psi(\omega)$ we may take the inverse of the reduced space of $\phi(\omega)$; viz. the space bounded by the semicircles

$$\begin{aligned} V_\nu^{-1} &= -[2, \ -1+4\nu, \ 8\nu^2-4\nu], \\ V_\nu \ &= \ \ [2, \ \ 1+4\nu, \ 8\nu^2+4\nu] \\ &\qquad (\nu = -3, \ -2, \ \dots, \ 2, \ 3), \\ V_4^{-1} &= -[2, 15, 112], \quad V_4 = [2, \ -15, 112], \\ V_0^{-1} &= \ \ [0, 1, 16], \qquad V_0 = [0, \ -1, 16]; \end{aligned}$$

these eighteen boundaries being equivalent in pairs, viz. $V_\nu^{-1} \times |V_\nu| = V_\nu$, where

$$V_\nu = |\psi| \times |U_\nu| \times |\psi|^{-1};$$

so that

$$V_0 = |\tau|^{-16} = \begin{vmatrix} 1, & -16 \\ 0, & 1 \end{vmatrix}, \quad V_4 = |\tau|^{-8} \times |\sigma|^2 \times |\tau|^{-8} = \begin{vmatrix} -15, & 112 \\ 2, & -15 \end{vmatrix}$$

$$V_\nu = |\tau|^{-2\nu} \times |\sigma|^2 \times |\tau|^{2\nu} = \begin{vmatrix} 1-4\nu, & -8\nu^2 \\ 2, & 1+4\nu \end{vmatrix}.$$

When the reduced space is mapped by the equation $\psi(\omega) = X + iY$, the

boundaries V_ν are represented by the same lines as the boundaries U_ν in the theory of the function $\phi(\omega)$; and the 'equation of passage' is $\omega = V_\nu \times \Omega$, ω and Ω being the reduced values of ω immediately before and immediately after the passage $X+iY$ from right to left (as seen from 0) across the boundary V_μ.

79. *Theory of* $\dfrac{\phi(\omega)}{\psi(\omega)}$.

The function $\dfrac{\phi(\omega)}{\psi(\omega)}$ is unchanged by the substitution of $\dfrac{c+d\omega}{a+b\omega}$, if

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = |\sigma|^{2\mu_1} \times |\tau|^{2\nu_1} \times \dots$$

is a unit matrix of type (1) satisfying the conditions

$$b+c \equiv 0, \text{ mod } 16, \quad \text{or} \quad \Sigma\mu + \Sigma\nu \equiv 0, \text{ mod } 8.$$

If $|A|$, $|B|$ are unit matrices appertaining respectively to the groups of $\phi(\omega)$, $\psi(\omega)$,

$$|\sigma| \times |A| \times |\sigma|^{-1}, \quad |\sigma|^{-1} \times |A| \times |\sigma|,$$
$$|\tau| \times |B| \times |\tau|^{-1}, \quad |\tau|^{-1} \times |B| \times |\tau|$$

appertain to the group of $\dfrac{\phi(\omega)}{\psi(\omega)}$. For the reduced space of $\dfrac{\phi(\omega)}{\psi(\omega)}$ we may take the reduced space of $\phi(\omega)$. The sixteen semicircular boundaries

$$[2\mu^2+2\mu, \ -2\mu-1, \ 2], \ \mu = -8, \dots, +7,$$

are equivalent in pairs; viz.

$$W_\nu^{-1} = -[8\nu^2+4\nu, \ -4\nu-1, \ 2], \qquad \mu = 2\nu,$$

is transformed into

$$W_\nu = [8\nu^2+12\nu+4, \ -4\nu-3, \ 2], \ \mu = 2\nu+1,$$

by $$|W_\nu| = |\sigma|^{2\nu} \times |\tau|^2 \times |\sigma|^{-2\nu-2}, \quad \nu = -4, \dots, +3.$$

Similarly the rectilineal boundary $W^{-1} = [16, 1, 0]$ is transformed into

$$W = -[16, \ -1, \ 0] \text{ by } |W| = |\sigma|^{16}.$$

In the figure defined by the equation

$$X+iY = \frac{\phi(\omega)}{\psi(\omega)} = e^{\frac{1}{8}i\pi} \times \phi(-1+\omega),$$

W, from ∞ to -8, is represented by the axis of X from 0 to ∞, and W, from $2\nu+1$ to $2\nu+2$, or W_ν^{-1}, from $2\nu+1$ to 2ν, is represented by $Re^{\frac{1}{8}(2\nu+1)i\pi}$ from $R=1$ to $R=\infty$: the 'equation of passage' when $X+iY$ traverses the boundary from right to left, as seen from 0, being $\omega=|W_\nu|\times\Omega$.

80. *Theory of* $\phi(\omega)$. *The Subclasses of Primarily Equivalent Quadratic Forms.*

The number of subclasses of quadratic forms contained in any given Gaussian class is ascertained as follows :

I. Properly Primitive Classes.

Let v_1 be the least number U which satisfies the equation $T^2-\Delta U^2=+1$, v^2 the least even number satisfying that equation, so that $v_2=v_1$ if v_1 is even, $v_2=2\tau_1 v_1$ if v_1 is uneven ; let also $\sigma=2$ or 1 according as v_1 is even or uneven ; let $\nu=2$, 4, 8, according as $\frac{1}{4}v_2$ is uneven, unevenly even, or evenly even. Then any Gaussian class contains $\sigma\times\nu$ subclasses of each of the types (A) and (C) of Art. 41, I. ; 4σ subclasses of the type (B), if $\Delta\equiv 1$, mod 4, $\nu=2$; and 8σ subclasses of the type (B) in every other case. When $\sigma=2$, the subclasses here considered are distributed equally between the two subclasses defined in Art. 41, I. (b) as appertaining to each of the three types.

II. Improperly Primitive Classes.

(1) If $\Delta\equiv 5$, mod 8, each Gaussian class contains 48 or 16 subclasses according as the least numbers satisfying the equation $T^2-\Delta U^2=4$ are even or uneven ; viz. each subclass of Art. 41, II. (2), contains eight of the subclasses considered here.

(2) If $\Delta\equiv 1$, mod 8, each Gaussian class contains sixteen subclasses of the type (C') of Art. 41, II. (1), and sixteen or eight of the types (A') and (B'), according as $v_1\equiv 0$, or $\equiv 4$, mod 8 : viz. each of the subclasses there considered contains eight of the subclasses here considered, if it is of type C', and eight or four (as the case may be) if it is of either of the types (A') or (B').

The demonstration of these assertions may be obtained by considering separately each subclass of Art. 41, II., and examining the character of its automorphics. For example, let $(A, B, C)=f$ be a properly primitive form of type (B). If v_1 is even, the automorphics $\begin{vmatrix} T-Bu, & -Cu \\ Au & , T+Bu \end{vmatrix}$ are all primary because $T\equiv\pm 1$, mod 8, $u\equiv 0$, mod 4, $B\equiv 1$, mod 2, $A\equiv 2$, mod 4, whence $Au\equiv(\tau+Bu)^2-1$, mod 16. The sixteen substitutions $|\sigma|^{2\mu}$, $|\tau|\times|\sigma|^{2\mu}$, which

alone transform f into forms of the same type (B), give in this case non-equivalent forms; if for example $f \times |\tau| \times |\sigma|^{2\mu}$ is transformed into $f \times |\tau| \times |\sigma|^{2\nu}$ by a primary matrix $|v|$, $|\tau| \times |\sigma|^{2\mu} \times |v| \times |\sigma|^{-2\nu} \times |\tau|^{-1}$ is an automorphic of f: this matrix is therefore primary, whence also $|\sigma|^{2\mu} \times |v| \times |\sigma|^{-2\nu}$ is primary, or $2\mu \equiv 2\nu$, mod 16. If v_1 is uneven, the fundamental automorphic is of the type τ, and may be represented by the formula $|\nu| \times |\sigma|^{2\lambda} \times |\tau|^{-1}$, $|\nu|$ being a primary unit and λ an uneven number determined by the congruence $A - 2\lambda B \equiv B^2 - 1$, mod 16, or $A + 6\lambda B \equiv B^2 - 1$, mod 16, according as Δ is $\equiv 3$, or $\equiv -5$, mod 16. In this case the eight forms $f \times |\sigma|^{2\mu}$ are non-equivalent to one another, but are respectively equivalent to the forms $f \times |\tau| \times |\sigma|^{2\mu - 2\lambda}$; because the form $f \times |\tau| \times |\sigma|^{2\mu - 2\lambda}$ is the same as the form $f \times |v| \times |\sigma|^{2\mu}$, which is transformed into $f \times |\sigma|^{2\mu}$ by the primary matrix $|\sigma|^{-2\mu} \times |v|^{-1} \times |\sigma|^{2\mu}$.

81. *Theory of $\phi(\omega)$. The Modular Equations.*

Let $|A|$ represent any term of a system of primary and primitive matrices of determinant Δ; and let $\Omega = |A| \times \omega$; any integral and symmetrical function Σ of the $\sigma'(\Delta)$ quantities $\phi(\Omega)$ is a rational and integral function of $\phi(\omega)$.

For if $\phi(\omega)$ describe any closed contour in the plane XY, traversing in succession the boundaries U, U', U'', ..., μ, μ', μ'' times, we have

$$\omega = U^{\mu} \times U'^{\mu'} \times U''^{\mu''} \times \ldots \times \omega_1, = |\alpha|^{-1} \times \omega_1,$$

ω_1 denoting the final value of ω, and $|\alpha|^{-1}$ a primary unit. Hence the $\sigma(\Delta)$ quantities $\phi(\Omega_1) = \phi(|A| \times |\alpha| \times \omega)$ are the same as the quantities $\phi(\Omega) = \phi(|A| \times \omega)$; *i. e.* Σ is a one-valued function of $\phi(\omega)$ throughout the whole plane; the demonstration may be completed as in Art. 52.

This theorem establishes the existence of a modular equation of the order $N = \sigma'(\Delta)$ between $v = \phi(\Omega)$ and $u = \phi(\omega)$; this equation, which we shall represent by

$$f(v, u) = \Pi(v - \phi(\Omega)) = 0,$$

is characterized by the following properties:

(i.) It may be written in the form $v^N + \Sigma a_{i,j} v^i u^j + \left(\frac{2}{\Delta}\right) u^N = 0$, the limits of i and j being 1 and $N-1$, and the coefficients $a_{i,j}$ satisfying the relations

$$a_{i,j} = \left(\frac{2}{\Delta}\right) a_{N-i,N-j} = \left(\frac{2}{\Delta}\right) a_{j,i}.$$

Thus the matrix $|a_{i,j}|$ is symmetrical or skew symmetrical with regard to both its principal diagonals according as $\Delta \equiv \pm 1$, or $\equiv \pm 3$, mod 8; and the equation is not changed by interchanging u and $\left(\frac{2}{\Delta}\right)v$, or by writing for u and v their reciprocals; viz.

$$f(v, u) = \left(\frac{2}{\Delta}\right) f\left(u, \left(\frac{2}{\Delta}\right)v\right) = \left(\frac{2}{\Delta}\right) u^N v^N f\left(\frac{1}{v}, \frac{1}{u}\right).$$

(ii.) The equation is not altered if we write $e^{\frac{1}{4}si\pi}u$ for u, and $e^{\frac{1}{4}\Delta si\pi}v$ for v: viz.

$$f(e^{\frac{1}{4}\Delta si\pi}v,\ e^{\frac{1}{4}si\pi}u) = e^{\frac{1}{4}\Delta si\pi} f(v, u).$$

Thus the coefficients $a_{i,j}$ are zero, except when $i + \Delta j \equiv \Delta i + j \equiv N$, mod 8.

The equation between $e^{\frac{1}{8}(2s+1)i\pi} \times u$ and $e^{\frac{1}{8}(2s+1)\Delta i\pi} \times v$ is obtained by changing in $f(u, v)$ the signs of those terms in which $\frac{1}{8}(\Delta i + j - N)$ is uneven.

If $u = e^{\frac{1}{4}si\pi}$ the values of v are all eighth roots of unity: viz. if g, g' are conjugate divisors of Δ, there are $f'(g')$ values of v (see Art. 63) equal to $\left(\frac{2}{g}\right) e^{\frac{1}{4}\Delta si\pi}$.

Of the $48 \times \sigma'(\Delta)$ values of $\phi(\Omega)$ determined by the equation

$$\Omega = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times \omega,$$

where $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ is any primitive matrix of determinant Δ, there are $8 \times \sigma'(\Delta)$, which correspond to matrices of the types $|\sigma|^{2\mu}$, $\mu = 0,\ 4,\ \pm 1,\ \pm 2,\ \pm 3$, and which satisfy, as we have seen, the eight equations $f(u, e^{-\frac{1}{4}\mu i\pi}v) = 0$. Again, $8 \times \sigma'(\Delta)$ of the matrices $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ are of the eight types $|\sigma|^{2\mu} \times |\tau|$; and the corresponding values of $\phi(\Omega)$ satisfy the eight equations $f\left(u, \frac{e^{\frac{1}{4}\mu i\pi}}{v}\right) = 0$. But the values of $\phi(\Omega)$, corresponding to each of the remaining sets of $8 \times \sigma'(\Delta)$ matrices, satisfy equations of the order $8 \times \sigma'(\Delta)$ which do not decompose into factors of the order $\sigma'(\Delta)$: viz. the values corresponding to matrices of the types $|\psi|$, $|\rho|$, $|\sigma|$, $|\rho|^2$, respectively satisfy the equations

$$F(\phi^8(\omega),\ 1 - \phi^8(\Omega)) = 0,$$

$$F\left(\phi^8(\omega),\ \frac{1}{1 - \phi^8(\Omega)}\right) = 0,$$

$$F\left(\phi^8(\omega), \frac{\phi^8(\Omega)}{\phi^8(\Omega)-1}\right)=0,$$

$$F\left(\phi^8(\omega), \frac{\phi^8(\Omega)-1}{\phi^8(\Omega)}\right)=0.$$

82. *Theory of $\phi(\omega)$. Representation of Rational Semicircles by the Modular Curves.*

We have next to consider the representation of the semicircles of determinant Δ by the modular curves.

I. The modular curves representing improperly primitive semicircles.

Theorem. If $a=e^{\frac{1}{4}i\pi}$, and $f(u, v)=0$ is the modular equation between $u=\phi(\omega)$ and $v=\phi(\Omega)$,

The eight equations $f\left(x+iy, \frac{x-iy}{a^\lambda}\right)=0$, $\lambda=0, \pm1, \pm2, \pm3, 4$ are real, if $\Delta\equiv1$, mod 8; and the curve $[\lambda+2]$ is the curve $[\lambda]$ turned through a negative angle $\frac{1}{4}\pi$.

The four equations $f\left(x+iy, \frac{x-iy}{a^{2\lambda+1}}\right)=0$, $\lambda=0, \pm1, 2$, are real if $\Delta\equiv5$, mod 8; and the curve $[2\lambda+1]$ is the curve $[2\lambda-1]$ turned through a positive angle $\frac{1}{4}\pi$.

The two equations $F\left(x+iy, \frac{x-iy}{\pm i}\right)=0$ if $\Delta\equiv3$, mod 8, are real; but the corresponding curves are imaginary.

The two equations $F\left(x+iy, \frac{x-iy}{\pm1}\right)=0$ are real, if $\Delta\equiv7$, mod 8; the corresponding curves are imaginary.

(i.) Let $\Delta\equiv1$, mod 8. The sixteen subclasses of semicircles of type (C') contained in any Gaussian class are represented by the four curves

$$f\left(x+iy, \frac{x-iy}{a^\lambda}\right)=0, \quad \lambda=1, 3, 5, 7;$$

viz. if C be a semicircle $[A, B, C]$ satisfying the congruence $A-2B\equiv B^2-1$, mod 16, the semicircles

C,	$C\times\lvert\sigma\rvert^8$,	$C\times\lvert\tau\rvert^{-1}\times\lvert\sigma\rvert^2$,	$C\times\lvert\tau\rvert^{-1}\times\lvert\sigma\rvert^{-6}$;
$C\times\lvert\sigma\rvert^2$,	$C\times\lvert\sigma\rvert^{-6}$,	$C\times\lvert\tau\rvert^{-1}\times\lvert\sigma\rvert^4$,	$C\times\lvert\tau\rvert^{-1}\times\lvert\sigma\rvert^{-4}$;
$C\times\lvert\sigma\rvert^4$,	$C\times\lvert\sigma\rvert^{-4}$,	$C\times\lvert\tau\rvert^{-1}\times\lvert\sigma\rvert^6$,	$C\times\lvert\tau\rvert^{-1}\times\lvert\sigma\rvert^{-2}$;
$C\times\lvert\sigma\rvert^6$,	$C\times\lvert\sigma\rvert^{-2}$,	$C\times\lvert\tau\rvert^{-1}\times\lvert\sigma\rvert^8$,	$C\times\lvert\tau\rvert^{-1}$;

are respectively represented by the curves

$$[\lambda = 1], \quad [\lambda = 3], \quad [\lambda = 5], \quad [\lambda = 7].$$

For any given semicircle $[A, B, C]$ of the type (C), the index of the equation by which the semicircle is represented is given by the congruence

$$A - 2\lambda B \equiv B^2 - 1, \text{ mod } 16.$$

The subclasses of semicircles of type (A), which are eight or sixteen in number according as $\frac{1}{4}\nu$ is uneven or even, are represented by the curves $[\lambda = 2\nu]$ in the following manner :—Let $(A, B, C) = C$ be a semicircle of type (A), satisfying the congruences

$$A \equiv B^2 - 1 \equiv C, \quad \text{mod } 16;$$

then the semicircles

C,	$C \times \lvert\psi\rvert$,	$C \times \lvert\sigma\rvert^8$,	$C \times \lvert\psi\rvert \times \lvert\sigma\rvert^8$,
$C \times \lvert\sigma\rvert^2$,	$C \times \lvert\psi\rvert \times \lvert\sigma\rvert^2$,	$C \times \lvert\sigma\rvert^{-6}$,	$C \times \lvert\psi\rvert \times \lvert\sigma\rvert^{-6}$,
$C \times \lvert\sigma\rvert^4$,	$C \times \lvert\psi\rvert \times \lvert\sigma\rvert^4$,	$C \times \lvert\sigma\rvert^{-4}$,	$C \times \lvert\psi\rvert \times \lvert\sigma\rvert^{-4}$,
$C \times \lvert\sigma\rvert^6$,	$C \times \lvert\psi\rvert \times \lvert\sigma\rvert^6$,	$C \times \lvert\sigma\rvert^{-2}$,	$C \times \lvert\psi\rvert \times \lvert\sigma\rvert^{-2}$,

are respectively represented by the curves

$$[\lambda = 0], \quad [\lambda = 2], \quad [\lambda = 4], \quad [\lambda = 6].$$

Lastly, let C be a semicircle of type (B), satisfying the congruences $A \equiv B^2 - 1$, mod 16, $2B + C \equiv 0$, mod 16, the semicircles

C,	$C \times \lvert\sigma\rvert$,	$C \times \lvert\sigma\rvert^8$,	$C \times \lvert\sigma\rvert^{-7}$,
$C \times \lvert\sigma\rvert^2$,	$C \times \lvert\sigma\rvert^3$,	$C \times \lvert\sigma\rvert^{-6}$,	$C \times \lvert\sigma\rvert^{-5}$,
$C \times \lvert\sigma\rvert^4$,	$C \times \lvert\sigma\rvert^5$,	$C \times \lvert\sigma\rvert^{-4}$,	$C \times \lvert\sigma\rvert^{-3}$,
$C \times \lvert\sigma\rvert^6$,	$C \times \lvert\sigma\rvert^7$,	$C \times \lvert\sigma\rvert^{-2}$,	$C \times \lvert\sigma\rvert^{-1}$,

are respectively represented by the curves

$$[\lambda = 0], \quad [\lambda = 2], \quad [\lambda = 4], \quad [\lambda = 6].$$

In these Tables the semicircles on the right of the vertical line are identical with the semicircles on the left, where $\frac{1}{4}\nu_1$ is uneven. In both cases the index of the equation satisfied by any semicircle $[A, B, C]$ is given by the congruence $A - 2\lambda B \equiv B^2 - 1$.

(ii.) Let $\Delta \equiv 5$, mod 8.

(1) Let the equation $T^2 - Du^2 = 4$ admit of uneven solutions; and let C be a form satisfying the congruences

$$A + 6B \equiv B^2 - 1, \qquad A + C \equiv 0, \mod 16;$$

then the subclasses

$$\begin{array}{llll} C, & C\times|\psi|, & C\times|\sigma|^8, & C\times|\psi|\times|\sigma|^8, \\ C\times|\sigma|^{-2}, & C\times|\psi|\times|\sigma|^{-2}, & C\times|\sigma|^6, & C\times|\psi|\times|\sigma|^6, \\ C\times|\sigma|^{-4}, & C\times|\psi|\times|\sigma|^{-4}, & C\times|\sigma|^4, & C\times|\psi|\times|\sigma|^4, \\ C\times|\sigma|^{-6}, & C\times|\psi|\times|\sigma|^{-6}, & C\times|\sigma|^2, & C\times|\psi|\times|\sigma|^2, \end{array}$$

are represented respectively by the curves

$$[\lambda = 1], \quad [\lambda = 3], \quad [\lambda = 5], \quad [\lambda = 7].$$

(2) Let the equation $T^2 - Du^2 = 4$ admit of no uneven solutions; and C be a form satisfying the same congruences as in (1); also let

$$|\chi| = |\rho|\times|\sigma|^{-2}\times|\tau|^2 \text{ or } |\rho|\times|\sigma|^2\times|\tau|^{-2}$$

according as $\Delta \equiv 5$, or 13, mod 16; then the forms symbolically represented by

$$\begin{aligned} &C\times[1+\psi]\times[1+\chi+\chi^2]\times[1+\sigma^8], \\ &C\times[1+\psi]\times[1+\chi+\chi^2]\times[1+\sigma^8]\times\sigma^{-2}, \\ &C\times[1+\psi]\times[1+\chi+\chi^2]\times[1+\sigma^8]\times\sigma^{-4}, \\ &C\times[1+\psi]\times[1+\chi+\chi^2]\times[1+\sigma^8]\times\sigma^{-6}, \end{aligned}$$

are respectively represented by the curves

$$[\lambda = 1], \quad [\lambda = 3], \quad [\lambda = 5], \quad [\lambda = 7].$$

In both cases the index of the equation satisfied by any semicircle $[A, B, C]$ is given by the congruence $A + 6\lambda B \equiv B^2 - 1$,

$$|\rho| = \begin{vmatrix} 1, & 1 \\ -1, & 0 \end{vmatrix}; \quad |\rho|\times|\sigma|^{-2}\times|\tau|^2 = -\begin{vmatrix} 1, & 1 \\ 1, & 2 \end{vmatrix}; \quad |\rho|\times|\sigma|^2\times|\tau|^{-2} = |\rho| = \begin{vmatrix} 0, & 1 \\ -1, & 1 \end{vmatrix}.$$

II. The modular curves representing properly primitive semicircles of type B, viz. $\equiv [0, 1, 1]$, mod 2.

Theorem. The equations

$$f\left(x+iy, \frac{a^\lambda}{x+iy}\right) = 0$$

are real (i.) if $\Delta \equiv 1$, $\lambda \equiv 4, 0$, mod 8; (ii.) if $\Delta \equiv 5$, $\lambda \equiv 2, 6$, mod 8; (iii.) if $\Delta \equiv 3$, $\lambda \equiv 1, 3, 5, 7$, mod 8; (iv.) if $\Delta \equiv 7$, then for all values of λ, mod 8.

(1) Let $\Delta \equiv 3$, mod 8.

Let $[A, B, C]$ be a semicircle satisfying the congruence

$$A + 6B \equiv B^2 - 1, \quad \text{mod } 16;$$

then the semicircles

$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{-2}$,	C,	$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{6}$,	$C\times\lvert\sigma\rvert^{8}$,
$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{4}$,	$C\times\lvert\sigma\rvert^{6}$,	$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{-4}$,	$C\times\lvert\sigma\rvert^{-2}$,
$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{-6}$,	$C\times\lvert\sigma\rvert^{-4}$,	$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{2}$,	$C\times\lvert\sigma\rvert^{4}$,
$C\times\lvert\tau\rvert$,	$C\times\lvert\sigma\rvert^{2}$,	$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{8}$,	$C\times\lvert\sigma\rvert^{-6}$,

are represented by the curves

$$[\lambda = 1], \quad [\lambda = 3], \quad [\lambda = 5], \quad [\lambda = 7]$$

respectively, the semicircles on the right of the vertical line being equivalent to those on the left, if the least solution of $T^2 - \Delta U^2 = 1$ is uneven; since in this case C has automorphics of the type $\lvert\sigma\rvert^{-\Delta-3}\times\lvert\tau\rvert^{-1}$.

(2) Let $\Delta \equiv 7$, mod 8, and let C or $[A, B, C]$ be a semicircle satisfying the congruence

$$A - 2B \equiv B^2 - 1, \quad \text{mod } 16;$$

then the semicircles

C,	$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{2}$,	$C\times\lvert\sigma\rvert^{8}$,	$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{-6}$,
$C\times\lvert\sigma\rvert^{2}$,	$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{4}$,	$C\times\lvert\sigma\rvert^{-6}$,	$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{-4}$,
$C\times\lvert\sigma\rvert^{4}$,	$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{5}$,	$C\times\lvert\sigma\rvert^{-4}$,	$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{-2}$,
$C\times\lvert\sigma\rvert^{6}$,	$C\times\lvert\tau\rvert\times\lvert\sigma\rvert^{8}$,	$C\times\lvert\sigma\rvert^{-2}$,	$C\times\lvert\tau\rvert$,

are represented respectively by the curves

$$[\lambda = 1], \quad [\lambda = 3], \quad [\lambda = 5], \quad [\lambda = 7].$$

The four remaining curves (though their equations are real) are imaginary; they have only a certain number of real conjugate points.

(3) Let $\Delta \equiv 1$, mod 8.

The curve $[\lambda = 0]$ is real, and the curve $[\lambda = 4]$ imaginary, or *vice versâ*, according as $\Delta \equiv 1$, or $\equiv 9$, mod 16; the semicircles of type (B) all satisfying

the congruence $A \equiv B^2 - 1$, mod 16, in the former case, and the congruence $A - 8B \equiv B^2 - 1$ in the later; in both cases alike they are all represented by one and the same curve. If C be any semicircle of a given Gaussian class, the others are symbolically given by the formula

$$C \times [1 + \tau] \times [1 + \sigma^2 + \sigma^4 + \sigma^6] \times [1 + \sigma^8],$$

the factor $1 + \sigma^8$ being omitted if $\frac{1}{4}v_1$ is even.

(4) Let $\Delta \equiv 5$, mod 8.

If C satisfies the congruence

$$A + 4B \equiv B^2 - 1, \quad \text{mod } 16,$$

the semicircles

$$C \times [1 + \tau\sigma^2] \times [1 + \sigma^4] \times [1 + \sigma^8]$$

satisfy the same congruence, and are represented by the curve $[\lambda = 2]$; the semicircles

$$C \times [\tau + \sigma^2] \times [1 + \sigma^4] \times [1 + \sigma^8]$$

satisfy the congruence

$$A - 4B \equiv B^2 - 1, \quad \text{mod } 16,$$

and are represented by the curve $[\lambda = 6]$.

III. The modular curve representing properly primitive semicircles of the type (A).

All these semicircles are represented by the equation

$$f(1 - (x + iy)^8, \quad (x + iy)^8) = 0,$$

which is derived from

$$f(1 - (X + iY), \quad X + iY) = 0,$$

by the transformation

$$R = r^8, \quad \Theta = 8\theta.$$

The above equation is irreducible, but breaks up into 8 factors if we adjoin the radical $\sqrt[8]{1 - (x + iy)^8}$.

IV. Lastly, the modular curve

$$f\left[(x + iy)^8, \quad \frac{(x - iy)^8}{(x - iy)^8 - 1}\right] = 0$$

represents all the properly primitive semicircles of type (B).

The limits of the present Memoir preclude any detailed description of the

geometric theory of the modular function $\chi(\omega)$. The reduced space corresponding to this function is the reduced space of Art. 36 repeated forty-eight times; and the general character of the theory closely resembles that of the functions $\phi(\omega)$ and $\psi(\omega)$.

Arts. 83–88. THEORY OF THE MODULAR FUNCTION $T(\omega)=[1-\chi^{24}(\omega)^3]\div\chi^{48}(\omega)$.

83. *Transformation of the Reduced Space.*

In the theory of the primary unit matrices of Art. 21 the reduced space consists of six regions (Art. 38); and any one of them may be taken as the reduced space when the unit matrices considered are no longer subject to the condition of being primary. We take as the reduced space the region BB' inverse to AA' (see fig. 2); we thus substitute the condition $c\leqq a$ for the condition $a\leqq c$ in the definition (Art. 36) of a reduced form of negative determinant. We include in the reduced space BB' the boundaries which lie to the right of the axis of y; and we exclude the equivalent boundaries which lie to the left of the axis of y.

The equation
$$T(|\alpha|\times\omega)=T(\omega), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$
where $|\alpha|$ is any unit matrix, shows that the reduced space is mapped homœomerically (see Art. 48) on the plane XY by the equation
$$T(\omega)=X+iY, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (ii)$$
the correspondence between the two spaces being one-to-one. For it follows from the theory of the functions $\Phi(\omega)$ and $\Psi(\omega)$ that for any given values of X and Y there always exists one, and only one, reduced value of ω satisfying the equation (ii).

It will be found that
$$\left.\begin{gathered} T(0)=T(1)=T(\infty)=\infty, \\ \operatorname{Lim} T(0)\times[1-\Phi^2(\omega)]=1, \quad \operatorname{Lim} T(1)\div\Phi^2(1)=1, \quad \operatorname{Lim} T(\infty)\times\Phi^2(\infty)=1; \end{gathered}\right\} \quad (iii)$$
$$T(i)=\frac{27}{4}, \quad T'(i)=0, \quad T''(i)=-\frac{3^4}{2^2}K^4(i); \quad . \quad . \quad . \quad . \quad (iv)$$
$$T(\rho)=T'(\rho)=T''(\rho)=0, \quad T'''(\rho)=\frac{2^6.3^{\frac{5}{2}}}{\pi^3}\times K^6(\rho); \quad . \quad . \quad . \quad (v)$$
where $\rho=\frac{1}{2}(\pm1+i\sqrt{3})$.

Since
$$\frac{T'(\omega)}{T(\omega)}=\frac{(2k^2-1)(2-k^2)(1+k^2)}{1-k^2+k^4}\times-\frac{4i}{\pi}K^2(\omega), \quad . \quad . \quad . \quad . \quad . \quad (vi)$$

we infer that $T'(\omega)$ is nowhere infinite in the reduced space, except at the point $\omega = \infty$, and nowhere zero, except at the points $\omega = i$, $\omega = \rho$. Thus the only points at a finite distance in the plane XY which present any singularity in the homœomeric transformation are the points $\frac{27}{4}$ and 0; angles at these points being respectively double and triple of the corresponding angles at i and ρ. The arc Q_1, from i to ρ, is represented on the axis of X by the segment from $\frac{27}{4}$ to 0; the line P_1, from ρ to ∞, by the segment from 0 to $-\infty$; the axis of y, from i to ∞, by the segment from $\frac{27}{4}$ to $+\infty$: the regions of the reduced space to the right and left of the axis of y are represented respectively by the regions below and above the axis of X: this is readily seen by considering values of ω lying near the cornicular point at an infinite distance. Hence, if in the plane XY we carry a slit along the axis of X from $-\infty$ to $\frac{27}{4}$, the plane so divided corresponds with continuity to the reduced space; but the continuity ceases if the point $X+iY$ traverses the slit. Thus, if $X+iY$ describes (either in the positive or negative direction) a closed contour round $\frac{27}{4}$, traversing the segment $(0, \frac{27}{4})$, we have $\omega_2 = -\dfrac{1}{\omega_1}$, ω varying continuously with $X+iY$, and ω_1, ω_2 being the initial and final values of ω; if $X+iY$ describes a closed contour round 0 and $\frac{27}{4}$ simultaneously, traversing the segment $(-\infty, 0)$, we have $\omega_2 = \pm 1 + \omega_1$, the upper or lower sign being taken according as the contour is positive or negative with regard to the infinite region of the plane XY; lastly, if $X+iY$ describes a closed contour round 0, traversing both the segments $(-\infty, 0)$ and $(0, \frac{27}{4})$, we have

$$\omega_2 = \begin{vmatrix} 0, & -1 \\ 1, & 0 \end{vmatrix} \times \begin{vmatrix} 1, & 0 \\ \epsilon, & 1 \end{vmatrix} \times \omega_1 = \begin{vmatrix} -\epsilon, & -1 \\ 1, & 0 \end{vmatrix} \times \omega_1,$$

or

$$\omega_2 = \begin{vmatrix} 1, & 0 \\ \epsilon, & 1 \end{vmatrix} \times \begin{vmatrix} 0, & -1 \\ 1, & 0 \end{vmatrix} \times \omega_1 = \begin{vmatrix} 0, & -1 \\ 1, & -\epsilon \end{vmatrix} \times \omega_1,$$

according as the segment first traversed is $(-\infty, 0)$ or $(0, \frac{27}{4})$; ϵ denoting in each case ∓ 1 according as the contour is positive or negative round 0.

84. *The Modular Equations and Curves.*

If $\Omega = |A|^{-1} \times \omega$, where $|A|$ represents any term of a system of primitive matrices of determinant n, non-equivalent by postmultiplication, we can prove, as in Art. 52, that any rational and integral symmetrical function of the $\sigma'(n)$

quantities $T(\Omega)$ is a rational and integral function of $T(\omega)$. Thus, if $x = T(\omega)$, the expression

$$F(y, x) = \Pi[y - T(\Omega)]$$

is rational and integral in x, and $F(y, x) = 0$ is the modular equation between $T(\omega)$ and $T(\Omega)$. This equation is irreducible; it is symmetrical with regard to x and y (see Art. 53); and the numerical coefficients are integral. For, if H_r is the coefficient of y^r in $F(y, x)$, it is evident, from the theory of the modular equation between k^2 and λ^2, that H can be expressed as a rational fraction in terms of k^2; and that the numerical coefficients in the numerator of this fraction are integral numbers, while the denominator is a power of $k^2(1-k^2)$, because $\Pi\lambda^2 = k^{2\sigma' n}$, $\Pi(1-\lambda^2) = (1-k^2)^{\sigma' n}$; whence it follows that when H_r is expressed in terms of $T(\omega)$ the numerical coefficients of the powers of $T(\omega)$ are integral.

If in the modular equation $F(T(\Omega), T(\omega), n) = 0$ we write $T(\omega) = X + iY$, $T(\Omega) = X - iY$, we obtain the equation of a curve in the plane XY, which is the image in that plane of the rational semicircles of determinant n in the plane xy (see Art. 65).

85. *The Rational Semicircles.*

As in Art. 42, every rational semicircle is equivalent to a series of arcs lying within the reduced space: these reduced arcs are of six different types, eight transitive, and two intransitive, characterized by the symbols

$$[P_{-1}, Q_{-1}],\ [P_{-1}, Q_1],\ [P_1, Q_{-1}],\ [P_1, Q_1],$$
$$[Q_{-1}, P_{-1}],\ [Q_1, P_{-1}],\ [Q_{-1}, P_1],\ [Q_1, P_1];$$

and

$$[P_{-1}, P_1],\ [P_1, P_{-1}].$$

The reduced arcs equivalent to any properly primitive semicircle of determinant $+1$ are the boundaries P_1, Q_1; similarly the arc equivalent to any improperly primitive semicircle of determinant $+1$ is the axis of y from i to ∞. Let the series of reduced arcs, equivalent to a given semicircle of determinant D, begin with a transitive arc C_1 of the type $[Q_{-\eta_0}, P_{\epsilon_1}]$; let C_1 be followed (1) by $\mu_1 - 1$ intransitive arcs of the type $[P_{-\epsilon_1}, P_{\epsilon_1}]$, (2) by a transitive arc C_1' of the type $[P_{-\epsilon_1}, Q_{\eta_1}]$, (3) by a transitive arc C_2 of the type $[Q_{-\eta_1}, P_{\epsilon_2}]$, and so on continually. For brevity, we shall attend only to the case in which D is not a square, so that none of the reduced arcs can pass through the cornicular

point at ∞. In this case the series of reduced arcs must form a period repeating itself for ever and beginning with C_1. For (1) the series can never terminate, (2) the number of reduced arcs appertaining to a given determinant is finite, (3) each reduced arc absolutely determines the reduced arcs which precede and follow it. We thus arrive at the automorphic equation

$$C_1 = |\sigma|^{\mu_1\epsilon_1} \times |\psi| \times |\sigma|^{\mu_2\epsilon_2} \times |\psi| \times \ \ldots \ \times |\psi| \times C_1, \quad \ldots \quad \text{(i)}$$

or, which is the same thing, at the development

$$\theta = [\mu_1\epsilon_1, \mu_2\epsilon_2, \ldots, \theta], \quad \ldots \quad \text{(ii)}$$

the continued fraction being subtractive, and θ being that root of the equation

$$a_0 + 2b_0\theta + a_1\theta^2 = 0,$$

associated to the semicircle $C_1 = [a_0, b_0, a_1]$, which is the greater in absolute magnitude.

To determine arithmetically the period of reduced arcs, when the symbol of one of them, which we may suppose to be the transitive semicircle

$$C_1 = [a_0, b_0, a_1],$$

is given, let ϵ_1 be a unit of the same sign as $-\dfrac{b_0}{a_1}$, and let $T_{-1}\,T_{+1}$ be the chord intercepted by C_1 on the indefinite line $\rho_{-1}\rho_{+1}$; if 2λ is the absolute length of this chord, we have

$$\lambda = \tfrac{1}{2}\sqrt{\left(\frac{4D}{a_1^2} - 3\right)}.$$

The figure shows that C_1 will not cease to be reduced if we shift it through as many unit spaces as there are units in the segment, the absolute value of which is

$$\tfrac{1}{2} - \epsilon_1\frac{b_0}{a_1} + \lambda :$$

hence

$$\mu_1 = E\left[\lambda - \epsilon_1\frac{b_0}{a_1}\right], \quad \ldots \quad \text{(1)}$$

E denoting the nearest integer. If $C_2 = [a_1, b_1, a_2]$, we have

$$-b_1 = \mu_1\epsilon_1 a_1 + b_0,$$

whence

$$b_1 + b_0 \equiv 0, \text{ mod } a_1. \quad \ldots \quad \text{(2)}$$

The centres of the semicircles C_1 and C_1' lie on opposite sides of the origin; thus

b_0 and b_1 have the same sign; again, the equation (1) shows that b_1 lies as near as it can to $\frac{1}{2}\sqrt{(4D-3a_1^2)}$, consistently with the congruence (2), the sign of the radical being the same as that of b_0 or b_1. We thus obtain the algorithm

$$b_1+b_0\equiv 0, \text{ mod } a_1; \quad \epsilon_1\mu_1=-\frac{b_0+b_1}{a_1}; \quad a_2=\frac{b_1^2-D}{a_1};$$

$$b_2+b_1\equiv 0, \text{ mod } a_2; \quad \epsilon_2\mu_2=-\frac{b_1+b_2}{a_2}; \quad a_3=\frac{b_2^2-D}{a_2};$$

. . . . . .

the numbers $b_0, b_1, \ldots, b_s$ having all the same sign, and b_s lying as near as it can to $\frac{1}{2}\sqrt{(4D-3a_s^2)}$ taken with that sign.

Example. Let $D=66$, $C_1=[5, 6, -6]$. We find

$C_1=[5, 6, -6]$, $[11, 0, -6]$, $[5, -6, -6]$;

$C_2=[-6, 6, 5]$, $[-13, 1, 5]$, $[-10, -4, 5]$, $[3, -9, 5]$;

$C_3=[5, 9, 3]$, $[-10, 6, 3]$, $[-19, 3, 3]$, $[-22, 0, 3]$,
$[-19, -3, 3]$, $[-10, -6, 3]$, $[5, -9, 3]$;

$C_4=[3, 9, 5]$, $[-10, 4, 5]$, $[-13, -1, 5]$, $[-6, -6, 5]$;

$C_5=C_1$;

and the continued fraction is

$$\frac{6+\sqrt{66}}{6}=2-\cfrac{1}{-3-\cfrac{1}{-6-\cfrac{1}{-3-\cfrac{1}{-2-\ldots}}}}$$

$$=2+\cfrac{1}{3+\cfrac{1}{-6+\cfrac{1}{3+\cfrac{1}{-2+\ldots}}}},$$

the reduced arcs being of the types indicated in the following scheme:

$C_1=[\,Q_1\,P_1\,]$, $[\,P_{-1}P_1\,]$, $C_1'=[P_{-1}Q_{-1}]$;

$C_2=[\,Q_1P_{-1}\,]$, $2\times[\,P_1P_{-1}\,]$, $C_2'=[\,P_1Q_1\,]$;

$C_3=[Q_{-1}P_{-1}]$, $5\times[\,P_1P_{-1}\,]$, $C_3'=[\,P_1Q_1\,]$;

$C_4=[Q_{-1}P_{-1}]$, $2\times[\,P_1P_{-1}\,]$, $C_4'=[\,P_1Q_{-1}\,]$.

The algorithm is of the same form as that of Gauss (Art. 43, *suprà*, Disq. Arith.

Art. 184), but the determination of the coefficients b is less simple ; viz. in the algorithm of Gauss these coefficients are all positive and less than $\sqrt{D}$, lying as close to it as they can consistently with the congruence $b_s + b_{s-i} \equiv 0$, mod a_s. The Gaussian period of forms equivalent to $(5, 6, -6)$ is

$$(5, 6, -6), \quad (-6, 6, 5), \quad (5, 4, -10), \quad (-10, 6, 3),$$
$$(3, 6, -10), \quad (-10, 4, 5), \quad (5, 6, -6), \ldots$$

giving the continued fraction

$$\frac{6+\sqrt{66}}{6} = [2, 2, 1, 4, 1, 2].$$

We may begin the period of reduced arcs with any arc of the period, and may trace it geometrically in either direction ; in the arithmetical theory there is the same freedom of choice. In the algorithm of this article we have begun, for simplicity, with a transitive arc, and have considered it as described in the direction QP. One half of the whole number of transitive arcs (viz. the arcs $C_1, C_2, C_3, \ldots$ one of which comes after each matrix $|\psi|$ in the automorphic (i), or at the beginning of each integral quotient in the development (ii)) are described in the same direction QP; the other half (viz. the arcs $C_1', C_2', \ldots$, which are the inverses of $C_2, C_3, \ldots$ and which immediately precede them) are described in the direction PQ. If we begin the arithmetical development with one of these inverse arcs $C_1', C_2' \ldots$ and consider it as described in the direction QP, we obtain the same series of semicircles in the reverse order, and a continued fraction having the same period of quotients, but also in the reverse order.

86. *Semicircles passing through a Singular Point.*

Some special considerations are needed in the cases in which one of the singular points ρ and i lies on a reduced arc.

(i.) Let $[a, b, c]$ be a semicircle passing through ρ_ϵ, then

$$a + \epsilon b + c = 0, \text{ and } D = a^2 + ac + c^2 :$$

there is therefore a primitive representation of $2D$ by the improperly primitive form $2x^2 + 2xy + 2y^2$. The only numbers D, of which the doubles admit of such representation, are of the type R or $3R$, R being divisible only by primes of the form $3n+1$; if R is divisible by σ different primes of that form, there are

2^σ sets of representations of $2D$ by (2, 1, 2), each set consisting, except when $R=1$, of six representations. Let then $D=a^2+ac+b^2$ be a given representation of $2D$ by (2, 1, 2); the six representations of the set are

$$(\pm a,\ \pm c), \qquad (\pm(a+c),\ \mp a), \qquad (\mp c,\ \pm(a+c)).$$

In one of these representations both the indeterminates are positive; we may therefore suppose that a and c are positive. With these six representations we shall also consider the six representations of the opposite set, viz.

$$(\pm c,\ \pm a), \qquad (\mp a,\ \pm(a+c)), \qquad (\pm(a+c),\ \mp c).$$

These twelve representations give rise to two systems of semicircles of determinant D, each containing six semicircles, viz. three passing through ρ and three passing through ρ_{-1}, the two systems being symmetrically placed with regard to the axis of x, and the six semicircles of each system being equivalent. The semicircles of one of the two systems are

$$[a+c,\quad a,\ -c], \qquad [-a,\ a+c,\ -c], \qquad [-c,\ -a-c,\ -a],$$
$$[a+c,\ -c,\ -a], \qquad [-c,\ -a,\ a+c], \qquad [-a,\quad c,\quad a+c];$$

we shall denote them by the symbols Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, Γ_6, taken in order. The semicircles of the other system are derived from these by changing the signs of the middle coefficients. The semicircles Γ_5 and Γ_6 have radii less than unity; they do not enter the reduced space, and do not belong to the period: Γ_2 and Γ_3 do not enter the reduced space, but they have radii greater than unity, and appear in the period as two consecutive transitive circles. Lastly, Γ_1 and Γ_4 actually enter the reduced space, and are to be regarded as intransitive circles of the period, respectively preceding Γ_2 and following Γ_3; the equations connecting these semicircles being

$$\Gamma_1=1+\Gamma_2, \qquad \Gamma_2=-\frac{1}{\Gamma_3}, \qquad \Gamma_3=1+\Gamma_4.$$

Whenever the determination of b_s, in the algorithm of Art. 85, is ambiguous, *i. e.* whenever the radical $\frac{1}{2}\sqrt{4D-3a_s^2}$ lies evenly between two values of b_s satisfying the congruence

$$b_s+b_{s-1}\equiv 0, \text{ mod } a_s,$$

the ambiguity is to be removed by taking for b_s the greater in absolute magnitude of the two values, and the resulting semicircle will pass through

the singular period ρ. Conversely, whenever the chain of reduced arcs passes through that singular point, the case of ambiguity presents itself in the determination of one of the circles $C_1, C_2, C_3, \ldots$. The four arcs $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are to be regarded as forming a chain which passes but once through the singular point ρ. Thus when these reduced arcs are mapped by $T(\omega)$ on the plane XY the evanescent arcs Γ_2 and Γ_3 contribute nothing to the trace in that plane, and the arcs Γ_1 and Γ_4 are mapped in one continuous line. For, if γ_1, γ_4, q_{-1}, q_{+1} are the tangents to Γ_1, Γ_4, Q_{-1}, Q_{+1} at the point ρ_{+1} or ρ_{-1}, we find from the equations of these semicircles

$$\gamma_1 \rho_1 q_1 + \gamma_4 \rho_{-1} q_{-1} = 60^\circ,$$

the angles considered being internal to the reduced space; whence, since angles at the points ρ are tripled in the plane XY, it follows that the tangents corresponding to γ_1 and γ_4 make at the point 0, with the line $(0, \frac{27}{4})$ and on opposite sides of the line, supplementary angles; so that these tangents are in a straight line.

Example. The transitive semicircle $[3, 8, -5]$ of determinant

$$79 = 7^2 + 3.7 + 3^2$$

gives rise to the period

$$[3, 8, -5], \quad [-5, 7, 6], \quad [6, 5, -9], \quad [-9, 4, 7], \quad [7, 10, 3],$$

the corresponding values of the radical being

$$\tfrac{1}{2}\sqrt{241}, \qquad \tfrac{1}{2}\sqrt{208}, \qquad \tfrac{1}{4}\sqrt{73}, \qquad 6\tfrac{1}{2}, \qquad 8\tfrac{1}{2}.$$

Here $6\frac{1}{2}$ lies evenly between the values 3 and 10, which alike satisfy the congruence $b + 4 \equiv 0$, mod 7. Thus $[7, 10, 3]$ is a transitive semicircle passing through ρ_{-1}; and the four semicircles

$$[-10, -3, 7], \qquad [3, -10, 7], \qquad [7, 10, 3], \qquad [-10, 7, 3]$$

form the chain $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$.

There are evidently as many chains of four semicircles passing through ρ as there are sets of representations of $2D$ by (2, 1, 2). More than one of these chains may belong to the same period, and in particular a symmetrical period must contain an even number (if any) of such chains.

Example. Each of the periods

$$(-4, 5, 3), \qquad (3, 7, 4), \qquad (4, 5, -3), \qquad (-3, 7, -4)$$

of determinant $37 = 4^2 + 4.3 + 3^2$, and

$$(1, 4, -3), \quad (-3, 5, -2), \quad (-2, 5, -3), \quad (-3, 4, 1)$$

of determinant $19 = 3^2 + 3.2 + 2^2$ contains two chains, symmetrical to one another, of the type (Γ). The latter of these periods is *ambiguous*, the former appertains to a determinant for which the equation $T^2 - DU^2 = -1$ is resoluble.

The case $D = 3$, which has been excluded in what precedes, forms no exception to the general theory. There is but one set of representations of 3 by $x^2 + xy + y^2$; and there is but one class of semicircles of determinant 3. The period of this class contains only one semicircle C, viz. [1, 2, 1], the complete series being

$$[1, 2, 1], \quad [-2, 1, 1], \quad [-3, 0, 1], \quad [-2, -1, 1], \quad [1, -2, 1],$$

which includes the chain

$$[-2, -1, -1], \quad [1, -2, 1], \quad [1, 2, 1], \quad [-2, 1, -1].$$

This chain is symmetrical to itself, and the only chain (Γ) possessing that property.

(ii.) Every semicircle passing through the point i is its own inverse: hence the arc of any transitive semicircle C' which arrives at this point is continued in the reduced space by itself described in the opposite direction. Thus every period which contains one semicircle passing through the point i also contains a second, but cannot contain more. The number of semicircles of determinant D which pass through the point i is 2θ, if θ is the number of compositions of D by the addition of two squares prime to one another; viz. if $D = a^2 + b^2$, the two compositions implied by this equation give the four semicircles

$$[a, b, -a], \quad [-a, b, a], \quad [b, a, -b], \quad [-b, a, b].$$

Hence the number of periods containing semicircles which pass through i is θ, of which, when D is uneven, $\frac{1}{2}\theta$ are properly and as many improperly primitive.

The semicircles $[\pm a, b, \mp a]$ respectively intersect the semicircles $[\mp b, a, \pm b]$ at right angles; and as angles at the point i are doubled when the figure is mapped by $T(\omega)$ on the plane YZ, the trace of the semicircle $[\pm a, b, \mp a]$ forms one continuous line with that of the semicircle $[\mp b, a, \pm b]$. But the two semicircles do not in general belong to the same period; viz. when D is uneven, one is properly and the other improperly primitive; and again, whether D is even or uneven, whenever there is an ambiguous period con-

taining a pair of semicircles passing through i, these are of the type $[\pm a, b, \mp a]$, and do not intersect at right angles when D is greater than 2. When, however, D is even, and the equation $t^2 - Du^2 = -1$ is irresoluble, the two extreme forms of the same period may be at right angles to one another, and indeed must be if D is the double of a prime.

87. *The number of Ovals of the Modular Curve.*

Let h, h' represent, as in Art. 41, the numbers of classes of properly and improperly primitive forms of determinant D; let H, H' be the numbers of classes of properly and improperly primitive semicircles; and let θ be the number of compositions of D by the addition of two squares prime to one another, so that $\theta = 0$, if D is divisible by 4 or by any prime of the form $4n+3$, and $\theta = 2^a$ if $D = R$ or $2R$, where R is uneven and divisible by a different primes of the form $4n+1$. Then, if D is uneven,

$$H = \tfrac{1}{2}h + \tfrac{1}{4}\theta,$$

$$H' = \tfrac{1}{2}h' + \tfrac{1}{4}\theta,$$

and, if D is even,

$$H = \tfrac{1}{2}h + \tfrac{1}{2}\theta.$$

For, in general, every class of semicircles $[a, b, c]$ corresponds to two classes of forms (a, b, c) and $(-a, -b, -c)$. But if (a, b, c) and $(-a, -b, -c)$ are equivalent there is only one class of forms corresponding to $[a, b, c]$. There are, when D is uneven, $\frac{1}{2}\theta$ properly primitive classes, and as many improperly primitive classes, which are equivalent to their own negatives. Hence

$$H = \tfrac{1}{2}(h - \tfrac{1}{2}\theta) + \tfrac{1}{2}\theta = \tfrac{1}{2}h + \tfrac{1}{4}\theta, \qquad H' = \tfrac{1}{2}(h' - \tfrac{1}{2}\theta) + \tfrac{1}{2}\theta = \tfrac{1}{2}h' + \tfrac{1}{4}\theta.$$

Similarly, when D is even, there are θ properly primitive classes which are equivalent to their own negatives; so that

$$H = \tfrac{1}{2}(h - \theta) + \theta = \tfrac{1}{2}h + \tfrac{1}{2}\theta.$$

Thus (1) when D is uneven, the whole number of distinct ovals comprised in the modular curve is $\frac{1}{2}h + \frac{1}{2}h'$; (2) when D is even, and the equation $t^2 - Du^2 = -1$ is resoluble, the number is $\frac{1}{2}h$; (3) when D is even, and the equation $t^2 - Du^2 = -1$ is irresoluble, the number is $\frac{1}{2} + h\frac{1}{2}\sigma$, if σ is the number of periods of semicircles containing orthogonal semicircles passing through i. In the two former cases $\frac{1}{2}\theta$ of the ovals have a double point at $(\frac{27}{4})$; in the last

case $\frac{1}{2}(\theta-\sigma)$ have a double point at $(\frac{27}{4})$, and σ pass through that point once and only once.

88. *The Cases of Symmetry.*

In certain cases the periods of semicircles are symmetrical with regard to the axis of y, and consequently the corresponding ovals symmetrical with regard to the axis of X. The principles upon which the discussion of these cases rests will be found in the memoir already cited in Art. 63.

I. Let the equation $t^2-Du^2=-1$ not be resoluble.

If the class of forms (a, b, c) is ambiguous the period of semicircles has a symmetry of the first species with regard to the axis of y. For if $[a, b, c]$ is a semicircle of the period, $[a, -b, c]$, which is equivalent to $[a, b, c]$, is also a semicircle of the period; *i. e.* the period is symmetrical. Let $[a, b, c]$ be a transitive semicircle C; then $[a, -b, c]$ cannot appear in the period as a semicircle C; for if it did so appear the form (a, b, c) would be equivalent to $(-a, b, -c)$, and the equation $t^2-Du^2=-1$ would be resoluble. Hence $[a, -b, c]$ must appear in the period as a semicircle C'; *i. e.* the two symmetrical semicircles $[a, b, c]$ and $[a, -b, c]$ are described in opposite directions.

Conversely, when the period of $[a, b, c]$ has a symmetry of the first species, the period of forms (a, b, c) is ambiguous. For, taking any two symmetrical semicircles of the period, and continuing simultaneously the chain of reduced arcs in opposite (and therefore symmetrical) directions from them, we shall arrive ultimately either at a semicircle $[a, 0, c]$ which is symmetrical to itself and also ambiguous; or at two consecutive circles, symmetrical to one another; in which case also it is evident that the period is ambiguous.

II. Let the equation $t^2-Du^2=-1$ be resoluble.

In this case the ambiguous periods of semicircles, and these only, pass through the point i. Then (1) all the periods have a symmetry of the second species; the ambiguous periods have also a symmetry of the first species, since each period consists of a chain of circles described first in one direction and then backward in the other. Conversely (2) whenever any period has a symmetry of the second species all the periods of that determinant have it, and the equation $t^2-Du^2=-1$ is resoluble. For (1) if $[a, b, c]$ is a transitive circle C, $[-a, b, -c]$ is also a transitive circle C, because the forms (a, b, c) and $(-a, b, -c)$ are properly equivalent. Hence these two symmetrical circles are described in the same direction; or the symmetry is of the second species. It

will be observed that when the period is ambiguous the half-period or chain running from one of the two extreme semicircles to the other has a symmetry of the first species, since the two extreme semicircles are certainly described in opposite directions. (2) Conversely, if two semicircles $[a, b, c]$, $[a, -b, c]$ described in the same direction both belong to the same period, the equation $t^2 - Du^2 = -1$ is resoluble. For, as before, we may suppose $[a, b, c]$ to be a transitive semicircle C; $[a, -b, c]$ is also a transitive semicircle, and as it is described in the same direction QP as $[a, b, c]$ it is a semicircle C; *i.e.* the two forms (a, b, c), $(-a, b, -c)$ are properly equivalent, and $t^2 - Du^2 = -1$ is resoluble.

Thus symmetry of the first species is the mark of ambiguity; symmetry of the second species is the mark of the resolubility of the equation $t^2 - Du^2 = -1$.

Arts. 89–90. The Differential Equation of the Modular Equations and Curves.

89. *Differential Equation of Jacobi.*

We have seen in Art. 12 that $k = \phi^4(\omega)$ satisfies the differential equation

$$2\frac{d^3k}{d\omega^3}\frac{dk}{d\omega} - 3\left(\frac{d^2k}{d\omega^2}\right) = -\frac{1}{k^2}\left(\frac{1+k^2}{1-k^2}\right)^2\left(\frac{dk}{d\omega}\right)^4, \quad \ldots\ldots \quad \text{(i)}$$

and that the complete solution of this equation is $\phi^4\left(\frac{\gamma+\delta\omega}{\alpha+\beta\omega}\right)$, α, β, γ, δ representing any constants whatever.

Hence if λ be any transformed modulus whatever, we have also

$$2\frac{d^3\lambda}{d\omega^3}\frac{d\lambda}{d\omega} - 3\left(\frac{d^2\lambda}{d\omega^2}\right)^2 = -\frac{1}{\lambda^2}\left(\frac{1+\lambda^2}{1-\lambda^2}\right)^2\left(\frac{d\lambda}{d\omega}\right)^4; \quad \ldots\ldots \quad \text{(ii)}$$

and if we eliminate ω from (i) and (ii) we shall obtain a differential equation between k and λ. To effect the elimination, we must write the equations (i) and (ii) in the form which they assume when $d\omega$ is not equicrescent; *i. e.* we must add the terms

$$-\frac{dk^2}{d\omega^2}\left[2\frac{d^3\omega}{d\omega^3} - 3\left(\frac{d^2\omega}{d\omega^2}\right)^2\right], \quad -\frac{d\lambda^2}{d\omega^2}\left[2\frac{d^3\omega}{d\omega^3} - 3\left(\frac{d^2\omega}{d\omega^2}\right)^2\right]$$

to the left-hand members of these equations respectively. On multiplying

(i) by $d\lambda^2$, (ii) by dk^2, and subtracting, the differentials of ω disappear, and we find

$$2\,dk\,d\lambda(d^3k\,d\lambda - d^3\lambda\,dk) - 3\{d\lambda^2(d^2k)^2 - dk(d^2\lambda)^2\}$$
$$+ dk^2\,d\lambda^2\left\{\left(\frac{1+k^2}{1-k^2}\right)^2\frac{dk^2}{k^2} - \left(\frac{1+\lambda^2}{1-\lambda^2}\right)^2\frac{d\lambda^2}{\lambda^2}\right\} = 0 \quad . \quad . \quad \text{(iii)}$$

which is the differential equation between k and λ given by Jacobi.

90. *Transformations of the Equation of Jacobi.*

If z is any function of y, and y any function of x, we have identically

$$\frac{2\frac{dz}{dx}\frac{d^3z}{dx^3} - 3\left(\frac{d^2z}{dx^2}\right)^2}{\left(\frac{dz}{dx}\right)^2} = \frac{2\frac{dz}{dy}\frac{d^3z}{dy^3} - 3\left(\frac{d^2z}{dy^2}\right)^2}{\left(\frac{dz}{dy}\right)^2} + \frac{2\frac{dy}{dx}\frac{d^3y}{dx^3} - 3\left(\frac{d^2y}{dx^2}\right)^2}{\left(\frac{dy}{dx}\right)^2}, \quad . \quad . \quad \text{(i)}$$

as may be verified without difficulty by means of the identity, already employed in Art. 12,

$$\frac{2\frac{dz}{dx}\cdot\frac{d^3z}{dx^3} - 3\left(\frac{d^2z}{dx^2}\right)^2}{\left(\frac{dz}{dx}\right)^2} = 2\frac{d^2}{dx^2}\left[\log\frac{dz}{dx}\right] - \left[\frac{d}{dx}\log\frac{dz}{dx}\right]^2.$$

Let $x=\omega$, $y=k$, $z=\mu$, μ being a given function of k; and let μ', μ'', μ''' be the derived functions of μ with respect to k. Combining the identity (iv) with the equation (i), we find that μ satisfies the differential equation

$$2\frac{d^3\mu}{d\omega^3}\frac{d\mu}{d\omega} - 3\left(\frac{d^2\mu}{d\omega^2}\right)^2 = -f(\mu)\left(\frac{d\mu}{d\omega}\right)^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

where $f(\mu)$ represents the quantity

$$\frac{1}{\mu'^4}\left[\left(\frac{1-k^2}{1+k^2}\right)^2\frac{\mu'^2}{k^2} - 2\mu'\mu''' + 3\mu''^2\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

which is given as a function of k, but which we may conceive to be expressed in terms of μ.

Again, writing in the identity (i), $x=\omega$, $y=\lambda$, $z=\nu$, ν being the same

function of λ that μ is of k, and combining the result with the equation (ii) of Art. 89, we find

$$2\frac{d^3\nu}{d\omega^3}\frac{d\nu}{d\omega}-3\left(\frac{d^2\nu}{d\omega^2}\right)^2=-f(\nu)\left(\frac{d\nu}{d\omega}\right)^4 \quad \text{(iv)}$$

Eliminating ω from (ii) and (iv) in the same manner as in Art. 89, we obtain a differential equation between μ and ν, of which the type is

$$(\mu,\nu)+d\mu^2\,d\nu^2\,[f(\mu)\,d\mu^2-f(\nu)\,d\nu^2]=0, \quad \text{(v)}$$

(μ,ν) representing the differential expression

$$2\,d\mu\,d\nu\,(d\nu\,d^3\mu-d\mu\,d^3\nu)-3[d\nu^2(d^2\mu)^2-d\mu^2(d^2\nu)^2].$$

The following are the forms of $f(\mu)$ in a few important cases

(a) $\mu=k$; $\quad f(\mu)=\frac{1}{\mu^2}\left(\frac{1+\mu^2}{1-\mu^2}\right)^2=\left(\frac{1}{\mu}-\frac{1}{\mu-1}-\frac{1}{\mu+1}\right)^2$;

(b) $\mu=k^2$; $\quad f(\mu)=\frac{1-\mu+\mu^2}{\mu^2(\mu-1)^2}=\frac{1}{\mu^2}-\frac{1}{\mu(\mu-1)}+\frac{1}{(\mu-1)^2}$;

(c) $\mu=k^2(1-k^2)$; $\quad f(\mu)=\frac{1}{\mu^2}-\frac{3}{4}\frac{1}{\mu(\mu-\frac{1}{4})}+\frac{3}{4}\frac{1}{(\mu-\frac{1}{4})^2}$;

(d) $\mu=\frac{(1-k^2+k^4)^3}{k^4(1-k^2)^2}$; $\quad f(\mu)=\frac{8}{9}\frac{1}{\mu^2}-\frac{23}{36}\frac{1}{\mu(\mu-\frac{27}{4})}+\frac{3}{4}\frac{1}{(\mu-\frac{27}{4})^2}$;

(e) $\mu=\sqrt[4]{k}$; $\quad f(\mu)=\frac{1}{\mu^2}\left[1+\frac{64\mu^8}{(1-\mu^8)^2}\right]$;

(f) $\mu=\sqrt[12]{kk'}$; $\quad f(\mu)=\frac{1}{\mu^2}+\frac{3}{4}\frac{24^2.\mu^{16}}{(\mu^{24}-\frac{1}{4})^2}-\frac{3}{4}.\frac{24^2\mu^{22}}{\mu^{24}-\frac{1}{4}}$.

If in the equations (v) we write $X+iY$ for μ and $X-iY$ for ν, we obtain the differential equations satisfied by the modular curves. These equations are, in the six cases just enumerated,

$$(a')\quad \frac{d\left(\frac{1}{\rho}\right)}{ds}+\left(\frac{\cos\alpha}{R}-\frac{\cos\alpha_1}{R_1}-\frac{\cos\alpha_2}{R_2}\right)\left(\frac{\sin\alpha}{R}-\frac{\sin\alpha_1}{R_1}-\frac{\sin\alpha_2}{R_2}\right)=0;$$

$$(b')\quad \frac{d\left(\frac{1}{\rho}\right)}{ds}+\frac{\sin\alpha_1\cos\alpha_1}{R_1^2}-\frac{1}{2}\frac{\sin(\alpha_1+\alpha_2)}{R_1R_2}+\frac{\sin\alpha_2\cos\alpha_2}{R_2^2}=0;$$

$$(c') \quad \frac{d\left(\frac{1}{\rho}\right)}{ds} + \frac{\sin\alpha\cos\alpha}{R^2} - \frac{3}{8}\frac{\sin(\alpha+\alpha')}{RR'} + \frac{3}{4}\frac{\sin\alpha'\cos\alpha'}{R'^2} = 0;$$

$$(d') \quad \frac{d\left(\frac{1}{\rho}\right)}{ds} + \frac{8}{9}\frac{\sin\alpha\cos\alpha}{R^2} - \frac{23}{72}\frac{\sin(\alpha+\alpha')}{RR'} + \frac{3}{4}\frac{\sin\alpha'\cos\alpha'}{R'^2} = 0;$$

$$(e') \quad \frac{d\left(\frac{1}{\rho}\right)}{ds} + \frac{\sin\alpha\cos\alpha}{R^2} + \sum_{j=0}^{j=7}(-1)^j\frac{\cos\alpha_j}{R_j} \times \sum_{j=0}^{j=7}(-1)^j\frac{\sin\alpha_j}{R_j} = 0;$$

$$(f') \quad \frac{d\left(\frac{1}{\rho}\right)}{ds} + \frac{\sin\alpha\cos\alpha}{R^2} + \frac{3}{4}\sum_{j=0}^{j=23}\frac{\cos\alpha_j}{R_j} \times \sum_{j=0}^{j=23}\frac{\sin\alpha_j}{R_j} - 18\sum_{j=0}^{j=11}\frac{\sin(\alpha_j+\alpha_{j+12})}{R_jR_{j+12}} = 0.$$

In these formulæ $\frac{1}{\rho}$ is the curvature and ds the element of the curve at any point P; the curvature at P is positive or negative according as the tangent at P revolves in the positive or negative direction (*i. e.* in the direction from 1 to i, or from i to 1) when we travel along the curve in the direction in which ds is measured; $R, R', R_1, \ldots$ are the absolute lengths of the radii vectores drawn from certain fixed points to P; if R is produced beyond P to Q, and if PT is the tangent at P drawn in the direction of ds, α is the angle QPT, and is positive or negative according as the rotation from PQ to PT is positive or negative; the angles $\alpha', \alpha_1, \ldots$ are similarly defined; R is drawn from the origin in all the formulæ; R_1 and R_2 are drawn from the points ± 1 in (a'), and from the points A_{-1}, A_{+1} of Art. 48 in (b'); R' is drawn from the point $\frac{1}{4}$ in (c') and from the point $\frac{27}{4}$ in (d'); lastly, in (e') and (f'), the points, from which the lines R_j are drawn, are respectively $\epsilon^{\frac{1}{4}ji\pi}$, and $2^{-\frac{1}{12}}\epsilon^{\frac{1}{12}ji\pi}$.

The formulæ (b) and (e) may be obtained immediately from (iii); but (c) is more easily deduced from (b), and (d) and (f) from (c). For example, to obtain the formula (d) we write $y = k^2(1-k^2)$, $\mu = \frac{(1-y)^3}{y^2}$, and we observe that by virtue of the formula (c) y satisfies the differential equation

$$\left.\begin{aligned} 2\frac{dy}{d\omega}\frac{d^3y}{d\omega^3} - 3\left(\frac{d^2y}{d\omega^2}\right)^2 &= -\left[\frac{1}{y^2} - \frac{3}{4}\frac{1}{y(y-\frac{1}{4})} + \frac{3}{4}\frac{1}{(y-\frac{1}{4})^2}\right]\left(\frac{dy}{y\omega}\right)^4 \\ &= -f_1(y)\left(\frac{dy}{d\omega}\right)^4 \end{aligned}\right\} \quad \ldots \quad \text{(vi)}$$

Again, combining the identity (i), in which we write $x=\omega$, $z=\mu$, with the equation (vi), and denoting by μ', μ'', μ''' the differential coefficients of μ with respect to y, we find

$$2\frac{d^3\mu}{d\omega^3}\frac{d\mu}{d\omega}-3\left(\frac{d^2\mu}{d\omega^2}\right)^2=-\frac{1}{\mu'^2}\left[f_1(y)-\frac{2\mu'\mu'''-3\mu''^2}{\mu'^2}\right]\left(\frac{d\mu}{d\omega}\right)^4=-f(\mu)\left(\frac{d\mu}{d\omega}\right)^4.$$

But $\mu'=-\dfrac{(y-1)^2(y+2)}{y^3}$, and $\dfrac{2\mu'\mu'''-3\mu''^2}{\mu'^2}=2\dfrac{d^2}{dy^2}[\log\mu']-\left[\dfrac{d}{dy}\log\mu'\right]^2$

$$=-\frac{12(1+2y+3y^2)}{y^2(y-1)^2(y+2)^2};$$

also $(y+2)^2(1-4y)=4(1-y)^3-27y^2$,

$$(1+2y+3y^2)(1-4y)^2+y^2(y-1)^2(y+2)^2=(1-y)^6-8(1-y)^3y^2+54y^4.$$

Hence

$$f(\mu)=\frac{1}{\mu'^2}\left[f_1(y)-\frac{2\mu'\mu'''-3\mu''^2}{\mu'^2}\right]$$

$$=\frac{y^6}{(y-1)^4(y+2)^2}\left[\frac{(1-y)(1-4y)+12y^2}{y^2(1-4y)^2}+\frac{12(1+2y+3y^2)}{y^2(y-1)^2(y+2)^2}\right]$$

$$=\frac{1}{\mu^2}\left[\frac{(1-y)^3(1-4y)(y+2)^2+12y^2(y-1)^2(y+2)^2+12(1+2y+3y^2)(1-4y)^2}{(y+2)^4(1-4y)^2}\right]$$

$$=\frac{1}{\mu^2}\frac{16(1-y)^6-123(1-y)^3y^2+648y^4}{[4(1-y)^3-27y^2]^2}$$

$$=\frac{16\mu^2-123\mu+648}{\mu^2(4\mu-27)^2}$$

$$=\frac{8}{9}\frac{1}{\mu^2}-\frac{23}{36}\frac{1}{\mu(\mu-\frac{27}{4})}+\frac{3}{4}\frac{1}{(\mu-\frac{27}{4})^2}.$$

To explain by an example the way in which the equations (α'), (b'), ..., are deduced from (a), (b), ... and (ν), we consider in particular the equation (f'). Here we have

$$(1)\quad (\mu,\nu)=4i\times ds^5\times d\left(\frac{1}{\rho}\right),\quad d\mu\,d\nu=ds^2;$$

$$(2)\quad \frac{d\mu^2}{\mu^2}-\frac{d\nu^2}{\nu^2}=\left(\frac{d\mu}{\mu}+\frac{d\nu}{\nu}\right)\left(\frac{d\mu}{\mu}-\frac{d\nu}{\nu}\right)=4i\times\frac{dR}{R}\times d\phi,$$

if $$\mu=Re^{i\phi},\quad \nu=Re^{-i\phi}.$$

$$(3)\quad \frac{24\mu^{23}}{\mu^{24}-\frac{1}{4}}=\sum_{j=0}^{j=23}\frac{1}{\mu-a\theta^{j}},\quad \frac{24\nu^{23}}{\nu^{24}-\frac{1}{4}}=\sum_{j=0}^{j=23}\frac{1}{\nu-a\theta^{-j}},$$

where $$a=2^{-\frac{1}{24}},\quad \theta=\epsilon^{\frac{1}{12}i\pi},$$

so that

$$\frac{24^2.\mu^{46}\,d\mu^2}{(\mu^{24}-\frac{1}{4})^2}-\frac{24^2\nu^{46}\,d\nu^2}{(\nu^{24}-\frac{1}{4})^2}=\sum_{j=0}^{j=23}\left(\frac{d\mu}{\mu-a\theta^{j}}+\frac{d\nu}{\nu-a\theta^{-j}}\right)\times\sum_{j=0}^{j=23}\left(\frac{d\mu}{\mu-a\theta^{j}}-\frac{d\nu}{\nu-a\theta^{-j}}\right)$$

$$=4i\Sigma\frac{dR_j}{R_j}\times\Sigma d\phi_j,\quad \text{if } \mu-a\theta^{j}=R_j\epsilon^{i\phi_j},\quad \nu-a\theta^{-j}=R_j\epsilon^{-i\phi_j};$$

$$(4)\quad \frac{24\mu^{22}}{\mu^{24}-\frac{1}{4}}=\frac{1}{a}\sum_{j=0}^{j=2}\frac{\theta^{-j}}{\mu-a\theta^{j}}=2\sum_{j=0}^{j=11}\frac{1}{\mu^2-a^2\theta^{2j}},\quad \frac{24\nu^{22}}{\nu^{24}-\frac{1}{4}}=2\sum_{j=0}^{j=11}\frac{1}{\nu^2-a^2\theta^{-2j}},$$

so that

$$\frac{24\mu^{22}\,d\mu^2}{\mu^{24}-\frac{1}{4}}-\frac{24\nu^{22}\,d\nu^2}{\nu^{24}-\frac{1}{4}}=2i\sum_{j=0}^{j=11}\frac{R_j.dR_{j+12}.d\phi_j+R_{j+12}.dR_j.d\phi_{j+12}}{R_jR_{j+12}}.$$

$$(5)\quad \frac{dR}{ds}=\cos\alpha,\quad \frac{Rd\phi}{ds}=\sin\alpha.$$

And by these substitutions the equations (v) and (f) are transformed into (f').

CONTENTS TO THE MEMOIR ON THE THETA AND OMEGA FUNCTIONS.

ARTS. 1–14. *Definitions and Elementary Properties of the Theta, Omega, and Elliptic Functions.*

ART.		PAGE
1.	The Theta Functions	415
2.	The Omega and Modular Functions	417
3.	The Theta and Omega Functions as Infinite Products	419
4.	Expressions for the Modular Functions $\phi(\omega)$, $\psi(\omega)$, $\chi(\omega)$ as Infinite Products	420
5.	Expansions of the Omega Functions in series proceeding by powers of q	423
6.	The Formula for the Multiplication of Four Theta Functions	424
7.	The Elliptic Functions of the First Species	425
8.	The Complete Elliptic Integrals	429
9.	The Partial Differential Equation of the Theta Functions	430
10.	The Elliptic Function of the Second Species	431
11.	The Differential Coefficients of the Omega Functions	433
12.	The Differential Equations satisfied by the Omega Functions	435
13.	The Abelian Functions	440
14.	Differential Coefficients of the Omega Functions expressed by means of the Abelian Coefficients	445

ARTS. 15–23. *Arithmetical Theory of Binary Matrices.*

15.	Composition of Matrices	447
16.	Unit Matrices, Primitive Matrices, Reciprocal Matrices	447
17.	Equivalence of Matrices	449
18.	Systems of Non-equivalent Matrices	449
19.	Composition of Systems of Matrices	451
20.	Reduction of any two Primitive Matrices of the same Determinant to one another	452
21.	The Six Types of Matrices of an Uneven Determinant	453
22.	Primary Matrices and Primary Equivalence	455
23.	The Nine Types of Primitive Matrices of an Even Determinant	457

ARTS. 24–34. *The Transformation of the Theta and Omega Functions.*

24.	Enunciation of the Problem of Transformation	460
25.	General Solution of the Problem of Transformation.—Method of M. Hermite	461
26.	The Multiplier	468
27.	Composition of Transformations	469

ART. PAGE
28. Linear Transformation of the Theta Functions 473
29. Linear Transformation.—Determination of the Multiplier 476
30. Linear Transformation of $\phi(\omega)$ and $\psi(\omega)$ 477
31. Linear Transformation of $\chi(\omega)$ 482
32. Linear Transformation of the Elliptic Functions 484
33. Transformation of any Uneven Order.—Development of the Solution 488
34. Quadratic Transformations of the Elliptic, Theta, and Modular Functions. . . . 494

ARTS. 35–45. *Geometrical Representation of Binary Quadratic Forms.*

35. Quadratic Forms of a Negative Determinant 504
36. The Reduced Space 505
37. The Circular Affinity of Moebius 507
38. The Subdivisions of the Reduced Space 509
39. Quadratic Forms of a Positive Determinant 509
40. Automorphics of a Quadratic Form 512
41. The Subclasses of Primarily Equivalent Quadratic Forms 514
42. Reduction of Quadratic Forms of a Positive Determinant 517
43. Case when the Determinant is not a Square 519
44. Comparison of the Geometrical and Arithmetical Reduction 521
45. Case when the Determinant is a Square 524

ARTS. 46–51. *Geometrical Representation of the Modular Functions* $\Phi(\omega)$ and $\Psi(\omega)[=\phi^8(\omega)$ and $\psi^8(\omega)]$.

46. Discussion of the Equation $\Phi(\omega) = A$ 529
47. Limiting Values of $\Phi(\omega)$ 533
48. Transformation of the Reduced Space by the Modular Functions $\Phi(\omega)$ and $\Psi(\omega)$ . . 534
49. Lines answering to the Semicircles of Determinant $+1$ 535
50. Discussion of the Correspondence between the Reduced Space and the Plane XY . . 538
51. Limiting Values of $\Phi(\Omega)$ 541

ARTS. 52–58. *The Modular Equation.*

52. Definition of the Modular Equation 544
53. Theorems relating to the Modular Equation 545
54. Ditto 547
55. Ditto 548
56. The Sums of the Powers of the Roots 549
57. Development of the Roots of the Modular Equation 551
58. The Analytical Parallelogram 552

ARTS. 59–62. *The Equation of the Multiplier.*

59. Theorems, and Linear Transformation 553
60. Case when n is a Prime.—Determination of the Coefficients 557
61. Theorem of Jacobi 558
62. Expression of the Multiplier as a Rational Function of k^2 and λ^2 558

ARTS. 63–73. *The Modular Curves.*

ART.		PAGE
63.	The Modular Curves of an Uneven Order	560
64.	The Modular Curves of an Even Order	563
65.	Representation of the Rational Semicircles by the Modular Curves	564
66.	Case when the Determinant is not a Square	566
67.	Case when the Determinant is a Square	568
68.	Geometrical Construction of the Transformed Modulus by means of the Modular Curve	570
69.	Theorems relating to the Multiplier	572
70.	Symmetry of the Modular Curves	573
71.	Transformation of a Modular Curve by Inversion with regard to another Modular Curve	578
72.	Multiple Points and Points of Intersection	580
73.	Intersections of the Modular Curves with the Axis	585

ARTS. 74–82. *Theory of the Modular Functions* $\phi(\omega)$ *and* $\psi(\omega)$.

74.	Theory of $\phi(\omega)$. Primary Matrices	587
75.	Theory of $\phi(\omega)$. The Reduced Space	589
76.	Transformation of the Reduced Space by the Modular Function $\phi(\omega)$	590
77.	Theory of $\phi(\omega)$. The Rational Semicircles	591
78.	Theory of $\psi(\omega)$	592
79.	Theory of $\phi(\omega) + \psi(\omega)$	593
80.	Theory of $\phi(\omega)$. The Subclasses of Primarily Equivalent Quadratic Forms	594
81.	Theory of $\phi(\omega)$. The Modular Equations	595
82.	Theory of $\phi(\omega)$. Representation of Rational Semicircles by the Modular Curves	597

ARTS. 83–88. *Theory of the Modular Function* $T(\omega) = [1-\chi^{24}(\omega)]^3 \div \chi^{48}(\omega)$.

83.	Transformation of the Reduced Space	602
84.	The Modular Equations and Curves	603
85.	The Rational Semicircles	604
86.	Semicircles passing through a Singular Point	607
87.	The number of Ovals of the Modular Curve	611
88.	The Cases of Symmetry	612

ARTS. 89–90. *The Differential Equation of the Modular Equations and Curves.*

89.	Differential Equation of Jacobi	613
90.	Transformations of the Equation of Jacobi	614

XLIV.

MÉMOIRE SUR LA REPRÉSENTATION DES NOMBRES PAR DES SOMMES DE CINQ CARRÉS.

[From the Mémoires présentés par divers savants à l'Académie des Sciences de l'Institut National de France, vol. xxix.]

Art. 1. SOIT

$$f_1 = \Sigma_i \Sigma_j a_{i,j} x_i x_j \ldots \ldots \ldots \ldots \quad (1)$$

une forme quadratique des n indéterminées $x_1, x_2, \ldots, x_n$, les coefficients $a_{i,j} = a_{j,i}$ étant des nombres entiers. Nous représenterons la *matrice* de cette forme (c'est-à-dire le système de ses coefficients rangés en carré) par

$$\| a_{i,j} \| = \overset{n \times n}{\| a_{i,j} \|} = \| a_{ij} \|, \quad \left.\begin{matrix} i = 1, 2, \ldots, n, \\ j = 1, 2, \ldots, n, \end{matrix}\right\} \ldots \ldots \quad (2)$$

Nous rappelons que, dans la notation des matrices, une équation telle que

$$\| a_{i,j} \| = \| \beta_{i,j} \|$$

équivaut aux n^2 équations

$$a_{ij} = \beta_{ij},$$

tandis que l'équation

$$| a_{ij} | = | \beta_{ij} |$$

exprime seulement l'égalité des deux déterminants.

Pour abréger, nous supposerons que f_1 soit une forme définie et positive, et que son déterminant soit différent de zéro.

Soit

$$\Delta_n = | a_{ij} | \ldots \ldots \ldots \ldots \quad (3)$$

ce déterminant, et désignons par Δ_s le plus grand diviseur commun des déter-

minants *mineurs* de l'ordre s, qu'on peut former avec les éléments de $||\, a_{ij} \,||$: en supposant (comme nous ferons toujours) que la forme f_1 soit primitive, on aura $\Delta_1 = 1$. On sait que les nombres

$$\Delta_0 = 1,\ \Delta_1,\ \Delta_2,\ \Delta_3,\ \ldots,\ \Delta_n,\ \ldots\ldots\ldots \quad (4)$$

ne changent pas quand les indéterminées se transforment par une substitution linéaire à coefficients entiers et au déterminant ± 1*.

On peut donc les regarder comme les invariants arithmétiques de f_1.

Mais il y a quelque avantage à considérer, au lieu des nombres Δ_s, la suite suivante :

$$I_1 = \frac{\Delta_2}{\Delta_1} \div \frac{\Delta_1}{\Delta_2}, \qquad I_2 = \frac{\Delta_3}{\Delta_2} \div \frac{\Delta_2}{\Delta_1},\ \ldots,\ I_{n-1} = \frac{\Delta_n}{\Delta_{n-1}} \div \frac{\Delta_{n-1}}{\Delta_{n-2}}. \quad \ldots \quad (5)$$

Les $n-1$ nombres $I_1,\ I_2,\ \ldots,\ I_{n-1}$, sont entiers†; nous les appellerons désormais le premier, le second, le $\ldots$ $(n-1)^{ième}$ invariant de f_1. D'ailleurs on aura

$$\frac{\Delta_s}{\Delta_{s-1}} = I_{s-1} \times I_{s-2} \times \ \ldots\ \times I_1, \quad \Delta_s = I_{s-1} \times I_{s-2}^2 \times I_{s-3}^3 \times\ \ldots \times I_1^{s-1}. \quad . \quad (6)$$

Définition des formes adjointes.—Soit $||\, \alpha_{i,j}^{(s)} \,||$ la $s^{ième}$ matrice dérivée de

---

* Si le déterminant de la substitution est un nombre autre que l'unité, mais premier à Δ_n, les invariants de la transformée s'obtiennent en multipliant les invariants de la proposée par les carrés des invariants de la matrice de la substitution. Mais si le déterminant de la substitution a un diviseur commun avec Δ_n, le résultat devient plus compliqué.

† *Philosophical Transactions*, vol. cli. p. 317 [vol. i. p. 396]; Dans ce mémoire, nous ferons usage des méthodes employées dans les écrits suivants :

1°. *On Systems of Linear Indeterminate Equations and Congruences* (*Phil. Trans.* vol. cli. p. 293), 1861 [vol. i. p. 367].

2°. *On the Orders and Genera of Ternary Quadratic Forms* (*Phil. Trans.* vol. clvii. p. 255), 1867 [vol. i. p. 455].

3°. *On the Orders and Genera of Quadratic Forms containing more than three Indeterminates* (première notice, *Proceedings of the Royal Society*, vol. xiii. p. 199; seconde notice, *ibid.*, vol. xvi. p. 197), 1867 [vol. i. pp. 412, 510].

On aurait surtout désiré de reproduire les résultats généraux renfermés dans la notice de 1867, en les soumettant à un examen attentif et en y ajoutant des démonstrations rigoureuses. Mais le temps ayant manqué pour un travail si étendu, on s'est restreint, autant que possible, dans les limites de la question proposée par l'Académie. Toutefois on a traité, dès le commencement, des formes quadratiques de n indéterminées, parce que les propriétés des formes quaternaires, dont dépend effectivement la solution du problème, sont plus faciles à saisir quand on les énonce d'une manière parfaitement générale.

$||\alpha_{i,j}||$, c'est-à-dire la matrice dont les éléments sont les déterminants mineurs de l'ordre s, appartenant à la matrice $||\alpha_{i,j}||$, et pris dans leur ordre naturel. Soit $S = \dfrac{\Pi . n}{\Pi . s \Pi . n-s}$; la matrice $||\alpha^{(s)}_{i,j}||$ est symétrique de l'ordre S.

La forme

$$f_s = \sum_{i=1}^{i=S} \sum_{j=1}^{j=S} \alpha^{(s)}_{i,j} x_i x_j$$

est la $(s-1)^{ième}$ *forme adjointe* de f_1. On remarquera (1) que f_{n-1}, la dernière des formes adjointes, est la contravariante, (2) que si f_1 se transforme en F_1 par une substitution linéaire $||\overset{n\times n}{\beta_{i,j}}||$, f_s se transforme en F_s par la substitution dérivée $||\overset{s\times s}{\beta^{(s)}_{i,j}}||$.

Les coefficients de f_s sont divisibles par Δ_s, et la forme $\theta_s = \dfrac{1}{\Delta_s} f_s$ est primitive (proprement ou improprement). Nous nommerons les formes

$$\theta_2, \quad \theta_3, \quad \ldots, \quad \theta_{n-1}$$

es formes adjointes *primitives;* nous écrirons quelquefois

$$\theta_1 = f_1.$$

Définition des ordres.—Deux formes f_1 et f_1', ayant les mêmes invariants, sont censées appartenir au même ordre, quand, pour toutes les valeurs de s, les formes θ_s et θ_s' sont en même temps paires ou impaires*.

Définition des genres.—Soit F_1 une forme quadratique de r indéterminées, F_2 la première forme adjointe de F_1; les caractères génériques du système

$$\theta_1, \quad \theta_2, \quad \ldots, \quad \theta_{n-1},$$

se déduisent de l'équation connue

$$F_1(x_1, x_2, \ldots, x_r) \times F_1(y_1, y_2, \ldots, y_r) - \left(\frac{1}{2} \Sigma y \frac{dF}{dx}\right)^2 = F_2(X_1, X_2, \ldots), \qquad (7)$$

dans laquelle $X_1, X_2, \ldots$, sont les déterminants de la matrice

$$\left\| \begin{matrix} x_1, x_2, \ldots, x_r \\ y_1, y_2, \ldots, y_r \end{matrix} \right\|$$

pris dans un ordre convenable.

---

* Pour abréger le discours, nous nous servirons assez souvent de ces expressions au lieu des 'formes improprement et proprement primitives' ou des 'formes de seconde et de première espèce' de Lejeune-Dirichlet.

Soit $\bar{\theta}_s$ la première forme adjointe de θ_s, q un diviseur premier impair de I_s; on vérifie sans peine que $\dfrac{\bar{\theta}_s}{I_s}$ est une forme entière et primitive, et que les nombres premiers à q, qui sont représentés par θ_s, sont ou tous résidus quadratiques de q, ou tous non résidus; et l'on attribuera à θ_s le *caractère générique particulier*

$$\left(\frac{\theta_s}{q}\right)=1 \quad \text{ou} \quad \left(\frac{\theta_s}{q}\right)=-1,$$

selon que le premier ou le second cas a lieu.

On peut appeler *caractères génériques principaux* les caractères que nous venons de définir, et qui dépendent des diviseurs premiers impairs des invariants.

Ces caractères ne sont pas, en général, les seuls qui existent. En effet, (1) quelques-unes des formes θ_s peuvent avoir des caractères (dits *supplémentaires*) par rapport aux nombres 4 et 8; (2) nous verrons ci-après qu'il peut exister, dans le système des θ, une suite de formes consécutives ayant un caractère *simultané*.

L'ensemble des caractères génériques particuliers (soit principaux, soit supplémentaires, soit simultanés) constitue le caractère complet du système θ.

L'étude des caractères génériques est beaucoup facilitée par l'emploi d'une forme *canonique* pour représenter la forme proposée. Nous allons nous occuper de la recherche d'une telle forme dans les articles suivants.

2. Lemme I.—Toute forme primitive peut représenter des nombres premiers à un nombre donné ou les doubles de tels nombres, selon que la forme est impaire ou paire.

Lemme II.—Soit, dans une matrice donnée de l'ordre n, p^{i_s} la plus haute puissance du nombre premier p qui divise Δ_s; tout déterminant d'un ordre $s<n$ qui appartient à la matrice et qui n'est pas divisible par p^{i_s+1} contient au moins un déterminant de l'ordre $s-1$, qui n'est pas divisible par $p^{i_{s-1}+1}$*.

Théorème.—Soit Δ un nombre composé avec des puissances quelconques des nombres premiers impairs qui divisent Δ_n; on peut trouver des formes ϕ_1, en nombre infini, équivalentes à f_1, et qui satisfont aux congruences

$$\left.\begin{aligned} \phi_1 &\equiv a_1 x_1^2 + a_2 I_1 x_2^2 + a_3 I_1 I_2 x_3^2 + \ldots + a_n I_1 I_2 \ldots I_{n-1} x_n^2, \bmod \Delta, \\ a_1 a_2 a_3 \ldots a_n &\equiv 1, \bmod \frac{\Delta}{\Delta_n}. \end{aligned}\right\} \quad \ldots \quad \text{(i)}$$

* *Phil. Trans.*, vol. cli. p. 320 [vol. i. p. 399].

Démonstration.—Puisque $\theta_{n-1} = \frac{1}{\Delta_{n-1}} f_{n-1}$ est une forme primitive, on peut supposer que le nombre $\gamma_n = \frac{1}{\Delta_{n-1}} \times [a_{n,n}]^*$, qui est représenté par θ_{n-1}, est premier à Δ. Donc les congruences simultanées

$$\left.\begin{aligned} a_{i,1}k_1 + a_{i,2}k_2 + \ldots + a_{i,n}k_n &\equiv 0, \\ i &= 1, 2, 3, \ldots, n-1, \\ a_{n,1}k_1 + a_{n,2}k_2 + \ldots + a_{n,n}k_n &\equiv \frac{1}{\gamma_n}\frac{\Delta_n}{\Delta_{n-1}} \end{aligned}\right\} \quad \ldots\ldots \text{(ii)}$$

sont résolubles pour le module Δ ; et l'on aura, en particulier, $k_n \equiv 1$, mod Δ.

Cela posé, la substitution

$$\begin{vmatrix} 1, & 0, & 0, & \ldots, & k_1, \\ 0, & 1, & 0, & \ldots, & k_2, \\ 0, & 0, & 1, & \ldots, & k_3, \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \ldots, & k_{n-1}, \\ 0, & 0, & 0, & \ldots, & 1, \end{vmatrix}$$

transformera la matrice $\overset{n\times n}{\| a_{ij} \|}$ dans une autre $\overset{n\times n}{\| b_{ij} \|}$, dans laquelle on aura

$$\left.\begin{aligned} \overset{(n-1)\times(n-1)}{\| b_{i,j} \|} &= \overset{(n-1)\times(n-1)}{\| a_{i,j} \|}; \\ i &= 1, 2, \ldots, n-1, \\ j &= 1, 2, \ldots, n-1, \end{aligned}\right\} \quad \ldots\ldots \text{(iii)}$$

et, en outre,

$$\left.\begin{aligned} b_{i,n} &\equiv 0, \mod \Delta, \quad i = 1, 2, \ldots, n-1, \\ b_{n,n} &\equiv \frac{1}{\gamma_n}\frac{\Delta_n}{\Delta_{n-1}}, \mod \Delta. \end{aligned}\right\} \quad \ldots\ldots \text{(iv)}$$

En appliquant à la matrice $\overset{(n-1)\times(n-1)}{\| a_{ij} \|}$, dont les invariants sont

$$I_1, I_2, \ldots, I_{n-3}, \gamma I_{n-2},$$

---

* Nous employons la notation $[a_{i,j}]$, $[a_{i,j}, a_{r,s}], \ldots$ pour signifier les déterminants

$$\frac{d}{d.a_{i,j}} |a_{i,j}|, \quad \frac{d^2}{d.a_{i,j}d.a_{r,s}} |a_{i,j}|, \ldots.$$

une transformation analogue à celle que nous venons d'employer, on parvient à une matrice $\overset{(n-1)\times(n-1)}{\| c_{i,j} \|}$, dans laquelle

$$c_{i,\,n-1} \equiv 0, \mod \Delta, \quad i = 1,\ 2,\ \ldots,\ n-2,$$

$$c_{n-1,\,n-1} \equiv \frac{1}{\gamma_{n-1}} \frac{\Delta_{n-1}}{\Delta_{n-2}}, \mod \Delta,$$

le mineur principal $\dfrac{1}{\Delta_{n-2}} \times \overset{(n-2)\times(n-2)}{| c_{ij} |}$ étant premier à Δ.

Cette seconde transformation peut bien changer les valeurs de

$$b_{1,\,n},\ b_{2,\,n},\ \ldots,\ b_{n-1,\,n},$$

mais les congruences (iv) subsisteront toujours. En continuant de la sorte, on parviendra enfin à une matrice $\overset{n\times n}{\| m_{i,j} \|}$ qui satisfera à la congruence

$$\| m_{i\,j} \| \equiv \left\| \begin{array}{ccccc} \dfrac{1}{\gamma_1}\dfrac{\Delta_1}{\Delta_0}, & 0, & 0, & \ldots, & 0 \\ 0, & \dfrac{1}{\gamma_2}\dfrac{\Delta_2}{\Delta_1}, & 0, & \ldots, & 0 \\ 0, & 0, & \dfrac{1}{\gamma_3}\dfrac{\Delta_3}{\Delta_2}, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \ldots, & \dfrac{1}{\gamma_n}\dfrac{\Delta_n}{\Delta_{n-1}} \end{array} \right\| \mod \Delta,$$

et l'on aura ainsi une forme

$$\phi_1 = \Sigma m_{i,j}\, x_i x_j,$$

équivalente à f_1 et satisfaisant aux congruences (I).

3. Lemme I.—Soit f_1 une forme impaire ; soient aussi impairs J_2, la première forme adjointe de f_1, et I_1 son premier invariant, on peut toujours trouver, si non dans la matrice de f_1, du moins dans une matrice équivalente, deux coefficients principaux*, tels que a_{ii} et $a_{ii}\, a_{jj} - a_{ij}^2$, dont l'un contient l'autre, et qui sont tous les deux impairs.

---

* Les coefficients des carrés dans les formes adjointes sont des mineurs *principaux* de la matrice, c'est-à-dire des mineurs symétriques par rapport à la diagonale principale.

Réciproquement, si θ_{n-1}, θ_{n-2} sont des formes impaires, le dernier invariant étant aussi impair, on peut toujours trouver, soit dans la matrice donnée, soit dans une matrice équivalente, deux coefficients principaux, tels que $|a_{i,j}|^{(n-1)\times(n-1)}$ et $|a_{i,j}|^{(n-2)\times(n-2)}$, dont le second est contenu dans le premier, et qui sont tous les deux impairs.

LEMME II.—Si θ_1 est une forme paire, I_1 et θ_2 sont impairs; et réciproquement, si θ_{n-1} est paire, θ_{n-2} et I_{n-1} sont impairs.

Théorème.—Supposons que I_s soit pair, I_{s+1}, I_{s+2}, ..., I_{n-1} impairs. Si $n-s$ est un nombre impair, les formes θ_s, θ_{s+1}, ..., θ_{n-1} sont toujours impaires; si $n-s$ est pair, ou ces formes sont toutes impaires, ou bien elles sont alternativement impairs et paires, θ_s étant impaire et θ_{n-1} paire.

Démonstration.—Soit δ_i la plus haute puissance de 2, qui divise Δ_i; δ une puissance quelconque de 2, qu'on peut supposer plus grande que δ_n.

A. Soit, en premier lieu, θ_{n-1} une forme impaire; nous ferons voir que l'imparité de I_{n-1} et I_{n-2}, c'est-à-dire l'égalité des trois nombres $\frac{\delta_n}{\delta_{n-1}}$, $\frac{\delta_{n-1}}{\delta_{n-2}}$, $\frac{\delta_{n-2}}{\delta_{n-3}}$, entraîne l'imparité de θ_{n-2}.

Supposons (ce qui est toujours admissible) que $\frac{1}{\delta_{n-1}} \times [a_{n,n}]$ soit un nombre impair, que nous désignerons par γ_n. Les congruences (ii) de l'article 2, en y écrivant $\frac{\delta_n}{\delta_{n-1}}$ au lieu de $\frac{\Delta_n}{\Delta_{n-1}}$, seront résolubles pour le module δ, et nous conduiront, comme dans l'article cité, à une matrice $||\, b_{i,j}\,||^{n\times n}$, équivalente à $||\, a_{i,j}\,||^{n\times n}$, et satisfaisant aux congruences

$$b_{n,n} \equiv \frac{1}{\gamma_n}\frac{\delta_n}{\delta_{n-1}}, \text{ mod } \delta; \quad b_{i,n} \equiv 0, \text{ mod } \delta, \quad i = 1, 2, \ldots, n-1;$$

on aura d'ailleurs $||\, b_{i,j}\,||^{(n-1)\times(n-1)} = ||\, a_{i,j}\,||^{(n-1)\times(n-1)}$, et les puissances de 2, qui divisent les déterminants mineurs des différents ordres dans cette matrice, seront $\delta_1, \delta_2, \ldots, \delta_{n-2}$. Soit $\psi_{n-2} \times \delta_{n-2}$ la forme contravariante de cette même matrice, $\psi_{n-3} \times \delta_{n-3}$ l'avant-dernière de ses formes adjointes; si ψ_{n-2} est une forme impaire, il est évident que θ_{n-2} est aussi impaire. Supposons donc que ψ_{n-2} soit paire; dans

ce cas ψ_{n-3} sera impaire (lemme II) ; par conséquent, on peut supposer que

$$h = \frac{1}{\delta_{n-3}} \times \overset{(n-3)\times(n-3)}{|\, a_{i,j}\,|}, \qquad \begin{matrix} i = 1,\ 2,\ \ldots,\ n-3, \\ j = 1,\ 2,\ \ldots,\ n-3, \end{matrix}$$

soit un nombre impair. Mais, dans la matrice $\overset{n\times n}{||\, b_{i,j}\,||}$, un des coefficients principaux de l'ordre $n-2$ est congru, pour le module δ, au nombre

$$h\delta_{n-3} \times \frac{1}{\gamma_n}\frac{\delta_n}{\delta_{n-1}} = \frac{h}{\gamma_n} \times \delta_{n-2}.$$

Donc θ_{n-2} est une forme impaire, puisque $\dfrac{h}{\gamma_n}$ est congru, pour le module $\dfrac{\delta_n}{\delta_{n-2}}$, à un nombre impair.

Maintenant, on conclut du lemme I qu'avant d'entreprendre la transformation de la matrice $\overset{n\times n}{||\, a_{i,j}\,||}$ dans la matrice $\overset{a\times n}{||\, b_{i,j}\,||}$, transformation que nous désignerons ci-après par (A), on peut supposer que la première de ces matrices ait été choisie parmi les matrices équivalentes, d'une telle manière que les nombres

$$\frac{1}{\delta_{n-1}} \overset{(n-1)\times(n-1)}{|\, a_{i,j}\,|} \text{ et } \frac{1}{\delta_{n-2}} \overset{(n-2)\times(n-2)}{|\, a_{i,j}\,|}$$

soient tous les deux impairs. Cela posé, on pourra employer la transformation (A) deux fois de suite, en opérant premièrement sur $\overset{n\times n}{||\, a_{i,j}\,||}$, et ensuite sur $\overset{(n-1)\times(n-1)}{||\, a_{i,j}\,||}$, qui fait partie de $\overset{n\times n}{||\, b_{i,j}\,||}$. En supposant que I_{n-3} soit impair, cette seconde transformation fera voir ; 1° que θ_{n-3} est une forme impaire ; 2° qu'on peut supposer $\dfrac{1}{\delta_{n-3}} \overset{(n-3)\times(n-3)}{|\, a_{i,j}\,|}$ impair, en même temps que $\dfrac{1}{\delta_{n-2}} \overset{(n-3)\times(n-3)}{|\, a_{i,j}\,|}$, de sorte que l'on puisse encore opérer la transformation (A) sur la matrice $\overset{(n-2)\times(n-2)}{||\, a_{i,j}\,||}$.

De cette manière, en supposant impairs les nombres I_{n-4}, I_{n-5}, ..., I_{s+1}, on démontrera successivement que θ_{n-4}, θ_{n-5}, ..., θ_{s+1} sont des formes impaires. Quant à θ_s, elle est certainement impaire, puisque I_s est pair.

On voit que les applications successives de la transformation (A), dont nous avons fait usage, conduisent à une forme ϕ_1, équivalente à f_1, et satisfaisant à la congruence

$$\phi_1 \equiv \sum_{i=1}^{i=s}\sum_{j=1}^{j=s} \beta_{i,j}\, x_i x_j + \frac{\delta_{s+1}}{\delta_s}\left[\beta_{s+1}x_{s+1}^2 + \beta_{s+2}x_{s+2}^2 + \beta_{s+3}x_{s+3}^2 + \ldots + \beta_n x_n^2\right], \quad . \quad \text{(i)}$$

dans laquelle $\frac{1}{\delta_s} \times |\overset{s\times s}{\beta_{i,j}}|$, β_{s+1}, β_{s+2}, ..., β_n sont des nombres impairs, et $\frac{\delta_{s+1}}{\delta_s} > \frac{\delta_s}{\delta_{s-1}}$.

B. Soit en second lieu θ_{n-1} une forme paire ; dans ce cas I_{n-1} est nécessairement impair ; en supposant que I_{n-2} soit aussi impair, nous allons démontrer que θ_{n-2} est impaire, θ_{n-3} paire, I_{n-3} impair. On pourra supposer (lemme I) que les nombres

$$\gamma_n = \frac{1}{2\delta_{n-1}}[\alpha_{n,n}] \text{ et } \frac{1}{\delta_{n-1}}[\alpha_{n-1,n}]$$

soient simultanément impairs.

Les congruences (ii) de l'article 2, dans lesquelles on écrira

$$\frac{\delta_n}{\delta_{n-1}} \text{ au lieu de } \frac{\Delta_n}{\Delta_{n-1}},$$

seront encore résolubles pour le module δ ; mais on aura

$$k_n \equiv 2, \text{ mod } \delta,$$

tandis que k_{n-1} sera impair. Soit

$$k_{n-1} \equiv 2\mu - 1, \text{ mod } \delta ;$$

la substitution

$$\begin{vmatrix} 1 & 0, & 0, & 0, & \ldots, & 0, & k_1 \\ 0, & 1, & 0, & 0, & \ldots, & 0, & k_2 \\ 0, & 0, & 1, & 0, & \ldots, & 0, & k_3 \\ . & . & . & . & . & . & . \\ 0, & 0, & 0, & 0, & \ldots, & \mu, & k_{n-1} \\ 0, & 0, & 0, & 0, & \ldots, & 1, & 2 \end{vmatrix}$$

conduit à une matrice $||\overset{n\times n}{c_{i,j}}||$, équivalente à $||\overset{n\times n}{\alpha_{i,j}}||$, qui satisfera aux congruences

$$b_{n,i} \equiv 0 ; \quad i = 1, 2, \ldots, n-2 ; \quad b_{n,n-1} \equiv \frac{1}{\gamma_n}\frac{\delta_n}{\delta_{n-1}},$$

$$b_{n,n} \equiv \frac{2}{\gamma_n}\frac{\delta_n}{\delta_{n-1}}, \text{ mod } \delta,$$

et dans laquelle on aura aussi

$$\overset{(n-2)\times(n-2)}{||\, b_{i,j}\,||} = \overset{(n-2)\times(n-2)}{||\, a_{i,j}\,||}.$$

Puisque θ_{n-1} est une forme paire, $\frac{1}{\delta_{n-1}} \times \overset{(n-1)\times(n-1)}{|\, a_{i,j}\,|}$ est pair ; mais $\frac{1}{\delta_{n-2}} \times \overset{(n-2)\times(n-2)}{|\, a_{i,j}\,|}$ est impair ; car $\frac{1}{\delta_n} \times \overset{n\times n}{|\, b_{i,j}\,|}$ étant impair, la valeur, mod δ, de

$$\frac{2}{\gamma_n\, \delta_{n-1}} \times \overset{(n-1)\times(n-1)}{|\, b_{i,j}\,|} - \frac{\delta_n}{\gamma_n^2\, \delta_{n-1}^2} \times \overset{(n-2)\times(n-2)}{|\, b_{i,j}\,|},$$

et, par conséquent, de

$$\frac{\delta_n \times \delta_{n-2}}{\gamma_n^2\, \delta_{n-1}^2} \times \frac{1}{\delta_{n-2}} \times \overset{(n-2)\times(n-2)}{|\, b_{i,j}\,|},$$

doit être impaire. On conclut de là que θ_{n-2} est une forme impaire ; pour démontrer que θ_{n-3} est paire, soit $\overset{(n-1)\times(n-1)}{|\, c_{i,j}\,|} = \omega_1 \times \delta_{n-1} \times \gamma_{n-1}$, $\omega_1 \times \delta_{n-1}$ étant la plus grande puissance de 2 qui divise le déterminant, et opérons une transformation (A) sur la matrice $\overset{(n-1)\times(n-1)}{||\, c_{i,j}\,||}$. On trouvera ainsi une transformée de $\overset{n\times n}{||\, a_{i,j}\,||}$, que nous désignerons par $\overset{n\times n}{||\, d_{i,j}\,||}$, dans laquelle on aura

$$\overset{(n-2)\times(n-2)}{||\, d_{i,j}\,||} = \overset{(n-2)\times(n-2)}{||\, a_{i,j}\,||},$$

$$d_{i,j} \equiv 0, \text{ mod } \delta\,; \quad i = n-1,\, n\,; \quad j = 1,\, 2,\, \ldots,\, n-2,$$

$$d_{n,n} \equiv \frac{2}{\gamma_n} \frac{\delta_n}{\delta_{n-1}}, \quad d_{n-1,n-1} \equiv \frac{\omega}{\gamma_{n-1}} \frac{\delta_n}{\delta_{n-1}},$$

$$d_{n,n-1} \equiv \frac{1}{\gamma_n} \frac{\delta_n}{\delta_{n-1}}, \quad \text{mod } \delta.$$

De plus, la contravariante réduite ψ_{n-3} de la matrice $\overset{(n-2)\times(n-2)}{||\, a_{i,j}\,||}$ sera une forme paire. En effet, s'il en était autrement, on pourrait supposer que $\frac{1}{\delta_{n-3}} \times \overset{(n-3)\times(n-3)}{|\, d_{i,j}\,|}$ fût impair ; mais alors $\frac{1}{\delta_{n-1}} \times [d_{n-1,\,n-1}]$, et, par conséquent, θ_{n-1}, serait impaire, car $\frac{1}{\delta_{n-1}} \times [d_{n-1,\,n-1}]$ est du même ordre de parité que $\frac{1}{\delta_{n-3}} \times \overset{(n-3)\times(n-3)}{|\, d_{i,j}\,|}$, ainsi qu'il résulte

de la congruence

$$[d_{n-1,\,n-1}] \equiv |\overset{(n-3)\times(n-3)}{d_{i,j}}| \times \frac{\delta_n^2}{\delta_{n-1}^2} \times \left(\frac{2\omega}{\gamma_n \gamma_{n-1}} - \frac{1}{\gamma_n^2}\right), \text{ mod } \delta$$

et de l'équation $\delta_n^2 \delta_{n-3} = \delta_{n-1}^3$. On voit facilement que la parité de ψ_{n-3} entraîne la parité de f_{n-3}, et par conséquent l'imparité de I_{n-3}.

Désignons par (B) une transformation analogue à celle par laquelle nous avons réduit la matrice $\|\overset{n\times n}{a_{i,j}}\|$ à la forme $\|\overset{n\times n}{d_{i,j}}\|$. Si I_{n-4} est impair, la matrice $\|\overset{(n-2)\times(n-2)}{d_{i,j}}\|$ admettra elle-même une transformation (B), puisque sa forme contravariante ψ_{n-3} est paire, et ses deux derniers invariants sont impairs. On aura donc ψ_{n-4} impaire, ψ_{n-5} paire, I_{n-5} impair ; d'où l'on conclut que θ_{n-4} sera paire, θ_{n-5} impaire.

En répétant le même raisonnement, on parvient à démontrer que si θ_{n-1} est une forme paire, les formes paires et impaires se succèdent alternativement dans la série des formes adjointes, et qu'il y a toujours un nombre impair d'invariants impairs avant d'arriver à un invariant pair. Si, comme ci-dessus, I_s est le dernier des invariants pairs, $n-s$ sera un nombre pair, et, après avoir opéré la transformation (B) $\frac{1}{2}(n-s)$ fois de suite, on sera conduit à une forme ϕ_1, équivalente à f_1, qui satisfera à la congruence

$$\phi_1 \equiv \sum_{i=1}^{i=s} \sum_{j=1}^{j=s} \beta_{i,j} x_i x_j + \frac{\delta_{s+1}}{\delta_s}[(x_{s+1}, x_{s+2})^2 + (x_{s+3}, x_{s+4})^2 + \ldots + (x_{n-1}, x_n)^2], \text{ mod } \delta, \quad \text{(ii)}$$

où l'on aura, comme dans la congruence (i), $\frac{1}{\delta_s} \times |\overset{s\times s}{\beta_{i,j}}|$ impair, $\frac{\delta_{s+1}}{\delta_s} > \frac{\delta_s}{\delta_{s-1}}$, et où les symboles $(x_{s+1}, x_{s+2})^2$ représentent des formes binaires paires, dans lesquelles on peut supposer que le premier coefficient est impairement pair.

4. Du théorème de l'article précédent, on déduit immédiatement cet autre plus général.

Théorème.—Soit $I_{\sigma+1}, I_{\sigma+2}, \ldots, I_{\sigma+\sigma'}$ une suite d'invariants impairs, précédée et suivie par des invariants pairs I_σ et $I_{\sigma+\sigma'+1}$. Si σ' est pair, les formes $\theta_\sigma, \theta_{\sigma+1}, \theta_{\sigma+2}, \ldots, \theta_{\sigma+\sigma'+1}$ seront toutes impaires, ou alternativement impaires et paires, la première et la dernière étant impaires.

[Dans l'énoncé de ce théorème, il faut regarder $I_0 = I_n = 0$ comme des invariants pairs, et $\theta_0 = \theta_n = 1$ comme des formes impaires.]

En effet, il résulte de l'inégalité $\frac{\delta_{s+1}}{\delta_s} > \frac{\delta_s}{\delta_{s-1}}$, qu'on suppose satisfaite dans les congruences (i) et (ii) de l'article précédent, que la parité ou imparité des formes $\theta_1, \theta_2, \ldots, \theta_{s-1}$ dépend uniquement des matrices $\| \overset{s\times s}{\beta_{i,j}} \|$ qui figurent dans ces congruences, et nullement des termes multipliés par $\frac{\delta_{s+1}}{\delta_s}$. Ainsi, pour avoir la démonstration du théorème énoncé, il suffit de traiter ces matrices par les procédés déjà développés (A et B).

On remarquera que l'emploi successif des transformations (A) et (B) conduit à une forme ϕ_1, équivalente à f_1, et qui satisfait à la congruence

$$\phi_1 \equiv \frac{\delta_1}{\delta_0}(1, \alpha) + \frac{\delta_{\alpha+1}}{\delta_\alpha}(\alpha+1, \beta) + \frac{\delta_{\beta+1}}{\delta_\beta}(\beta+1, \gamma) + \ldots + \frac{\delta_{s+1}}{\delta_s}(s+1, n), \text{ mod } \delta, \quad \text{(i)}$$

dans laquelle $I_\alpha, I_\beta, I_\gamma, \ldots, I_s$, $\alpha < \beta < \gamma < \ldots < s$ sont les invariants pairs, et les expressions $(1, \alpha)$, $(\alpha+1, \beta)$, ... représentent des sommes de α, $\beta-\alpha$, $\gamma-\beta$, ..., carrés avec des coefficients impairs, ou (lorsque quelques-uns des nombres $\alpha, \beta-\alpha, \gamma-\beta, \ldots, n-s$ sont pairs) des sommes de formes binaires impaires.

Soit, par exemple,

$$1, 1, 1, 2, 4, 2, 1, 2, 1, 1, 4, 1,$$

la suite des invariants ; ϕ_1 satisfera à une congruence de la forme

$$\phi_1 \equiv (1, 4) + 2\beta_5 x_5^2 + 2^3\beta_6 x_6^2 + 2^4(7, 8) + 2^5(\beta_9 x_9^2 + \beta_{10} x_{10}^2 + \beta_{11} x_{11}^2) + 2^7(12, 13), \text{ mod } \delta,$$

en représentant par l'expression $(1, 4)$ soit la somme $(x_1, x_2)^2 + (x_3, x_4)^2$, soit la somme $\sum_1^4 \beta_i x_i^2$; et par $(7, 8)$, $(12, 13)$, soit les sommes $\beta_7 x_7^2 + \beta_8 x_8^2$, $\beta_{12} x_{12}^2 + \beta_{13} x_{13}^2$, soit les formes $(x_7, x_8)^2$, $(x_{12}, x_{13})^2$.

Ajoutons qu'on peut trouver des formes ϕ_1 équivalentes à f_1, et qui satisfont en même temps à la congruence (i) de cet article et à la congruence (i) de l'article (2). C'est ce qui résulte des propositions suivantes.

Lemme.—Soit $\| \overset{n\times n}{a_{i,j}} \|$ une matrice donnée, symétrique ou non, dont les éléments soient des nombres entiers ; soit aussi, dans cette matrice, D_s le plus grand diviseur commun des déterminants mineurs de l'ordre s ; on peut toujours trouver deux matrices du $n^{ième}$ ordre, $\| u \|$ et $\| v \|$, dont le déterminant est

l'unité positive, et qui satisfont à l'équation*

$$\| a_i \| = \| u \| \times \left\| \frac{D_n}{D_{n-1}}, \frac{D_{n-1}}{D_{n-2}}, \ldots, \frac{D_1}{D_0} \right\| \times \| v \| \ldots \quad \text{(ii)}$$

On remarquera (1) que les nombres entiers $J_s = \frac{D_{s+1}}{D_s} \div \frac{D_s}{D_{s-1}}$ sont les invariants arithmétiques de la forme bilinéaire $\Sigma a_{i,j} x_i y_j$, (2) que l'équation (ii) admet une infinité de solutions $\| u \|$ et $\| v \|$.

Pour abréger, nous désignons par $\| a_1, a_2, \ldots \|$ une matrice, dont tous les éléments sont zéro, excepté ceux $(a_1, a_2, \ldots)$ qui se trouvent sur la diagonale principale.

Problème.—Soit $\| \overset{n \times n}{a_{i,j}} \|$ une matrice donnée, qui satisfait à la congruence $\| \overset{n \times n}{a_{i,j}} \| \equiv 1$, pour un module quelconque P; trouver une matrice $\| \overset{n \times n}{x_{i,j}} \|$ qui satisfasse à l'équation $\| \overset{n \times n}{x_{i,j}} \| = 1$ et à la congruence $\| \overset{n \times n}{x_{i,j}} \| \equiv \| \overset{n \times n}{a_{i,j}} \|$, mod P.

Solution.—Remplaçons $\| \overset{n \times n}{a_{i,j}} \|$ par l'expression équivalente donnée par la formule (ii). Si $D_{n-1} = 1$, la matrice $\| x_{i,j} \| = \| u \| \times \| v \|$ satisfera aux conditions du problème. Si $D_{n-1} > 1$, écrivons $\frac{D_n}{D_{n-1}} - P$ à la place de $\frac{D_n}{D_{n-1}}$ dans la formule (ii), et désignons la matrice résultante par $\| \beta_{i,j} \|$. Le plus grand commun diviseur des mineurs d'ordre $n-1$ dans $\| \beta_{i,j} \|$ est D_{n-2}, puisque les invariants de cette matrice sont les mêmes que les invariants de la matrice

$$\left\| \frac{D_n}{D_{n-1}} - P, \frac{D_{n-1}}{D_{n-2}}, \ldots, \frac{D_1}{D_0} \right\|.$$

Mais l'inégalité $D_{n-1} > 1$ entraîne celle-ci, $D_{n-1} > D_{n-2}$; donc on aura remplacé la matrice donnée $\| a_{i,j} \|$ par une autre $\| \beta_{i,j} \|$, qui sera toujours congrue à $\| a_{i,j} \|$ pour le module P, mais dans laquelle on aura diminué le plus grand diviseur commun des déterminants mineurs d'ordre $n-1$. En continuant cette suite d'opérations, on parviendra enfin à une matrice dans laquelle ce diviseur sera l'unité, et qui donnera, comme ci-dessus, la solution du problème.

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* Voir, pour la démonstration de ce lemme, les *Philosophical Transactions*, vol. cli. p. 314 [vol. i. p. 391].

Corollaire.—Étant données des matrices $\| \overset{n-n}{a_{i,j}} \|$, $\| \overset{n-n}{\beta_{i,j}} \|$, $\| \overset{n-n}{\gamma_{i,j}} \|$, ..., en nombre quelconque, qui satisfont aux congruences

$$| a_{i,j} | \equiv 1, \text{ mod } P, \quad | \beta_{i,j} | \equiv 1, \text{ mod } Q, \quad | \gamma_{i,j} | \equiv 1, \text{ mod } R, \ \ldots,$$

les nombres P, Q, R, ..., étant premiers entre eux deux à deux, on peut toujours trouver une matrice $\| x_{i,j} \|$ qui satisfasse aux congruences

$$\| x_{i,j} \| \equiv \| a_{i,j} \|, \text{ mod } P, \quad \equiv \| \beta_{i,j} \|, \text{ mod } Q, \quad \equiv \| \gamma_{i,j} \|, \text{ mod } R, \ \ldots,$$

et dont le déterminant soit l'unité positive. En effet, on déterminera les n^2 nombres $m_{i,j}$ de sorte qu'on ait

$$m_{i,j} \equiv a_{i,j}, \text{ mod } P, \quad \equiv \beta_{i,j}, \text{ mod } Q, \quad \equiv \gamma_{i,j}, \text{ mod } R, \ \ldots;$$

il en résultera

$$| \overset{n \times n}{m_{i,j}} | \equiv 1, \text{ mod } P \times Q \times R \times \ldots,$$

et la matrice cherchée sera déterminée par les conditions

$$\| \overset{n \times n}{x_{i,j}} \| \equiv | \overset{n \times n}{m_{i,j}} \|, \text{ mod } P \times Q \times R \ldots; \quad | x_{i,j} | = 1,$$

qui sont précisément celles du problème précédent.

Maintenant, soient $\| \beta \|$ et $\| \alpha \|$ des substitutions qui transforment f_1 en des formes qui satisfont respectivement à la congruence (i) de cet article et à la congruence (i) de l'article 2. Pour avoir une transformée de f_1 qui satisfasse simultanément aux deux congruences, on n'aura qu'à prendre une substitution de déterminant $+1$, dont la matrice soit congrue à $| a_{i,j} \|$ pour le module Δ et à $\| \beta_{i,j} \|$ pour le module δ.

5. *Définition.*—Soit $\| \Gamma_{i,j} \|$ une matrice équivalente à $\| a_{i,j} \|$; nous dirons que les nombres

$$\frac{\Gamma_{1,1}}{\Delta_1}, \quad \frac{1}{\Delta_2} \times | \overset{2 \times 2}{\Gamma_{i,j}} |, \quad \frac{1}{\Delta_3} \times | \overset{3 \times 3}{\Gamma_{i,j}} |, \ \ldots, \quad \frac{1}{\Delta_{n-1}} \times | \overset{(n-1) \times (n-1)}{\Gamma_{i,j}} |,$$

que, pour abréger, nous désignerons par

$$m_1, \ m_2, \ m_3, \ \ldots, \ m_{n-1},$$

sont représentés *simultanément* et *primitivement* par les formes θ_1, θ_2, ..., θ_{n-1}. Si $\| \Gamma_{i,j} \|$ était dérivé de $\| a_{i,j} \|$ par une substitution dont le déterminant fût plus grand que l'unité, la représentation serait simultanée mais non pas primitive.

Si les nombres m_i sont impairs ou impairement pairs, selon que les formes θ_i sont paires ou impaires, et si, de plus, deux nombres consécutifs m_i et m_{i+1} sont toujours premiers entre eux, nous dirons que le système des nombres m_i est un système *caractéristique* de nombres représentés simultanément par les formes $\theta_1, \theta_2, \ldots, \theta_{n-1}$, et que la matrice $\| \Gamma_{i,j} \|$ est une matrice *caractéristique*.

On peut toujours trouver en nombre infini des matrices caractéristiques, équivalentes à une matrice donnée, même parmi celles qui satisfont aux congruences (i) de l'article 2 et (i) de l'article 4.

Soit $\overset{n\times n}{\| \Gamma_{i,j} \|}$ la matrice d'une forme Γ équivalente à f_1 et satisfaisant à ces deux congruences.

Désignons par

$$M_1, \ M_2, \ M_3, \ \ldots, \ M_{n-1}$$

les nombres

$$\frac{1}{\Delta_1}\Gamma_{1,1}, \ \frac{1}{\Delta_2}\left| \overset{2\times 2}{\Gamma_{i,j}} \right|, \ \ldots ;$$

ces nombres seront représentés simultanément par $\theta_1, \theta_2, \ldots, \theta_{n-1}$; ils seront premiers à tout diviseur impair de Δ_n ; de plus, ils seront impairs ou impairement pairs, selon que les formes correspondantes sont paires ou impaires. Mais le système (M) ne sera pas nécessairement un système caractéristique, puisque deux nombres consécutifs peuvent avoir un diviseur commun. Considérons la matrice $\overset{(n-1)\times(n-1)}{\| \Gamma_{i,j} \|}$ dont le déterminant est $\Delta_{n-1} \times M_{n-1}$, et supposons que M_{n-1} et $M_{n-2} = \dfrac{1}{\Delta_{n-2}} \times \left| \overset{(n-2)\times(n-2)}{\Gamma_{i,j}} \right|$ aient un diviseur commun μ. Ce diviseur est impair, puisque M_{n-1} et M_{n-2} ne sauraient être pairs en même temps ; il est aussi premier à Δ_n, et par conséquent à Δ_{n-2}, qui est le plus grand diviseur commun de tous les déterminants mineurs d'ordre $n-2$, qui appartiennent à la matrice $\overset{(n-1)\times(n-1)}{\| \Gamma_{i,j} \|}$. Donc on peut transformer cette matrice par une substitution linéaire $\overset{(n-1)\times(n-1)}{\| \alpha_{i,j} \|}$ telle que, dans la matrice transformée, la valeur de M_{n-2} soit première à M_{n-1}. Après cette transformation la forme Γ pourrait, à la vérité, ne pas satisfaire aux deux congruences citées ; pour éviter cet inconvénient on se servira, au lieu de $\overset{(n-1)\times(n-1)}{\| \alpha_{i,j} \|}$, d'une substitution $\overset{(n-1)\times(n-1)}{\| \beta_{i,j} \|}$, qui satisfait aux congruences $\| \beta_{i,j} \| \equiv \| \alpha_{i,j} \|$, mod M_{n-1}, ou mod $\frac{1}{2} M_{n-1}$ si M_{n-1} est impairement pair, et

$$\| \beta_{i,j} \| \equiv \| \ 1, \ 1, \ 1 \ \ldots \ \|, \ \text{mod} \ \delta \times \Delta.$$

On aura ainsi une forme Γ dans laquelle M_{n-1} et M_{n-2} seront premiers entre eux ; si M_{n-2} et M_{n-3} ont un diviseur commun, on s'en débarrassera en opérant une transformation analogue sur la matrice $\overset{(n-2)\times(n-2)}{\| \Gamma_{i,j} \|}$. On arrivera donc finalement à une forme Γ, dont la matrice sera caractéristique, et qui satisfera en même temps aux deux congruences.

Une telle forme sera pour nous désormais une forme canonique.

6. *Énumération des caractères supplémentaires.*—Il sera facile maintenant de faire une énumération complète des caractères supplémentaires d'un système donné. Soit I'_s le plus grand diviseur impair de I_s, μ_s l'exposant de la plus haute puissance de 2 qui divise I_s, augmenté d'une unité quand une des formes θ_{s-1}, θ_{s+1} est paire et de deux unités quand ces formes sont paires toutes les deux. Alors on a les règles suivantes pour former les caractères supplémentaires.

I. Si $\mu_s \geqq 2$, θ_s aura le caractère

$$(-1)^{\frac{1}{2}(\theta_s-1)}.$$

II. Si $\mu_s \geqq 3$, θ_s aura le caractère

$$(-1)^{\frac{1}{8}(\theta^2-1)}.$$

III. Si $\mu_s = 1$, $\mu_{s-1} \geqq 2$, $\mu_{s+1} \geqq 2$, θ_s aura le caractère

$$(-1)^{\frac{1}{8}(\theta_s^2-1)} \text{ ou } (-1)^{\frac{1}{2}(\theta_s-1)+\frac{1}{8}(\theta_s^2-1)},$$

selon qu'on aura

$$(-1)^{\frac{1}{2}(\theta_{s-1}-1)+\frac{1}{2}(\theta_{s+1}-1)} = (-1)^{\frac{1}{2}(I'_s+1)} \text{ ou } = (-1)^{\frac{1}{2}(I'_s-1)}.$$

IV. Si $\mu_s = 0$, $\mu_{s-1} \geqq 2$, $\mu_{s+1} \geqq 2$, et si, de plus, θ_s est impaire, cette forme aura le caractère

$$(-1)^{\frac{1}{2}(\theta_s-1)}$$

ou n'aura aucun caractère supplémentaire, selon qu'on aura

$$(-1)^{\frac{1}{2}(\theta_{s-1}-1)+\frac{1}{2}(\theta_s-1)} = (-1)^{\frac{1}{2}(I_s-1)} \text{ ou } = (-1)^{\frac{1}{2}(I_s+1)}.$$

La démonstration de ces règles s'obtient immédiatement, en se servant des formes canoniques. Ainsi, par exemple, pour démontrer la quatrième règle, observons qu'on a la congruence

$$\theta_s \equiv \prod_{s=1}^{s=s-1} \beta_s [\beta_s x_s^2 + I_s \beta_{s+1} x_{s+1}^2], \text{ mod } 4.$$

Donc θ_s aura le caractère $(-1)^{\frac{1}{2}(\theta_s-1)}$ ou n'aura aucun caractère, mod 4, selon que $I_s\beta_s\beta_{s+1}\equiv 1$ ou $\equiv -1$, mod 4, ce qui s'accorde avec la règle, puisque $\theta_{s-1}\times\theta_{s+1}\equiv\beta_s\times\beta_{s+1}$, mod 4.

Les suites de formes adjointes alternativement paires et impaires méritent une attention spéciale. Soit I_σ, $I_{\sigma+1}$, ..., $I_{\sigma+2\sigma'}$ une suite en nombre impair d'invariants impairs, $I_{\sigma-1}$ et $I_{\sigma+2\sigma'+1}$ étant pairs ; soient aussi paires les formes θ_σ, $\theta_{\sigma+2}$, ..., $\theta_{\sigma+2\sigma'}$. Aucune de ces formes n'aura un caractère supplémentaire, tandis que chacune des formes impaires $\theta_{\sigma-1}$, $\theta_{\sigma+1}$, $\theta_{\sigma+3}$, ..., $\theta_{\sigma+2\sigma'+1}$ aura un caractère par rapport à 4 (première règle). Mais ces caractères ne sont pas indépendants les uns des autres.

En effet, en se servant de la forme canonique, on établit l'équation

$$(-1)^{\frac{1}{2}(\theta_{\sigma+2s-1}-1)+\frac{1}{2}(\theta_{\sigma+2s+1}-1)}=(-1)^{\frac{1}{2}(I_{\sigma+2s}+1)},$$

d'où l'on tire cette autre

$$(-1)^{\frac{1}{2}(\theta_{\sigma+2s-1}-1)}=(-1)^s\times(-1)^{\frac{1}{2}(I_\sigma\times I_{\sigma+2}\times\ldots\times I_{\sigma+2s-2}-1)}\times(-1)^{\frac{1}{2}(\theta_{\sigma-1}-1)},$$

s désignant un quelconque des nombres 0, 1, 2, ..., $\sigma'+1$.

7. *Définition des caractères simultanés.*—Soit

$$\theta_{\sigma+1},\ \theta_{\sigma+2},\ \ldots,\ \theta_{\sigma+\sigma'},$$

une suite de σ' formes impaires ; supposons aussi qu'on ait

$$\mu_\sigma\geqq 2,\ \mu_{\sigma+\sigma'+1}\geqq 2,\ \mu_{\sigma+1}<2,\ \mu_{\sigma+2}<2,\ \ldots,\ \mu_{\sigma+\sigma'}<2\ ;$$

alors θ_σ et $\theta_{\sigma+\sigma'+1}$ auront des caractères supplémentaires ; mais aucune des formes $\theta_{\sigma+1}$, $\theta_{\sigma+2}$, ..., $\theta_{\sigma+\sigma'}$ ne pourra avoir un tel caractère. Désignons par $\psi(\sigma,\sigma')$ l'expression

$$\tfrac{1}{4}\Big[\sum_{s=\sigma}^{s=\sigma+\sigma'}(\theta_s-1)(\theta_{s+1}-1)+\sum_{s=\sigma+1}^{s=\sigma+\sigma'}(I_s+1)(\theta_s-1)+\tfrac{1}{2}\sum_{s=\sigma+1}^{s=\sigma+\sigma'}\mu_s(\theta_s^2-1)\Big],\quad\ldots\quad\text{(i)}$$

dans laquelle θ_σ, $\theta_{\sigma+1}$, ..., $\theta_{\sigma+\sigma'+1}$ sont des nombres représentés simultanément par ces mêmes formes et faisant partie d'un système caractéristique pour les formes θ_1, ..., θ_{n-1}. L'unité

$$(-1)^{\psi(\sigma,\sigma')}$$

aura la même valeur pour tous les systèmes de nombres

$$\theta_\sigma,\ \ldots,\quad\theta_{\sigma+\sigma'+1},$$

et cette valeur sera pour nous désormais le *caractère simultané* des formes $\theta_{\sigma+1}, \theta_{\sigma+2}, \ldots, \theta_{\sigma+\sigma'}$.

Pour démontrer que l'équation

$$(-1)^{\psi(\sigma,\sigma')} = \pm 1$$

jouit de la propriété *invariantive* que nous avons énoncée, considérons d'abord un système de nombres caractéristiques

$$1, \omega_1 m_1, \omega_2 m_2, \ldots, \omega_{n-1} m_{n-1}, 1,$$

m_s étant impair, et $\omega_s = 2$, ou 1, selon que θ_s est paire ou impaire. Ces nombres sont liés entre eux par les équations suivantes :

$$\begin{aligned}
\omega_1 m_1 \times m'_1 &= J_1^2 + I_1 \omega_2 m_2 \times 1, \\
\omega_2 m_2 \times m'_2 &= J_2^2 + I_2 \omega_3 m_3 \times \omega_1 m_1, \\
\omega_3 m_3 \times m'_3 &= J_3^2 + I_3 \omega_4 m_4 \times \omega_2 m_2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\omega_{n-1} m_{n-1} \times m'_{n-1} &= J_{n-1}^2 + I_{n-1} 1 \times \omega_{n-2} m_{n-2},
\end{aligned}$$

qui se déduisent de l'équation (7), article 1er, et dans lesquelles m'_s représente $\frac{1}{\Delta_s}[\alpha_{s,s}]$, le symbole $[\alpha_{s,s}]$ se rapportant à la matrice

$$\| a_{i,j} \|, \quad \begin{matrix} i = 1, 2, \ldots, s+1, \\ j = 1, 2, \ldots, s+1 \end{matrix}$$

De ces équations on tire les suivantes :

$$\left(\frac{\omega_2 m_2}{m_1}\right) = \left(\frac{-I_1}{m_1}\right),$$

$$\left(\frac{\omega_1 m_1}{m_2}\right) \times \left(\frac{\omega_3 m_3}{m_2}\right) = \left(\frac{-I_2}{m_2}\right),$$

$$\ldots\ldots\ldots$$

$$\left(\frac{\omega_{n-3} m_{n-3}}{m_{n-2}}\right) \times \left(\frac{\omega_{n-1} m_{n-1}}{m_{n-2}}\right) = \left(\frac{-I_{n-2}}{m_{n-2}}\right),$$

$$\left(\frac{\omega_{n-2} m_{n-2}}{m_{n-1}}\right) = \left(\frac{-1_{n-1}}{m_{n-1}}\right),$$

qu'on peut ensuite remplacer par celles-ci :

$$\left(\frac{m_2}{m_1}\right) = \left(\frac{2^{\mu_1}}{m_1}\right) \times \left(\frac{-I'_1}{m_1}\right),$$

$$\left(\frac{m_1}{m_2}\right) \times \left(\frac{m_3}{m_2}\right) = \left(\frac{2^{\mu_2}}{m_2}\right) \times \left(\frac{-I'_2}{m_2}\right),$$

. . . . . .

$$\left(\frac{m_{n-3}}{m_{n-2}}\right) \times \left(\frac{m_{n-1}}{m_{n-2}}\right) = \left(\frac{2^{\mu_{n-2}}}{m_{n-2}}\right) \times \left(\frac{-I'_{n-2}}{m_{n-2}}\right),$$

$$\left(\frac{m_{n-2}}{m_{n-1}}\right) = \left(\frac{2^{\mu_{n-1}}}{m_{n-1}}\right) \times \left(\frac{-I'_{n-1}}{m_{n-1}}\right).$$

Multiplions ensemble les équations de ce dernier système et transformons les deux nombres par la loi de réciprocité ; il en résulte

$$(-1)^{\frac{1}{4}\sum_{s=1}^{s=n-2}(m_s-1)(m_{s+1}-1)} \times (-1)^{\frac{1}{4}\sum_{s=1}^{s=n-1}(I'_s+1)(m_s-1)} \times (-1)^{\frac{1}{8}\sum_{s=1}^{s=n-1}\mu_s(m_s^2-1)} = \prod_{s=1}^{s=n-1}\left(\frac{m_s}{I'_s}\right),$$

relation qu'on peut aussi écrire

$$(-1)^{\psi(0,n-1)} = \prod_{s=1}^{s=n-1}\left(\frac{\theta'_s}{I'_s}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

en observant que $\theta_0 = 1$, $\theta_n = 1$, et en admettant que lorsque θ_s est une forme paire, on remplace θ_s par $\theta'_s = \frac{1}{2}\theta_s$. Il importe d'observer que les symboles θ'_s disparaissent d'eux-mêmes de l'expression $(-1)^{\psi(0,\,n-1)}$. En effet, les termes de $\psi(0, n-1)$, où figure θ'_s, sont les suivants :

$$\left[\tfrac{1}{2}(\theta_{s-1}) + \tfrac{1}{2}(\theta_{s+1}-1) + \tfrac{1}{2}(I_s+1)\right] \times \tfrac{1}{2}(\theta'_s - 1) + \mu_s \times \tfrac{1}{8}(\theta_s^2 - 1).$$

Mais $\mu_s = 0$, et nous avons déjà établi que le coefficient de $\frac{1}{2}(\theta'_s - 1)$ est pair.

On conclut de ce que nous venons de dire que l'expression $(-1)^{\psi(0,n-1)}$ dépend uniquement des caractères supplémentaires et simultanés des formes θ, et qu'il n'y entre aucun autre élément.

Supposons maintenant que les formes $\theta_1, \theta_2, \ldots, \theta_{n-1}$ soient toutes impaires, et que les invariants soient tous impairs ou impairement pairs. Dans ce cas,

aucune des formes $\theta_1, \theta_2, \ldots, \theta_{n-1}$ n'aura un caractère supplémentaire ; mais l'équation (ii) fait voir que l'ensemble de ces formes aura un caractère simultané

$$(-1)^{\psi(0,n-1)} = \pm 1,$$

puisque, en effet, la valeur de cette unité coïncide avec celle de $\Pi\left(\frac{\theta_s}{I_s}\right)$. Donc, dans ce cas particulier, l'existence d'un caractère simultané est démontrée. Pour étendre cette conclusion au cas général, revenons à la définition de $\psi(\sigma, \sigma')$, et remplaçons les formes θ par un système canonique équivalent.

On aura ainsi

$$\theta_{\sigma+1} \equiv a[\beta_{\sigma+1} x_{\sigma+1}^2 + \beta_{\sigma+2} I_{\sigma+1} x_{\sigma+2}^2 + \ldots + \beta_{\sigma+\sigma'+1} I_{\sigma+1} \times I_{\sigma+2} \times \ldots \times I_{\sigma+\sigma'} x_{\sigma+\sigma'+1}^2], \text{ mod } 4,$$

en désignant, pour abréger, par a le produit $\prod_{s=1}^{s=\sigma} \beta_s$. Dans l'expression à droite, substituons à $\beta_{\sigma+1}, \beta_{\sigma+2}, \ldots$, des nombres $b_{\sigma+1}, b_{\sigma+2}, \ldots$, positifs et premiers entre eux, respectivement congrus à $\beta_{\sigma+1}, \beta_{\sigma+2}, \ldots$, pour le module 8. Soit

$$\chi_1 = b_{\sigma+1} x_{\sigma+1}^2 + b_{\sigma+2} I_{\sigma+1} x_{\sigma+2}^2 + \ldots + b_{\sigma+\sigma'+1} I_{\sigma+1} \times I_{\sigma+2} \times \ldots \times I_{\sigma+\sigma'} x_{\sigma+\sigma'+1}^2,$$

les invariants de χ_1 seront $I_{\sigma+1}, I_{\sigma+2}, \ldots, I_{\sigma+\sigma'-1}, C \times I_{\sigma+\sigma'}$, en désignant par C le produit $\prod_{s=\sigma+1}^{s=\sigma+\sigma'+1} \beta_s$. Soient encore $\chi_2, \chi_3, \ldots$, les formes primitives adjointes à χ_1 ; on se convaincra sans peine que les congruences

$$\theta_{\sigma+s} \equiv a \times \chi_s, \quad \chi_s \equiv a \times \theta_{\sigma+s}, \text{ mod } 4, \quad \ldots\ldots \quad \text{(iii)}$$

ont lieu pour toutes les valeurs de s depuis 1 jusqu'à $\sigma'+1$; et, de plus, que lorsque $I_{\sigma+s}$ est pair, la congruence correspondante subsiste encore pour le module 8. Donc les nombres de tout système caractéristique pour les formes

$$\chi_1, \chi_2, \ldots, \chi_{\sigma'},$$

seront congrus, pour le module 4 ou 8, après avoir été multipliés par a, aux nombres correspondants d'un système caractéristique pour les formes θ.

Mais il a été démontré que les formes χ ont un caractère simultané ; soit

$(-1)^\rho$ ce caractère ; on aura

$$\rho \equiv \tfrac{1}{4} \sum_{s=1}^{s=\sigma'-1} (\chi_s - 1)(\chi_{s+1} - 1) + \tfrac{1}{4} \sum_{s=1}^{s=\sigma'} (I'_{\sigma+s} + 1)(\chi_s - 1)$$
$$+ \tfrac{1}{4}(C-1)(\chi_{\sigma'} - 1) + \sum_{s=1}^{s=\sigma'} (-1)^{\frac{\mu_{\sigma+s}}{8}(\chi_s^2 - 1)}, \text{ mod } 2.$$

Substituons, pour χ_1, χ_2, ..., leurs valeurs données par les congruences (iii), et faisons

$$b \equiv aC, \; ab \equiv C, \text{ mod } 4.$$

En observant que

$$\theta_\sigma \equiv a, \; \theta_{\sigma+\sigma'+1} \equiv b, \text{ mod } 4,$$

on aura la congruence

$$\rho = \psi(\sigma, \sigma') + \tfrac{1}{4}(a-1) \sum_{s=1}^{s=\sigma'} (I'_{\sigma+s} - 1) + \tfrac{1}{2}(a-1) + \tfrac{1}{8}\gamma(a^2-1),$$

où nous avons désigné par γ le nombre des invariants $I_{\sigma+1}$, ..., $I_{\sigma+\sigma'}$, qui ont une valeur paire. Donc, enfin, $(-1)^{\psi(\sigma,\sigma')}$ aura la même valeur pour tous les systèmes caractéristiques appartenant aux formes θ. C.Q.F.D.

Remarquons que si μ_σ est $\geqq 3$, θ_σ qui a certainement un caractère, mod 4, aura aussi un caractère, mod 8. Dans ce cas, l'unité $(-1)^\rho$ est elle-même invariantive ; la même chose a lieu si le nombre γ est pair.

L'équation (ii) est d'une grande importance dans la théorie qui nous occupe. Elle exprime une condition de possibilité pour l'existence des genres, et fait voir que de tous les caractères complets qu'on peut former pour un système donné d'invariants, la moitié correspond à des genres qui n'existent pas.

Nous verrons plus tard que les genres, dont les caractères satisfont à la condition de possibilité, existent toujours.

La considération des caractères simultanés est très utile en beaucoup de questions, qui regardent les formes quadratiques ; cependant, il faut admettre qu'on pourrait à la rigueur s'en passer dans la théorie des formes ternaires, quaternaires et quinaires. En effet, pour ces formes il n'y a jamais qu'un seul caractère simultané, dont par conséquent la valeur est déterminée par celle des autres caractères. Mais déjà, pour certaines classes de formes sénaires, il y en a deux dont le produit est déterminé par la condition de possibilité.

8. *Théorème I.*—Soit p un nombre premier impair, qui ne divise aucun des nombres $a_1, a_2, \ldots, a_n$; soit aussi

$$K = (-1)^i a_1 a_2 \ldots a_n,$$

n étant $= 2i+1$ ou $= 2i$; et désignons le nombre des solutions de la congruence

$$\sum_{s=1}^{s=n} a_s x_s^2 \equiv \mu, \mod p,$$

par

$$p^n \times \Phi(\mu) ;$$

on aura :

si $n = 2i+1$, ... (i)

$$\Phi(\mu) = \frac{1}{p} \times \left[1 + \left(\frac{K\mu}{p}\right)\frac{1}{p^i}\right];$$

si $n = 2i$, ... (ii)

$$\Phi(\mu) = \frac{1}{p} \times \left[1 + \left(\frac{K}{p}\right)\frac{1}{p^{i-1}} - \left(\frac{K}{p}\right)\frac{1}{p^i}\right]$$

ou

$$\Phi(\mu) = \frac{1}{p}\left[1 - \left(\frac{K}{p}\right)\frac{1}{p^i}\right],$$

selon que μ est congru, ou non, à zéro pour le module p.

Ce théorème est compris, comme cas très particulier, dans les résultats obtenus par M. Camille Jordan*.

Théorème II.—Soit f_1 une forme quadratique donnée, p_s un nombre premier qui divise I_s, mais ne divise aucun des nombres $I_1, I_2, \ldots, I_{s-1}$; le nombre $p^n \times \Phi(\mu)$ des solutions de la congruence

$$f_1 \equiv \mu, \mod p,$$

est donné par les formules suivantes, dans lesquelles on a fait

$$\Omega_0 = 1,$$

$$\Omega_1 = -I_1,$$

$$\Omega_2 = -I_2,$$

* *Traité des substitutions*, p. 156–160 ; *Journal de mathématiques*, 2e série, vol. xvii. p 368–402.

$$\Omega_3 = -I_3 \times -I_1,$$

. . . . . .

$$\Omega_\sigma = -I_\sigma \times -I_{\sigma-2} \times -I_{\sigma-4} \times \ldots$$

I. μ premier à p.

	$n = 2i$,	$n = 2i+1$,
$p =$	$\Phi(\mu) =$	$\Phi(\mu) =$
p_{n-1},	$\frac{1}{p}\left[1+\left(\frac{\mu\Omega_{n-2}\theta_{n-1}}{p}\right)\frac{1}{p^{i-1}}\right]$,	$\frac{1}{p}\left[1-\left(\frac{\Omega_{n-2}\theta_{n-1}}{p}\right)\frac{1}{p^{i}}\right]$,
p_{n-2},	$\frac{1}{p}\left[1-\left(\frac{\Omega_{n-3}\theta_{n-2}}{p}\right)\frac{1}{p^{i-1}}\right]$,	$\frac{1}{p}\left[1+\left(\frac{\mu\Omega_{n-3}\theta_{n-2}}{p}\right)\frac{1}{p^{i-1}}\right]$,
p_{n-3},	$\frac{1}{p}\left[1+\left(\frac{\mu\Omega_{n-4}\theta_{n-3}}{p}\right)\frac{1}{p^{i-2}}\right]$,	$\frac{1}{p}\left[1-\left(\frac{\Omega_{n-4}\theta_{n-3}}{p}\right)\frac{1}{p^{i-1}}\right]$,
. . . .	. . . . .	. . . . .
$p_{n-2\sigma}$,	$\frac{1}{p}\left[1-\left(\frac{\Omega_{n-2\sigma-1}\theta_{n-2\sigma}}{p}\right)\frac{1}{p^{i-\sigma}}\right]$,	$\frac{1}{p}\left[1+\left(\frac{\mu\Omega_{n-2\sigma-1}\theta_{n-2\sigma}}{p}\right)\frac{1}{p^{i-\sigma}}\right]$,
$p_{n-2\sigma-1}$,	$\frac{1}{p}\left[1+\left(\frac{\mu\Omega_{n-2\sigma-2}\theta_{n-2\sigma-1}}{p}\right)\frac{1}{p^{i-\sigma-1}}\right]$,	$\frac{1}{p}\left[1-\left(\frac{\Omega_{n-2\sigma-2}\theta_{n-2\sigma-1}}{p}\right)\frac{1}{p^{i-\sigma}}\right]$,
. . . .	. . . . .	. . . . .
p_1,	$\frac{1}{p}\left[1+\left(\frac{\mu\Omega_0\theta_1}{p}\right)\right]$,	$\frac{1}{p}\left[1-\left(\frac{\mu\Omega_0\theta_1}{p}\right)\right]$.

II. $\mu \equiv 0 \mod p$; $p = p_s$.

$$\Phi(0) = \frac{1}{p},$$

si s est impair,

$$\Phi(0) = \frac{1}{p}\left[1+\left(\frac{\Omega_{s-1}\theta_s}{p}\right)\frac{1}{p^{s-1}} - \left(\frac{\Omega_{s-1}\theta_s}{p}\right)\frac{1}{p^s}\right],$$

si s est pair.

La démonstration de ces formules se déduit immédiatement du théorème (I), en considérant, au lieu de la forme f_1, une forme canonique équivalente.

On voit que dans tous les cas le nombre $\Phi(\mu)$ s'exprime par les caractères quadratiques des invariants et par les caractères génériques principaux du système de formes adjointes à f_1. Si, au lieu de p, on prend une puissance

quelconque de ce nombre, le nombre des solutions s'exprime toujours par les mêmes caractères. Lorsqu'on prend pour module une puissance de 2, les caractères supplémentaires et simultanés figurent dans les résultats, qui deviennent par conséquent plus compliqués. Même pour les modules 4 et 8 (qui sont les seuls qu'il nous est nécessaire de considérer ici) il faut distinguer plusieurs cas. Pour abréger, nous ne nous occuperons que de quelques cas particuliers, qui nous seront utiles plus tard.

Théorème III.—Soit f_1 une forme impaire, dont tous les invariants sont impairs ; soit aussi

$$(-1)^{\sigma} = (-1)^{\psi(0,n-1)}$$

le caractère simultané des formes adjointes à f_1 ; ϵ une unité donnée positive ou négative ; Δ_s l'unité définie par la congruence

$$\Delta_s \equiv I_s \times I_{s-2} \times I_{s-4} \times \ldots, \text{ mod } 4,$$

$$s = 1,\ 2,\ \ldots,\ n-1 ;$$

enfin soit ω le résidu pour le module 2 du nombre

$$\tfrac{1}{4}(\Delta_{n-1}+1)(\Delta_{n-2}-1)+\tfrac{1}{4}(\Delta_{n-2}+1)(\Delta_{n-3}-1)+\ldots$$
$$+\tfrac{1}{4}(\Delta_3+1)(\Delta_2-1)+\tfrac{1}{4}(\Delta_2+1)(\Delta_1-1) ;$$

le nombre $2^{2n} \times \Phi(\epsilon)$ des solutions de la congruence

$$f_1 \equiv \epsilon, \text{ mod } 4,$$

sera donné par les formules

I. $n = 4i$, $\quad \Phi(\epsilon) = \tfrac{1}{4}\left[1 + (-1)^{i+\omega+\sigma} \dfrac{\epsilon(\Delta_{n-1}-1)}{2^{2i}}\right]$;

II. $n = 4i+1$, $\quad \Phi(\epsilon) = \tfrac{1}{4}\left[1 + (-1)^{i+\omega+\sigma} \dfrac{\epsilon\Delta_{n-1}}{2^{2i}}\right]$;

III. $n = 4i+2$, $\quad \Phi(\epsilon) = \tfrac{1}{4}\left[1 + (-1)^{i+\omega+\sigma} \dfrac{\epsilon(\Delta_{n-1}+1)}{2^{2i+1}}\right]$;

IV. $n = 4i+3$, $\quad \Phi(\epsilon) = \tfrac{1}{4}\left[1 + (-1)^{i+\omega+\sigma} \dfrac{\epsilon}{2^{2i+1}}\right]$.

Démonstration.—Soit

$$\lambda \equiv \beta_1 x_1^2 + \beta_2 I_1 x_2^2 + \beta_3 I_1 I_2 x_3^2 + \ldots + \beta_n I_1 I_2 I_3 \ldots I_{n-1} x_n^2$$

une forme canonique équivalente à f_1, le module étant une puissance quelconque de 2 ; écrivons aussi

$$\lambda \equiv \chi_1, \text{ mod } 4,$$

$$\chi_1 = \eta_1 x_1^2 + \eta_2 x_2^2 + \ldots + \eta_n x_n^2,$$

$\eta_1, \eta_2, \ldots, \eta_n$ étant des unités positives ou négatives.

Soit $\mu + \nu = n$, ν étant le nombre des unités négatives η ; $h = \frac{1}{2}(\nu - 1)$ ou $= \frac{1}{2}(\nu - 2)$, selon que ν est impair ou pair. En observant que le nombre des solutions de la congruence

$$\sum_{s=1}^{s=s} x_s^2 \equiv \epsilon, \text{ mod } 4,$$

est donné par la formule

$$2^{2s} \times \tfrac{1}{4}\left[1 + \frac{i^{-\epsilon}}{2^s}(1+i)^s + \frac{i^{\epsilon}}{2^s}(1-i)^s\right],$$

i étant l'unité imaginaire, on trouve sans peine que les formules du théorème sont vérifiées lorsqu'on y écrit $(-1)^h$ à la place de $(-1)^{\omega+\sigma}$. Il faut donc démontrer l'égalité de ces deux unités.

Soit

$$\begin{aligned}
\Sigma_n \equiv \tfrac{1}{4}\big[&(\eta_1 - 1)(\eta_1\eta_2 - 1) + (\eta_1\eta_2 - 1)(\eta_1\eta_2\eta_3 - 1) \\
&+ \ldots + (\eta_1\eta_2 \ldots \eta_{n-2} - 1)(\eta_1\eta_2 \ldots \eta_{n-1} - 1)\big] \\
+ \tfrac{1}{2}\big[&(\eta_1 - 1) + (\eta_1\eta_2 - 1) + (\eta_1\eta_2\eta_3 - 1) + \ldots \\
&+ (\eta_1\eta_2 \ldots \eta_{n-2} - 1) + \tfrac{1}{2}(\eta_1\eta_2 \ldots \eta_n + 1)(\eta_1 \ldots \eta_{n-1} - 1)\big] \\
\equiv \tfrac{1}{4}\big[&(\eta_1 - 1)(\eta_1\eta_2 + 1) + (\eta_1\eta_2 - 1)(\eta_1\eta_2\eta_3 + 1) + \ldots \\
&+ (\eta_1\eta_2 \ldots \eta_{n-1} - 1)(\eta_1\eta_2 \ldots \eta_n + 1)\big], \text{ mod } 2^*.
\end{aligned}$$

On trouve d'abord $\Sigma_n \equiv h$, mod 2 ; car la valeur de Σ_n (mod 2) ne change pas lorsqu'on y permute entre elles deux unités consécutives qui ont des signes

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* On peut regarder $(-1)^{\Sigma_n}$ comme le caractère simultané des formes adjointes à χ ;

$$\eta_1,\ \eta_1 \times \eta_2,\ \eta_1 \times \eta_2 \times \eta_3,\ \ldots,\ \eta_1 \times \eta_2 \times \ldots \times \eta_{n-1}$$

étant des nombres représentés simultanément par ces formes, et tous les invariants étant des unités positives, excepté le dernier, qui aura pour valeur

$$\eta_1 \times \eta_2 \times \ldots \times \eta_n = \Delta_{n-1}.$$

contraires ; de plus, la congruence $\Sigma_n \equiv h$, mod 2, se vérifie immédiatement si l'on suppose que les μ premières unités soient positives, les ν dernières négatives. Il reste donc à démontrer que $\Sigma_n \equiv \omega + \sigma$, mod 2. Écrivons, dans l'expression congruentielle de Σ_n, pour $\eta_1, \eta_2, \ldots, \eta_n$ les valeurs équivalentes

$$\eta_1 \equiv \beta_1, \quad \eta_2 \equiv \beta_2 I_1, \quad \eta_3 \equiv \beta_3 I_1 I_2, \ \ldots, \ \eta_n \equiv \beta_n I_1 I_2 \ldots I_{n-1}, \text{ mod } 4,$$

on aura

$$\begin{aligned}\Sigma_n \equiv \tfrac{1}{4}[&(\theta_1 - 1)(\Delta_1 \theta_2 + 1) + (\Delta_1 \theta_2 - 1)(\Delta_2 \theta_3 + 1) + \ldots \\ &+ (\Delta_{n-3}\theta_{n-2} - 1)(\Delta_{n-2}\theta_{n-1} + 1) + (\Delta_{n-2}\theta_{n-1} - 1)(\Delta_{n-1} + 1)], \text{ mod } 2 ;\end{aligned}$$

d'où l'on tire

$$\begin{aligned}\Sigma_n - \Sigma_{n-1} &\equiv \tfrac{1}{4}[(\Delta_{n-3}\theta_{n-2} - 1)(\Delta_{n-2}\theta_{n-1} + 1) \\ &\qquad + (\Delta_{n-2}\theta_{n-1} - 1)(\Delta_{n-1} + 1) - (\Delta_{n-3}\theta_{n-2} - 1)(\Delta_{n-2} + 1)] \\ &\equiv \tfrac{1}{4}(\Delta_{n-2} - 1)(\Delta_{n-1} + 1) \\ &\qquad + \tfrac{1}{4}[(\theta_{n-2} - 1)(\theta_{n-1} - 1) + (I_{n-1} + 1)(\theta_{n-1} - 1)], \text{ mod } 2 ;\end{aligned}$$

et, par conséquent, en ajoutant les valeurs de $\Sigma_n - \Sigma_{n-1}$, $\Sigma_{n-1} - \Sigma_{n-2}$, $\ldots$,

$$\Sigma_n \equiv \omega + \sigma, \text{ mod } 2. \qquad \text{C. Q. F. D.}$$

Voici des formules plus générales.

Soient $a \equiv \epsilon$, mod 4, un nombre quelconque impair, 2^δ une puissance quelconque de 2, $2^{n\delta}\Phi(a)$ le nombre des solutions de la congruence

$$f_1 \equiv a, \text{ mod } 2^\delta.$$

On aura, dans les mêmes cas que ci-dessus :

I. $$\Phi(a) = \frac{1}{2^\delta}\left[1 + (-1)^{i+\omega+\sigma}\frac{\epsilon(\Delta_{n-1} - 1)}{2^{2i}}\right].$$

II. Si $\epsilon \equiv \Delta_{n-1}$, mod 4.

$$\Phi(a) = \frac{1}{2^\delta}\left[1 + (-1)^{i+\omega+\sigma}\left[\frac{1}{2^{2i}} - \left(\frac{2}{a}\right)\frac{\Delta_{n-1}}{2^{2i-1}}\right]\right].$$

Si $\epsilon \equiv -\Delta_{n-1}$, mod 4.

$$\Phi(a) = \frac{1}{2^\delta}\left[1 - (-1)^{i+\omega+\sigma}\frac{1}{2^{2i}}\right].$$

III $$\Phi(a) = \frac{1}{2^\delta}\left[1 + (-1)^{i+\omega+\sigma}\frac{\epsilon(\Delta_{n-1} + 1)}{2^{2i+1}}\right].$$

IV. Si $\epsilon \equiv \Delta_{n-1}$, mod 4

$$\Phi(a) = \frac{1}{2^\delta}\left[1 + (-1)^{i+\omega+\sigma}\frac{\Delta_{n-1}}{2^{2i+1}}\right].$$

Si $\epsilon \equiv -\Delta_{n-1}$, mod 4

$$\Phi(a) = \frac{1}{2^\delta}\left[1 - (-1)^{i+\omega+\sigma}\left[\frac{\Delta_{n-1}}{2^{2i+1}} - \left(\frac{2}{a}\right)\frac{1}{2^{4i+1}}\right]\right].$$

Il existe des formules analogues pour les valeurs paires de a.
Ainsi, par exemple, en ne considérant que le module 4, on aura les formules

I. $$\Phi(2a) = \tfrac{1}{4}\left[1 + (-1)^{i+a+\omega+\sigma}(\Delta+1) \times \frac{1}{2^{2i}}\right].$$

II. $$\Phi(2a) = \tfrac{1}{4}\left[1 + (-1)^{i+a+\omega+\sigma} \times \frac{1}{2^{2i}}\right].$$

III. $$\Phi(2a) = \tfrac{1}{4}\left[1 + (-1)^{i+a+\omega+\sigma}(\Delta-1) \times \frac{1}{2^{2i+1}}\right].$$

IV. $$\Phi(2a) = \tfrac{1}{4}\left[1 + (-1)^{i+a+\omega+\sigma}\Delta \times \frac{1}{2^{2i+1}}\right],$$

le nombre des solutions de la congruence $f_1 \equiv 2a$, mod 4, étant toujours désigné par $2^{2n}\Phi(2a)$.

Pour les puissances plus élevées de 2, il faudrait distinguer entre les solutions *primitives* (celles dans lesquelles il y a au moins une des indéterminées dont la valeur soit impaire) et les solutions dérivées. Mais ces formules ne nous étant point nécessaires, nous pouvons les passer sous silence ici.

Théorème IV.—Soit f_1 une forme paire, dont tous les invariants sont impairs, on aura pour le nombre $2^n \times \Phi(1)$ des solutions de la congruence

$$\tfrac{1}{2}f_1 \equiv 1, \text{ mod } 2,$$

l'expression

$$\Phi(1) = \tfrac{1}{2}\left[1 - \left(\frac{2}{\Delta}\right)\frac{1}{2^{\frac{1}{2}n}}\right],$$

le nombre n des indéterminées étant nécessairement pair.

On tire la démonstration de cette formule de l'observation qu'une forme binaire improprement primitive représente des nombres impairement pairs pour

trois des quatre systèmes de valeurs (mod 2) des indéterminées, ou pour un seulement, selon que le déterminant de la forme est $\equiv 3$, ou $\equiv 7$, mod 8.

Supposons, par exemple, qu'on ait $n \equiv 0$, mod 4 ; $\Delta \equiv 1$, mod 8. Considérons une forme canonique équivalente à f_1, et supposons que les congruences

$$\alpha\alpha' - b^2 \equiv 3, \text{ mod } 8,$$

$$\alpha\alpha' - b^2 \equiv 7, \text{ mod } 8,$$

soient satisfaites pour μ et ν respectivement des form s binaires improprement primitives, dont la forme canonique se compose. Les deux nombres μ et ν seront pairs, et le nombre des solutions de la congruence proposée sera

$$\begin{aligned} &\tfrac{1}{4}[(1+3)^\mu - (1-3)^\mu] \times [(1+3)^\nu + (1-3)^\nu] \\ &\quad + \tfrac{1}{4}[(1+3)^\mu + (1-3)^\mu] \times [(1+3)^\nu - (1-3)^\nu] \\ &\quad = \tfrac{1}{2}[2^n - 2^{\frac{1}{2}n}], \end{aligned}$$

ce qui s'accorde avec la formule du théorème.

On démontre sans difficulté que la congruence

$$\tfrac{1}{2} f_1 \equiv a, \text{ mod } 2^\delta$$

admet le même nombre de solutions pour toute valeur impaire de a. Donc, en désignant, comme ci-dessus, par $2^{n\delta}\,\Phi(a)$ le nombre des solutions de cette congruence, on aura

$$\Phi(a) = \frac{1}{2^\delta}\left[1 - \left(\frac{2}{\Delta_{n-1}}\right)\frac{1}{2^{\frac{1}{2}n}}\right].$$

Théorème V.—Soit f_1 une forme dont tous les invariants sont impairs, excepté le premier I_1, qu'on suppose impairement pair ; soient aussi impaires les formes $f_2, f_3, \ldots$; en désignant par a un nombre impair et conservant les mêmes notations que ci-dessus, on aura pour le module 2^δ ($\delta \geqq 3$)

I. $$\Phi(a) = \frac{1}{2^\delta}\left[1 + (-1)^{i+\omega+\sigma}\left(\frac{2}{a}\right)\Delta^{\frac{1}{2}(a+1)} \times \frac{1}{2^{2i-1}}\right].$$

II. $$\Phi(a) = \frac{1}{2^\delta}\left[1 + (-1)^{i+\omega+\sigma}\left(\frac{-2}{a}\right)(\Delta + (-1)^{\frac{1}{2}(a-1)}) \times \frac{1}{2^{2i}}\right].$$

III. $$\Phi(a) = \frac{1}{2^\delta}\left[1 + (-1)^{i+\omega+\sigma}\left(\frac{2}{a}\right)\Delta^{\frac{1}{2}(a-1)} \times \frac{1}{2^{2i}}\right].$$

IV. $$\Phi(a) = \frac{1}{2^\delta}\left[1 + (-1)^{i+\omega+\sigma}\left(\frac{-2}{a}\right)(\Delta - (-1)^{\frac{1}{2}(a-1)}) \times \frac{1}{2^{2i+1}}\right].$$

Il est bien entendu qu'on doit remplacer I_1 par $\frac{1}{2} I_1$ dans les définitions des unités $(-1)^\omega$ et Δ, et que $(-1)^\sigma$ est toujours le caractère simultané de $f_1, f_2, \ldots$. La démonstration de ce théorème peut s'obtenir par le moyen des formules précédentes, en considérant la forme canonique

$$f_1 \equiv a_1 x_1^2 + I_1 [a_2 x_2^2 + a_3 I_2 x_2^3 + \ldots], \text{ mod } 2^\delta,$$

et en déterminant le nombre des solutions de la congruence

$$a_2 x_2^2 + a_3 I_2 x_3^2 + \ldots \equiv \mu, \text{ mod } 4,$$

dans laquelle il faut remarquer qu'on a

$$a_2 a_3 \ldots a_n \equiv a_1, \text{ mod } 4.$$

9. *Théorème I.*—Soit ϕ_1 une forme quadratique de $n-1$ indéterminées, représentée primitivement par f_1; soient aussi D_{n-1} le déterminant et ϕ_{n-2} la contravariante de ϕ_1 divisée par Δ_{n-2}; la forme

$$-I_{n-1} \times \phi_{n-2}$$

sera congrue à un carré pour le module $\dfrac{D_{n-1}}{\Delta_{n-1}}$.

En effet, en supposant que la matrice de ϕ_1 soit

$$\overset{(n-1)\times(n-1)}{\| a_{i,j} \|} \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

il y aura toujours une forme équivalente à f_1, dont la matrice sera

$$\overset{n\times n}{\| \alpha_{i,j} \|}, \quad \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

et l'on établira facilement l'équation

$$-\Delta_n \times [\alpha_{i,j}, \alpha_{n,n}] = [\alpha_{i,n}] \times [\alpha_{j,n}] - [\alpha_{i,j}] \times [\alpha_{n,n}], \quad \ldots\ldots \quad \text{(iii)}$$

qui subsiste pour toutes les valeurs de i et j depuis 1 jusqu'à $n-1$, et dont les deux membres deviennent identiquement zéro, si l'un de ces deux indices est égal à n. En observant que $[\alpha_{n,n}] = D_{n-1}$, on en conclut la congruence

$$-I_{n-1} \times \phi_{n-2} \equiv [\beta_{n,1} x_1 + \beta_{n,2} x_2 + \ldots + \beta_{n,n-1} x_{n-1}]^2, \text{ mod } \frac{D_{n-1}}{\Delta_{n-1}},$$

dans laquelle nous avons écrit

$$\beta_{n,i} = \frac{1}{\Delta_{n-1}} \times [\alpha_{n,i}].$$

Il faut remarquer (1) que si $\frac{D_{n-1}}{\Delta_{n-1}}$ est premier à Δ_{n-1}, la forme ϕ_{n-2} est primitive, (2) que même quand $\frac{D_{n-1}}{\Delta_{n-1}}$ est pair, la congruence (iv) subsiste pour les coefficients des *doubles* rectangles $2x_1 x_2$, $2x_1 x_3$, ...

Théorème II.—Soient f une forme primitive, p un nombre premier dont la puissance p^s divise le premier invariant de f, M un nombre premier à p, qui satisfait à la condition

$$\left(\frac{M}{p}\right) = \left(\frac{f}{p}\right),$$

la congruence

$$M \times f \equiv \square, \text{ mod } p^s,$$

sera résoluble, et admettra deux solutions.

Théorème III.—Soient f une forme proprement primitive, 2^s une puissance de 2 qui divise le premier invariant de f.

Si $s = 1$, la congruence

$$M \times f \equiv \square, \text{ mod } 2^s,$$

sera toujours résoluble, et admettra une seule solution.

Si $s = 2$, cette congruence admettra deux solutions, ou sera irrésoluble, selon que la condition

$$(-1)^{\frac{1}{2}(M-1)} = (-1)^{\frac{1}{2}(f-1)}$$

est satisfaite ou non.

Si $s \geqslant 3$, les conditions de résolubilité seront

$$(-1)^{\frac{1}{2}(M-1)} = (-1)^{\frac{1}{2}(f-1)}, \quad (-1)^{\frac{1}{8}(M^2-1)} = (-1)^{\frac{1}{8}(f^2-1)},$$

et lorsqu'elles sont satisfaites il y aura quatre solutions.

Problème.—Soit ϕ_1 une forme quadratique de $n-1$ indéterminées, ayant pour invariants I_1, I_2, ..., MI_{n-2}, et dont la contravariante primitive ϕ_{n-2} satisfait à la congruence

$$-I_{n-1} \times \phi_{n-2} \equiv \square, \text{ mod } M,$$

I_{n-1} aussi bien que I_{n-2}, I_{n-3}, ..., I_1, étant premier à M; trouver toutes les

formes de n indéterminées, ayant pour invariants I_1, I_2, ..., I_{n-2}, I_{n-1}, qui contiennent ϕ_1 comme partie.

La solution de ce problème pour $n=3$ a été donnée par Gauss dans les *Disquisitiones Arithmeticae*. Dans un mémoire sur les formes ternaires inséré dans les *Transactions de la Société royale de Londres* (vol. CLVII, 1867, p. 269), cette solution est présentée dans une forme qui admet une extension facile au cas général.

Soit comme ci-dessus

$$\overset{n-1\times n-1}{\| a_{i,j} \|}$$

la matrice de ϕ_1, et posons

$$-I_{n-1}\times\phi_{n-2}\equiv[\beta_{n,1}x_1+\beta_{n,2}x_2+\ldots+\beta_{n,n-1}x_{n-1}]^2, \text{ mod } M. \quad . \quad . \quad \text{(iv)}$$

Cette congruence indéterminée équivaut à $\frac{1}{2}(n^2-n)$ congruences numériques, qu'on peut écrire comme il suit :

$$\left.\begin{aligned} -I_{n-1}\times[a_{i,j},\ a_{n,n}] &= \beta_{i,j}M-\beta_{i,n}\beta_{j,n}, \\ i &= 1, 2, \ldots, n-1, \\ j &= 1, 2, \ldots, n-1, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

et qui fournissent un système de nombres entiers, avec lesquels nous formons la matrice symétrique

$$\overset{n\times n}{\| \beta_{i,j} \|},$$

en attribuant à $\beta_{n,n}$ [qui ne figure pas dans les équations (v)] la valeur M.

Cela posé, désignons par $a_{i,j}\times\Delta_{n-2}$ les déterminants mineurs de la matrice de ϕ_1, de sorte que

$$\Delta_{n-2}\times a_{i,j}=[a_{i,j},\ a_{n,n}],$$

et déterminons les nombres $a_{n,1}$, $a_{n,2}$, ..., $a_{n,n-1}$ par les équations

$$\left.\begin{aligned} a_{1,j}\,a_{n,1}+a_{2,j}\,a_{n,2}+\ldots+a_{n-1,j}\,a_{n,n-1} &= \frac{\Delta_{n-1}}{\Delta_{n-2}}\beta_{j,n}, \\ j &= 1, 2, 3, \ldots, n-1. \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(vi)}$$

Cette détermination est toujours possible, puisque le déterminant $\overset{n-1\times n-1}{\| a_{i,j} \|}$ est différent de zéro ; quant au nombre $a_{n,n}$, nous supposerons que sa valeur soit

donnée par l'équation

$$\left.\begin{aligned}\beta_{1,n}\,a_{n,1}+\beta_{2,n}\,a_{n,2}+\ldots+\beta_{n,n}\,a_{n,n}\\ =I_1\times I_2\times\ldots\times I_{n-2}\times I_{n-1}=\frac{\Delta_n}{\Delta_{n-1}},\end{aligned}\right\}\quad\ldots\quad\text{(vii)}$$

dans laquelle $\beta_{n,n}=M$ est différent de zéro. Les nombres $a_{n,1}$, $a_{n,2}$, ..., $a_{n,n}$, nous serviront pour compléter la matrice

$$\overset{n\times n}{\|\,a_{i,j}\,\|};\quad\ldots\quad\text{(viii)}$$

mais il reste à faire voir : (1) que ces éléments nouveaux sont entiers ; (2) que les invariants de la matrice ainsi complétée sont I_1, I_2, ..., I_{n-1}. Pour cela, observons d'abord que la matrice réciproque de la matrice (viii) est précisément

$$\overset{n\times n}{\|\,\Delta_{n-1}\times\beta_{i,j}\,\|}.$$

En effet, les équations (vi) peuvent s'écrire

$$[a_{n,j}]=\Delta_{n-1}\beta_{n,j},$$

$$j=1,\ 2,\ 3,\ \ldots,\ n-1\ ;$$

de plus, on a évidemment

$$[a_{n,n}]=\Delta_{n-1}\times M=\Delta_{n-1}\beta_{n,n}\ ;$$

et, par conséquent, en substituant les valeurs trouvées dans l'équation (vii),

$$\overset{n\times n}{|\,a_{i,j}\,|}=\Delta_n.$$

Il reste encore à démontrer les $\frac{1}{2}(n^2-n)$ égalités

$$[a_{i,j}]=\Delta_{n-1}\beta_{i,j},$$

$$i=1,\ 2,\ \ldots,\ n-1,\quad j=1,\ 2,\ \ldots,\ n-1\ :$$

on y parvient aussitôt en comparant les deux systèmes d'équations (iii) et (v), auxquels satisfont respectivement les nombres $[a_{ij}]$ et $[\beta_{ij}]$.

Maintenant, on conclut de la congruence (iv) que les nombres

$$\beta_{n,i},\quad i=1,\ 2,\ 3,\ \ldots,\ n,$$

sont premiers entre eux, puisque, s'il y avait un diviseur commun de tous ces nombres, I_{n-1} et $M=\beta_{n,n}$ ne seraient pas premiers entre eux.

Cela posé, les n équations

$$a_{i,1}\beta_{j,1}+a_{i,2}\beta_{j,2}+\ldots+a_{i,n}\beta_{j,n}=0,$$
$$j\gtrless i,$$
$$a_{i,1}\beta_{i,1}+a_{i,2}\beta_{i,2}+\ldots+a_{i,n}\beta_{i,n}=\frac{\Delta_n}{\Delta_{n-1}},$$

font voir que $a_{i,n}$ $(i<n)$ est entier, puisque tous les nombres qui entrent dans ces équations (hormis $a_{i,n}$) sont entiers ; et, de plus, les coefficients de $a_{i,n}$ dans les différentes équations sont premiers entre eux. La même démonstration réussit pour $i=n$, quand on suppose que la proposition est déjà démontrée pour les valeurs de i inférieures à n.

Nous avons déjà vu que le déterminant de la matrice (viii) est Δ_n ; on sait aussi que le plus grand diviseur des mineurs du $n^{ième}$ ordre est Δ_{n-1}, puisque $[a_{i,j}]=\Delta_{n-1}\times\beta_{i,j}$, les n nombres $\beta_{i,n}$ étant premiers entre eux. Mais par hypothèse le déterminant

$$\frac{1}{\Delta_{n-1}}\times\left|a_{i,j}\right|^{n-1\times n-1},$$

dont la valeur est M, est premier à Δ_{n-1}.

Donc, par un résultat connu (lemme II, art. 2), le plus grand diviseur des mineurs d'un ordre quelconque dans la matrice de Φ_1 est égal au plus grand diviseur commun des mineurs du même ordre dans la matrice (viii) ; d'où l'on conclut que les invariants de cette matrice sont $I_1, I_2, \ldots, I_{n-1}$, ainsi que nous l'avons énoncé.

Il est bon de remarquer que si l'on supposait que le nombre M fût premier à I_{n-1} seulement, sans être premier aux autres invariants $I_1, I_2, \ldots, I_{n-2}$, on pourrait déterminer comme ci-dessus la matrice (viii), dont les coefficients seraient entiers, et qui aurait Δ_n pour déterminant et Δ_{n-1} pour le plus grand diviseur commun de ses mineurs de l'ordre $n-1$. Mais il ne s'ensuivrait pas que $\Delta_{n-2}, \Delta_{n-3}, \ldots$, fussent les plus grands diviseurs communs des mineurs des ordres inférieurs, et par conséquent les invariants de la matrice (viii) pourraient être différents de $I_1, I_2, \ldots, I_{n-1}$.

Pour faire voir quelle est la relation entre les différentes formes f_1 qu'on peut construire, par le procédé que nous venons de décrire, en partant d'une même forme ϕ_1, remarquons que si l'on prenait comme solution de la congruence (iv) le système

$$\beta_{n,1}-k_1M,\quad \beta_{n,2}-k_2M,\quad \beta_{n,3}-k_3M,\quad \ldots,$$

au lieu du système

$$\beta_{n,1}, \quad \beta_{n,2}, \quad \beta_{n,3}, \quad \ldots,$$

la nouvelle forme f_1 serait équivalente à l'ancienne et s'en déduirait par la substitution

$$\left|\begin{array}{ccccc} 1, & 0, & 0, & \ldots, & k_1 \\ 0, & 1, & 0, & \ldots, & k_2 \\ 0, & 0, & 1, & \ldots, & k_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \ldots, & k_{n-1} \\ 0, & 0, & 0, & \ldots, & 1 \end{array}\right| \quad \ldots\ldots\ldots \quad \text{(ix)}$$

Réciproquement, si f_1' est une forme quelconque *contiguë* à f_1 [c'est-à-dire équivalente à f_1 par une substitution telle que (ix)], on démontrerait sans peine que ces deux formes conduisent à des solutions identiques (par rapport au module M) de la congruence (iv). On conclut de là qu'on peut construire un système de $2^{\mu+j}$ formes f_1 différentes entre elles, dont aucune n'est contiguë à une autre, et ayant cette propriété que toute forme qui contient ϕ_1 comme partie sera contiguë à une d'entre elles.

10. La forme ϕ_1 aux invariants $[I_1, I_2, \ldots, MI_{n-2}]$ étant représentée primitivement par la forme f_1 aux invariants $[I_1, I_2, \ldots, I_{n-1}]$, et le nombre M étant toujours assujetti à être premier aux invariants, les caractères ordinaux et génériques de f_1 (sauf un cas d'exception dont nous ferons mention plus tard) sont déterminés par ceux de ϕ_1.

(1) Soit M impair ; il s'ensuit que θ_{n-1} est une forme impaire ; si ϕ_{n-2} est une forme impaire, ou si, ϕ_{n-2} étant paire, I_{n-1} est aussi pair, on voit sans peine que tout système de nombres, caractéristique pour les formes $\phi_1, \phi_2, \ldots$, devient un système caractéristique pour $\theta_1, \theta_2, \ldots$, en y joignant le nombre M. Mais si ϕ_{n-2} est pair, I_{n-1} impair, on a d'abord I_{n-2} impair, puis θ_{n-2} impaire, parce qu'une suite de formes alternativement paires et impaires ne peut pas se terminer par une forme impaire θ_{n-1} ayant un invariant impair I_{n-1}. Donc, dans ce cas, les nombres caractéristiques pour $\ldots, \phi_{n-3}, \phi_{n-2}$, ne peuvent pas, à la rigueur, être regardés comme caractéristiques pour les formes $\ldots, \theta_{n-3}, \theta_{n-2}$. Cependant ces nombres suffisent pour déterminer les caractères génériques de $\ldots, \theta_{n-3}, \theta_{n-2}, \theta_{n-1}$, puisque, les invariants $\ldots, I_{n-2}, I_{n-1}$ étant impairs, il

n'est question que des caractères principaux ; le caractère simultané de ..., θ_{n-2}, θ_{n-1} se déduisant de la condition de possibilité.

(2) Soit M pair, ce qui implique que tous les invariants sont impairs ; si n est pair, les formes ϕ_1, ϕ_2, ..., θ_1, θ_2, ... sont nécessairement impaires, puisque dans les deux cas il y a une suite d'un nombre pair d'invariants impairs ; la même chose a lieu si n est pair et ϕ_1 impaire. Mais si n est pair et la forme ϕ_1 aussi paire, il peut y avoir une ambiguïté dans la détermination de l'ordre de θ_1, puisque les formes θ_1, θ_2, ..., peuvent être ou impaires ou alternativement paires et impaires. Dans chacun de ces deux ordres comme dans les cas où il n'y a qu'un seul ordre admissible, les caractères génériques des formes θ_1, θ_2, ... sont absolument déterminés.

Le cas d'ambiguïté que nous avons signalé mérite un examen plus attentif ; cependant nous le laissons de côté ici, puisqu'il ne nous est pas nécessaire pour la théorie de la décomposition des nombres en cinq carrés.

Réciproquement, le caractère générique de ϕ_1 est déterminé par celui de f_1. On a, en effet,

$$\left(\frac{\phi_s}{q}\right) = \left(\frac{\theta_s}{q}\right), \quad s = 1, 2, \ldots, n-2,$$

si q est un diviseur premier de I_s ; de même, si I_s est divisible par 4 ou 8, les caractères de θ_s et ϕ_s par rapport à ces modules sont coïncidents ; enfin, les caractères de ϕ_{n-2} par rapport aux diviseurs de M se déterminent d'après la congruence

$$-I_{n-1} \times \phi_{n-2} \equiv \square, \text{ mod } M.$$

Quant aux caractères simultanés, ceux qui se rapportent à des suites de formes qui se terminent avant d'arriver à θ_{n-2} ou ϕ_{n-2} ont la même valeur dans les deux systèmes ; tandis que la valeur du caractère simultané qui peut exister pour les dernières formes de chaque système est déterminée par la condition de possibilité.

Cependant, il faut bien remarquer qu'ici encore il y a un cas d'ambiguïté. Si M est impair, et si la série ..., I_{n-3}, I_{n-2}, I_{n-1} se termine par une suite d'un nombre pair d'invariants impairs, les formes ..., θ_{n-3}, θ_{n-2}, θ_{n-1} seront nécessairement impaires, mais les formes ..., ϕ_{n-3}, ϕ_{n-2} pourront être ou impaires ou alternativement impaires et paires. Donc, dans ce cas, il y a deux genres de formes des invariants $[I_1, I_2, \ldots, MI_{n-2}]$, qui peuvent être représentés par un

genre donné de formes des invariants $[I_1, I_2, \ldots, I_{n-2}, I_{n-1}]$; mais ces deux genres appartiennent à des ordres différents. Ce cas est très important pour notre but actuel, puisqu'il se présente dans la théorie de la décomposition des formes quaternaires en cinq carrés.

11. D'après la définition d'Eisenstein, la *densité* d'une classe de formes quadratiques est la fraction qu'on obtient en divisant l'unité par le nombre des substitutions linéaires, à déterminant $+1$, qui transforment en elle-même une forme quelconque de la classe. La densité d'un genre est la somme des densités des classes qui appartiennent au genre. Ajoutons à cette définition que la densité de la représentation primitive d'une forme de $n-s$ indéterminées par une forme de n indéterminées est le produit des densités des deux classes auxquelles ces formes appartiennent ; en particulier, la densité de la représentation d'un nombre par une forme est la densité de cette forme, ou plus exactement de la classe à laquelle la forme appartient.

Soit

$$\overset{n\times n-1}{\| \delta_{i,j} \|}, \quad \left.\begin{matrix} i=1,\ 2,\ \ldots,\ n-1, \\ j=1,\ 2,\ \ldots,\ n \end{matrix}\right\} \quad \ldots\ldots\ldots \quad \text{(i)}$$

une matrice de $n-1$ colonnes verticales qui sert à représenter primitivement la forme ϕ_1 par f_1, et désignons par $[\delta_{n,1}]$, $[\delta_{n,2}]$, $\ldots$, $[\delta_{n,n}]$ les déterminants de cette matrice, qui seront premiers entre eux, puisque la représentation est primitive. Alors M sera représenté par θ_{n-1}, en attribuant aux indéterminées les valeurs $[\delta_{n,1}]$, $[\delta_{n,2}]$, $\ldots$. Nous dirons que ces deux représentations sont *contravariantes* ou *simultanées*.

Étant donnée une représentation de ϕ_1 par θ_1, il n'y a évidemment qu'une seule représentation de M par θ_{n-1}. Mais si $\frac{1}{\delta}$ est la densité de ϕ_1 ou de ϕ_{n-1}, à une représentation donnée de M par θ_{n-1}, il correspond δ représentations différentes de ϕ_1 par θ_1. En effet, les matrices (i) qui ont $[\delta_{n,1}]$, $[\delta_{n,2}]$, $\ldots$, pour déterminants sont comprises dans la formule

$$\overset{n\times n-1}{\| \delta_{i,j} \|} \times \overset{n-1\times n-1}{\| \sigma_{i,j} \|},$$

la seconde matrice ayant $+1$ pour son déterminant ; d'où l'on voit qu'il y en a précisément δ qui transforment f_1 en ϕ_1. On conclut que la densité d'une

représentation donnée de M par θ_{n-1} est toujours égale à la densité des représentations contravariantes de ϕ_1 par θ_1.

On peut se servir des considérations précédentes pour démontrer la proposition importante, 'que tout genre dont le caractère complet satisfait à la condition de possibilité, existe effectivement.' Pour cela, considérons un caractère générique donné, qui satisfait à la condition de possibilité, pour les invariants $I_1, I_2, \ldots, I_{n-1}$. Soit M un nombre, dont les caractères par rapport aux diviseurs de I_{n-1} coïncident avec les caractères génériques proposés; il faudra que M soit premier aux invariants et impair ou impairement pair, selon que la forme contravariante θ_{n-1} est impaire ou paire. En supposant que la proposition soit vraie pour les formes de $n-1$ indéterminées, imaginons un genre auxiliaire des invariants $[I_1, I_2, \ldots, MI_{n-2}]$, dont les caractères, par rapport aux diviseurs de $I_1, I_2, \ldots, I_{n-2}$, soient les mêmes que les caractères proposés, et dont les caractères, par rapport aux diviseurs premiers impaires de M, soient les mêmes que ceux de $-I_{n-1}$. Pour compléter ce caractère générique, on lui assignera les mêmes caractères simultanés qui font partie du caractère générique proposé, exception faite du caractère simultané des *dernières* formes adjointes, si toutefois il y a un tel caractère. Or, le caractère qu'on a supposé au genre auxiliaire satisfait à la condition de possibilité. En effet, si $\omega = 1$ est cette condition et $\Omega = 1$ l'équation correspondante qui par hypothèse est satisfaite par le caractère générique proposé, les facteurs de Ω qui ne se présentent pas en ω sont

$$\left.\begin{aligned}\left(\frac{\theta_{n-1}}{I'_{n-1}}\right) \times (-1)^{\frac{1}{4}(\theta_{n-2}-1)(\theta^{n-1}-1)} \times (-1)^{\frac{1}{4}(I'_{n-1}+1)(\theta_{n-1}-1)} \\ \times (-1)^{\mu_{n-1}\frac{1}{8}(M^2-1)},\end{aligned}\right\} \quad \ldots \quad \text{(ii)}$$

et les facteurs qui se présentent en ω sans être facteurs de Ω sont

$$\left(\frac{\phi_{n-2}}{M}\right) \times (-1)^{\frac{1}{4}(M-1)(\phi_{n-2}-1)} \quad \ldots \quad \text{(iii)}$$

En faisant attention à l'équation

$$\left(\frac{\phi_{n-2}}{M}\right) = \left(\frac{-I_{n-1}}{M}\right) = (-1)^{\frac{1}{4}(I_{n-1}+1)(M-1)} \times (-1)^{\mu_{n-1}\frac{1}{8}(M^2-1)} \times \left(\frac{M}{I'_{n-1}}\right),$$

et en observant que, dans les formules (ii) et (iii), on peut remplacer θ_{n-1} par M et $\frac{1}{2}(\phi_{n-2}-1)$ par $\frac{1}{2}(\theta_{n-2}-1)$, on voit que les deux unités exprimées par ces

formules sont égales, et que, par conséquent, l'une des équations $\Omega = 1$, $\omega = 1$, entraîne nécessairement l'autre. Donc le genre auxiliaire existe actuellement, et une forme quelconque de ce genre servira pour construire une forme du genre proposé, dont l'existence est ainsi établie.

Pour abréger, nous avons supposé M impair ; mais la démonstration n'exige que de légers changements pour être applicable au cas où M serait pair.

12. *Théorème.*—Soient $f_1, f_1', f_1'', \ldots$, des formes représentant les classes différentes d'un genre donné (Γ) aux invariants $[I_1, I_2, \ldots, I_{n-1}]$; soient de même $\phi_1, \phi_1', \phi_1'', \ldots$ les formes représentant les classes, aux invariants $[I_1, I_2, \ldots, MI_{n-2}]$, qui admettent une représentation par les formes de (Γ) ; en supposant toujours que M soit premier à tous les invariants de (Γ), soit μ le nombre des diviseurs premiers impairs de M ; soit $j = 0, 1, 2$, selon que M est (1) impair ou impairement pair ; $(2) \equiv 0$, mod 4 ; $(3) \equiv 0$, mod 8 ; soit enfin γ la densité totale des classes $(\phi_1), (\phi_1'), \ldots$; la somme des densités des représentations primitives, soit des formes $(\phi_1), (\phi_1'), \ldots$ par les formes $(f_1), (f_1'), \ldots$, soit du nombre M par les formes contravariantes $(f_{n-1}), (f_{n-1}'), \ldots$, sera égale à $2^{\mu+j} \times \gamma$.

On se rappellera que les formes $(\phi_1), (\phi_1'), \ldots$, représentent les classes, soit d'un genre, soit de deux genres qui appartiennent à des ordres différents. Mais cette remarque, quoique très importante pour les applications, n'a aucune influence sur la démonstration du théorème. Pour abréger, nous supposerons qu'on ait exclu le cas dans lequel n et M sont tous les deux pairs ; cette exclusion faite, on peut s'assurer que toutes les formes $(\phi_1), (\phi_1'), \ldots$, sont représentées par un seul genre (Γ).

Pour le cas $n = 3$, la démonstration du théorème se trouve dans le mémoire *Sur les formes quadratiques ternaires* (*Philosophical Transactions*, vol. CLVII). Le même raisonnement, avec de légers changements, s'applique au cas général, et il suffira de la reproduire très succinctement ici.

Par le procédé de l'article (9), de chaque forme ϕ_1 on déduit $2^{\mu+j}$ formes F_1, qui contiennent ϕ_1 comme partie et qui appartiennent au genre (Γ) ; ces formes sont toutes différentes entre elles et il ne se trouve pas parmi elles deux formes contiguës. Considérons une quelconque des formes représentatives f_1 ; on voit facilement qu'il y a autant de représentations de ϕ_1 par f_1 qu'il y a de transformations de f_1 dans des formes F_1. Soient $\frac{1}{d}$ la densité de f_1, s le nombre des formes F_1 qui sont équivalentes à f_1 ; le nombre des transformations de f_1 en des

formes F_1 sera $d \times s$; par conséquent, si $\frac{1}{\delta}$ est la densité de ϕ_1, la densité des représentations de ϕ_1 par f_1 sera $d \times s \times \frac{1}{d\delta} = \frac{s}{\delta}$. Étendons cette conclusion à toutes les formes $f_1, f_1', \ldots$; la densité des représentations de ϕ_1 par les formes de (Γ) sera $\frac{1}{\delta} \times \Sigma s$; mais $\Sigma s = 2^{\mu+j}$, parce qu'il y a $2^{\mu+j}$ formes F_1, dont chacune est équivalente à une, et à une seulement des formes f_1. Donc enfin la densité des représentations de toutes les formes $\phi_1, \phi_1', \ldots$, par les formes de Γ sera

$$2^{\mu+j} \times \Sigma \frac{1}{\delta} = 2^{\mu+j} \times \gamma. \qquad \text{C. Q. F. D.}$$

13. Soient I_1, I_2, I_3 les invariants d'un système de formes de quatre indéterminées x_1, x_2, x_3, x_4; (Γ) un genre donné de ces formes. Nous nous proposons de déterminer la densité W de (Γ), en supposant d'abord que les formes $\theta_1, \theta_2, \theta_3$ et les invariants I_1, I_2, I_3 soient tous impairs.

Nous désignerons par q les nombres premiers qui divisent les invariants; q_i ne divisera qu'un seul de ces nombres, $q_{i,j}$ en divisera deux, $q_{1,2,3}$ divisera tous les trois. Soient L un nombre positif qu'on peut supposer aussi grand qu'on voudra, M un nombre impair et premier à Πq_1 qui ne surpasse pas L, qui est représenté (primitivement ou non) par θ_3, et qui en outre satisfait aux conditions $M \equiv 1$, mod 4, $\left(\frac{M}{q_1}\right) \equiv 1$, mod q_1, pour toute valeur de q_1 (on verra plus tard l'avantage qu'il y a à introduire ces restrictions). Soit enfin T la somme des densités des représentations de tous les nombres M par les contravariantes $\theta_3, \theta_3', \theta_3'', \ldots$, des formes $\theta_1, \theta_1', \theta_1'', \ldots$, comprises dans (Γ). On parvient à déterminer la densité de (Γ) en égalant deux expressions différentes qu'on peut trouver pour la limite de $\frac{T}{L^2}$, lorsqu'on fait croître L indéfiniment.

I. Les valeurs positives ou négatives de x_1, x_2, x_3, x_4, pour lesquelles une forme donnée θ_3 acquiert une des valeurs désignées par M, sont évidemment distribuées dans un certain nombre de progressions arithmétiques

$$\left.\begin{array}{ll} x_1 = \Delta y_1 + \lambda_1^{(\mu)}, & x_3 = \Delta y_3 + \lambda_3^{(\mu)}, \\ x_2 = \Delta y_2 + \lambda_2^{(\mu)}, & x_4 = \Delta y_4 + \lambda_4^{(\mu)}, \end{array}\right\} \quad \ldots\ldots\ldots \quad (1)$$

dont la différence commune est $4\Pi q = \Delta$.

Nous désignerons par $\Delta^4 \times \phi(\Delta)$ le nombre de ces progressions; d'après les

théorèmes de l'article (8), on aura pour $\phi(\Delta)$ l'expression

$$\left.\begin{aligned}\phi(\Delta) = \tfrac{1}{4} \times \tau \times \Pi\left(1-\frac{1}{q}\right) \times \Pi\tfrac{1}{2}\left[1+\left(\frac{-I_2\theta_1}{q_1}\right)\frac{1}{q_1}\right] \\ \times \Pi\left[1-\left(\frac{-I_3\theta_2}{q_{1,2}}\right)\frac{1}{q_{1,2}}\right] \times \Pi\left[1-\left(\frac{-I_3\theta_2}{q_2}\right)\frac{1}{q_2}\right],\end{aligned}\right\} \quad \ldots \quad (2)$$

où l'on a

$$\tau = 1 \quad \text{si} \quad I_1 \times I_3 \equiv 1, \text{ mod } 4,$$

et $$\tau = \tfrac{1}{2}\left[2+(-1)^{\frac{1}{4}(I_2+1)(I_3-1)+\sigma}\right] \quad \text{si} \quad I_1 \times I_3 \equiv -1, \text{ mod } 4,$$

en désignant par $(-1)^\sigma$ le caractère simultané des formes θ_1, θ_2, θ_3, c'est-à-dire de (Γ).

Soit ν_i le nombre des systèmes de valeurs de x_1, x_2, x_3, x_4 comprises dans la $i^{ième}$ des formules (1), qui satisfont à l'inégalité

$$\theta_3(x_1, x_2, x_3, x_4) < L;$$

on a, d'après les principes connus,

$$Lim\frac{\Delta^4\nu_i}{L^2} = \frac{1}{2}\frac{\pi^2}{I_2 I_3\sqrt{I_1 I_3}},$$

et, par conséquent,

$$Lim\frac{\Sigma\nu_i}{L^2} = \frac{1}{2} \times \frac{\pi^2}{I_2 I_3\sqrt{I_1 I_3}} \times \phi(\Delta).$$

Multiplions chaque membre de cette équation par la densité de la classe θ_1 ou θ_3, et prenons la somme des produits pour toutes les classes de (Γ) ; on trouve immédiatement

$$Lim\frac{T}{L^4} = \frac{1}{2}W \times \frac{\pi^2}{I_2 I_3 \times \sqrt{I_1 I_3}} \times \phi(\Delta).$$

II. On obtient une seconde détermination de la même limite en se servant du théorème de l'article (10).

Soit (γ) le genre de formes ternaires aux invariants (I_1, MI_2), qui admet une représentation primitive par les formes de Γ ; ce genre est certainement impair, puisque les deux invariants sont impairs. Soient ϕ_1 une forme ternaire appartenant au genre (γ), et qui peut être représentée primitivement par θ_1 ; ϕ_2 la contravariante de ϕ_1. Les caractères principaux de (γ) se déterminent par les équations suivantes, dans lesquelles μ désigne un diviseur quelconque impair de M :

$$\left.\begin{aligned}
&\left(\frac{\phi_1}{q_1}\right)=\left(\frac{\theta_1}{q_1}\right),\quad \left(\frac{\phi_1}{q_{1,2}}\right)=\left(\frac{\theta_1}{q_{1,2}}\right),\quad \left(\frac{\phi_1}{q_{1,2,3}}\right)=\left(\frac{\theta_1}{q_{1,2,3}}\right),\\
&\left(\frac{\phi_2}{q_2}\right)=\left(\frac{\theta_2}{q_2}\right),\quad \left(\frac{\phi_2}{q_{1,2}}\right)=\left(\frac{\theta_2}{q_{1,2}}\right),\quad \left(\frac{\phi_2}{q_{1,2,3}}\right)=\left(\frac{\theta_2}{q_{1,2,3}}\right),\\
&\left(\frac{\phi_2}{\mu}\right)=\left(\frac{-I_3}{\mu}\right).
\end{aligned}\right\} \quad . \quad . \quad (3)$$

Il n'y a aucun caractère supplémentaire.

Soit $(-1)^{\sigma'}$ le caractère simultané de ϕ_1, ϕ_2 ; on aura

$$(-1)^{\sigma'}=(-1)^{\sigma}.$$

En effet, les valeurs de σ' et σ sont données respectivement par les congruences

$$\left.\begin{aligned}
&\sigma'\equiv\tfrac{1}{4}[(\phi_1-1)(\phi_2-1)+(I_1+1)(\phi_1-1)+(I_2+1)(\phi_2-1)],\\
&\sigma\equiv\tfrac{1}{4}[(\theta_1-1)(\theta_2-1)+(\theta_2-1)(\theta_3-1)+(I_1+1)(\theta_1-1)\\
&\qquad+(I_2+1)(\theta_2-1)+(I_3+1)(\theta_3-1)],\ \text{mod } 2\ ;
\end{aligned}\right\} \quad . \quad . \quad (4)$$

d'ailleurs, si (m_1, m_2) est un système caractéristique pour les formes (ϕ_1, ϕ_2), (m_1, m_2, M) est un système caractéristique pour $(\theta_1, \theta_2, \theta_3)$, de sorte que dans les congruences (ii) on puisse écrire m_1, m_2 à la place de ϕ_1, ϕ_2, et m_1, m_2, M à la place de θ_1, θ_2, θ_3 ; cette substitution met en évidence la congruence $\sigma'\equiv\sigma$, mod 2.

Cela posé, on a, d'après une formule connue*, l'expression suivante pour la densité w du genre (γ) :

$$\begin{aligned}
w=\tfrac{1}{12}M\times I_1I_2\times\tfrac{1}{2}[2+(-1)^{\frac{1}{4}(I_1+1)(MI_2+1)+\sigma'}]\\
\times\Pi\tfrac{1}{4}\left(1-\frac{1}{q_{1,2}^2}\right)\times\Pi\tfrac{1}{4}\left(1-\frac{1}{q_{1,2,3}^2}\right)\\
\times\Pi\tfrac{1}{2}\left[1+\left(\frac{-MI_2\phi_1}{q_1}\right)\frac{1}{q_1}\right]\times\Pi\tfrac{1}{2}\left[1+\left(\frac{-MI_2\phi_1}{q_{1,3}}\right)\frac{1}{q_{1,3}}\right]\\
\times\Pi\tfrac{1}{2}\left[1+\left(\frac{-I_1\phi_2}{q_2}\right)\frac{1}{q_2}\right]\times\Pi\tfrac{1}{2}\left[1+\left(\frac{-I_1\phi_2}{q_{2,3}}\right)\frac{1}{q_{2,3}}\right]\\
\times\Pi\tfrac{1}{2}\left[1+\left(\frac{-I_1\phi_2}{\mu}\right)\frac{1}{\mu}\right].
\end{aligned}$$

* *Phil. Trans.*, vol. clvii. p. 291 [vol. i. p. 499].

En multipliant par 2^μ et faisant attention aux équations (3) et aux conditions satisfaites par M, cette formule devient

$$\left.\begin{aligned} 2^\mu \times w = {} & \tfrac{1}{4} \times I_1 I_2 \times \tfrac{1}{6}\left[2 + (-1)^{\frac{1}{4}(I_1+1)(I_2+1)+\sigma}\right] \\ & \times \Pi \tfrac{1}{4}\left(1 - \frac{1}{q_{1,2}^2}\right) \times \Pi \tfrac{1}{4}\left(1 - \frac{1}{q_{1,2,3}^2}\right) \\ & \times \Pi \tfrac{1}{2}\left[1 + \left(\frac{-I_2\theta_1}{q_1}\right)\right] \times \Pi \tfrac{1}{2}\left[1 + \left(\frac{-I_2\theta_1\theta_3}{q_{1,3}}\right)\frac{1}{q_{1,3}}\right] \\ & \times \Pi \tfrac{1}{2}\left[1 + \left(\frac{-I_1\theta_2}{q_2}\right)\frac{1}{q_2}\right] \times \Pi \tfrac{1}{2}\left[1 + \left(\frac{-I_1\theta_2}{q_{2,3}}\right)\frac{1}{q_{2,3}}\right] \\ & \times M \times \Pi\left[1 + \left(\frac{I_1 I_2}{\mu}\right)\frac{1}{\mu}\right] \\ = {} & \rho \times M \times \Pi\left[1 + \left(\frac{I_1 I_2}{\mu}\right)\frac{1}{\mu}\right], \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

ρ étant un coefficient qui reste le même pour tous les nombres M et pour tous les genres ternaires (γ), qui ont (I_1, MI_2) pour invariants, et qui peuvent être représentés primitivement par les formes de (Γ).

D'après le théorème de l'article (10), la formule (5) exprime la densité des représentations primitives du nombre donné M par les formes de Γ. Donc la densité totale des représentations, tant dérivées que primitives, de M par ces mêmes formes est exprimée par la formule

$$\rho \times \Sigma\left[\frac{M}{d^2} \times \Pi\left[1 + \left(\frac{I_1 I_2}{\mu}\right)\frac{1}{\mu}\right]\right],$$

dans laquelle le signe de sommation se rapporte à tous les diviseurs carrés de M, μ désignant maintenant un diviseur premier de $\frac{M}{d^2}$.

On en déduit l'équation

$$\frac{T}{L^2} = \frac{\rho}{L^2} \times \sum_{M \gtrless L}^{M \gtrless 1} \Sigma\left[\frac{M}{d^2} \times \Pi\left[1 + \left(\frac{I_1 I_2}{\mu}\right)\frac{1}{\mu}\right]\right],$$

qu'on peut aussi écrire

$$\frac{T}{L^2} = \frac{\rho}{L^2} \times \sum_{M \gtrless L}^{M \gtrless 1} \Sigma\left(\frac{I_1 I_2}{d}\right)\frac{M}{d},$$

le nombre d désignant, dans cette dernière formule, un diviseur quelconque

de M. Pour passer aux limites, posons $\frac{M}{d} = \delta$, et effectuons en premier lieu la sommation par rapport à δ. Le nombre d peut avoir une valeur quelconque première à $4\Pi q$ et comprise entre 1 et L; tandis que, pour une valeur donnée de d, δ peut avoir seulement des valeurs comprises entre 1 et $\frac{L}{d}$, et appartenant à certaines progressions arithmétiques dont la différence commune est $\Delta = 4\Pi q$. Ces progressions ne sont pas les mêmes pour toutes les valeurs de d; toutefois, leur nombre est toujours le même; en effet, ce nombre est représenté par la formule $\Delta \times \chi(\Delta)$, si l'on écrit, pour abréger,

$$\begin{aligned}\chi(\Delta) = \tfrac{1}{4} \times \Pi\tfrac{1}{2}\left(1-\frac{1}{q_1}\right) \times \Pi\left(1-\frac{1}{q_2}\right) \\ \times \Pi\tfrac{1}{2}\left(1-\frac{1}{q_3}\right) \times \Pi\left(1-\frac{1}{q_{1,2}}\right) \\ \times \Pi\tfrac{1}{2}\left(1-\frac{1}{q_{1,3}}\right) \times \Pi\tfrac{1}{2}\left(1-\frac{1}{q_{2,3}}\right) \\ \times \Pi\tfrac{1}{2}\left(1-\frac{1}{q_{1,2,3}}\right),\end{aligned}$$

où l'on remarquera que les facteurs $\frac{1}{2}$ se présentent ou non, selon que $\delta \times d = M$ doit avoir un caractère quadratique par rapport à q ou doit seulement être premier par rapport à ce nombre. Il est vrai que si $d > \frac{L}{d}$, quelques-unes de ces progressions ne contiendraient qu'un seul terme; mais on peut sans erreur sensible supprimer les termes qui se rapportent à ces valeurs de d.

Soit

$$a, \quad a+\Delta, \quad a+2\Delta, \quad \ldots, \quad a+(x-1)\Delta, \quad \ldots\ldots \quad (6)$$

une des progressions dans lesquelles se trouvent les valeurs de δ correspondant à une valeur donnée de $d < \frac{L}{\Delta}$, et posons

$$b = \frac{L}{d^2} - a - (x-1)\Delta,$$

b étant positif et moindre que Δ. On aura l'équation

$$\begin{aligned}\frac{1}{L^2}\left[a + (a+\Delta) + \ldots + (a+\overline{x-1}\,\Delta)\right] \\ = \frac{1}{2\Delta d^2} + \frac{1}{2\Delta L} \times \frac{2b-\Delta}{d} + \frac{(a-b)(\Delta-a-b)}{2\Delta L^2},\end{aligned}$$

d'où l'on voit que la partie de $\frac{T}{L^2}$ dans laquelle d a la valeur donnée sera comprise entre les limites

$$\tfrac{1}{2}\rho\left(\frac{I_1 I_2}{d}\right)\chi(\Delta)\left[\frac{1}{d^2} \pm \left(\frac{\Delta}{L}\times\frac{1}{d}+\frac{\Delta^2}{L^2}\right)\right].$$

On aura donc

$$Lim\,\frac{T}{L^2}=\tfrac{1}{2}\rho\chi(\Delta)\Sigma\left(\frac{I_1 I_2}{d}\right)\frac{1}{d^2},$$

la sommation se rapportant à tous les nombres d premiers à Δ, depuis 1 jusqu'à ∞, ou, ce qui revient au même,

$$Lim\,\frac{T}{L^2}=\tfrac{1}{2}\rho\chi(\Delta)\times\Pi\left[1-\left(\frac{I_1 I_2}{q}\right)\frac{1}{q^2}\right]\times\Sigma\left(\frac{I_1 I_2}{m}\right)\frac{1}{m^2},$$

la sommation se rapportant à tous les nombres impairs.

En égalant cette valeur de la limite de $\frac{T}{L^2}$ à celle que nous avons déjà trouvée, on a finalement

$$\left.\begin{aligned} W=\tfrac{1}{12}\times\zeta\times\left(\tfrac{1}{2}\right)^{J_1}\times\left(\tfrac{1}{2}\right)^{J_3} & \\ \times\Pi\tfrac{1}{2}\left[1+\left[\left(\frac{-I_1\theta_2}{q_2}\right)+\left(\frac{-I_3\theta_2}{q_3}\right)\right]\frac{1}{q_2}+\frac{1}{q_2^2}\right] & \\ \times\Pi\tfrac{1}{4}\left[1+\left(\frac{-I_2\theta_1\theta_3}{q_{1,3}}\right)\frac{1}{q_{1,3}}\right]\times\Pi\tfrac{1}{8}\left(1-\frac{1}{q_{1,2,3}^2}\right) & \\ \times\Pi\tfrac{1}{4}\left[1+\left(\frac{-I_3\theta_2}{q_{,2}}\right)\frac{1}{q_{1,2}}\right]\times\Pi\tfrac{1}{4}\left[1+\left(\frac{-I_1\theta_2}{q_{2,3}}\right)\frac{1}{q_{2,3}}\right] & \\ \times I_1^{\frac{3}{2}}I_2^2 I_3^{\frac{3}{2}}\times\frac{1}{\pi^2}\Sigma\left(\frac{I_1 I_3}{m}\right)\frac{1}{m^2}. & \end{aligned}\right\} \quad . \quad . \quad . \quad (7)$$

On remarquera que cette expression est symétrique par rapport à I_1, θ_1, et I_3, θ_3, comme cela devait être. La valeur de ζ est $+1$, si $I_1\times I_3\equiv -1$, mod 4, et

$$\tfrac{1}{2}\left[2+(-1)^{\frac{1}{4}(I_1+1)(I_2+1)+\sigma}\right], \text{ si } I_1\times I_3\equiv 1, \text{ mod } 4.$$

J_1 est le nombre des nombres premiers q_1 et J_3 a une pareille signification.

14. La formule que nous venons de trouver, pour le cas où les formes et les invariants sont impairs, s'applique avec de légères modifications à tous les autres cas ; et, en effet, il n'y a que la valeur du coefficient ζ qui doit être changée.

Soient $2^3 \times \eta'$ le nombre des formes linéaires (mod 8) dans lesquelles sont contenus les nombres auxiliaires M, qui, dans chaque cas, sont employés dans l'investigation ; $2^{4k} \times \eta$ le nombre des systèmes des indéterminées (mod 2^k) qui satisfont à la congruence

$$f_3 \equiv M \quad \text{ou} \quad \tfrac{1}{2} f_3 \equiv M, \text{ mod } 2^k,$$

l'exposant k étant 1, 2, 3, selon les cas ; enfin, λ le coefficient * propre au genre de formes ternaires, qui peut être représenté par le genre donné de formes quaternaires ; on aura

$$\zeta = \tfrac{3}{2} \times \frac{\eta' \lambda}{\eta} \quad \text{ou} \quad = \tfrac{3}{4} \times \frac{\eta' \lambda}{\eta},$$

selon que la forme θ_3 est impaire ou paire.

Voici le tableau de ces coefficients :

A. $[I_1, I_2, I_3] \equiv [1, 1, 1]$, mod 2.

(*a*) $[\theta_1, \theta_2, \theta_3] \equiv [1, 1, 1]$, mod 2 ;

$\zeta = 1$, si $I_1 \times I_3 \equiv -1$, mod 4 ;

$$\zeta = \tfrac{1}{2}\left[2 + (-1)^{\frac{1}{4}(I_1+1)(I_2+1)+\sigma}\right],$$

si $I_1 \times I_3 \equiv +1$, mod 4, $(-1)^\sigma$ étant le caractère simultané de $\theta_1, \theta_2, \theta_3$.

(*b*) $[\theta_1, \theta_2, \theta_3] \equiv [0, 1, 0]$, mod 2 ;

$$\zeta = \tfrac{1}{15}\left[4 + \left(\frac{2}{I_1 I_3}\right)\right].$$

B. $[I_1, I_2, I_3] \equiv [1, 1, 0]$, mod 2.

Dans ce cas, les formes $\theta_1, \theta_2, \theta_3$ sont impaires.

(*a*) $I_3 \equiv 2$, mod 4 ;

$$\zeta = +1.$$

(*b*) $I_3 \equiv 4$, mod 8 ;

$$\zeta = \tfrac{1}{4}\left[2 + (-1)^{\frac{1}{4}(I_1+1)(I_2\theta_3+1)+\sigma}\right],$$

$(-1)^\sigma$ étant le caractère simultané de θ_1, θ_2.

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* *Phil. Trans.*, vol. clvii. p. 291, où ce coefficient s'appelle ζ [vol. i. p. 499].

(*c*) $I_3 \equiv 0$, mod 8 ;

$$\zeta = \tfrac{1}{8}[2+(-1)^{\frac{1}{4}(I_1+1)(I_2\theta_3+1)+\sigma}].$$

Ici encore $(-1)^\sigma$ est le caractère simultané de θ_1, θ_2.

B′. $[I_1, I_2, I_3] \equiv [0, 1, 1]$, mod 2.

On échange entre eux I_1 et I_3, θ_1 et θ_3 dans les formules *B*.

C. $[I_1, I_2, I_3] \equiv [1, 0, 1]$, mod 2.

(*a*) $[\theta_1, \theta_2, \theta_3] \equiv [1, 1, 1]$, mod 2.

(1) $I_2 \equiv 2$, mod 4 ;

$$\zeta = \tfrac{3}{4}.$$

(2) $I_2 \equiv 4$, mod 8 ;

$$\zeta = \tfrac{3}{64}[3+(-1)^{\frac{1}{2}(I_1\theta_2+1)}] \times [3+(-1)^{\frac{1}{2}(I_3\theta_2+1)}].$$

(3) $I_2 \equiv 0$, mod 8 ;

$$\zeta = \tfrac{3}{128}[3+(-1)^{\frac{1}{2}(I_1\theta_2+1)}] \times [3+(-1)^{\frac{1}{2}(I_3\theta_2+1)}].$$

(*b*) $[\theta_1, \theta_2, \theta_3] \equiv [0, 1, 1]$, mod 2.

(1) $I_2 \equiv 2$, mod 4 ;

$$\zeta = \tfrac{1}{8}[3+(-1)^{\frac{1}{2}(I_3 f_2+1)}].$$

(2) $I_2 \equiv 0$, mod 4 ;

$$\zeta = \tfrac{1}{32}[3+(-1)^{\frac{1}{2}(I_3 f_2+1)}] \times [2+(-1)^{\frac{1}{8}(I_1^2 f_2^2-1)}].$$

(*b*′) $[\theta_1, \theta_2, \theta_3] \equiv [1, 1, 0]$, mod 2.

Ce cas est réciproque de (*b*).

(*c*) $[\theta_1, \theta_2, \theta_3] \equiv [0, 1, 0]$, mod 2 ;

$$\zeta = \tfrac{1}{24}[2+(-1)^{\frac{1}{8}(I_1^2 f_2^2-1)}] \times [2+(-1)^{\frac{1}{8}(I_3^2 f_2^2-1)}].$$

D. $[I_1, I_2, I_3] \equiv [0, 0, 1]$, mod 2.

(*a*) $[\theta_1, \theta_2, \theta_3] \equiv [1, 1, 1]$, mod 2.

(1) $I_2 \equiv 2$, mod 4,

$$\zeta = \frac{3}{2^{i+2}},$$

où $i = 0, 1, 2$, représente le nombre des congruences

$$I_1 \equiv 4, \text{ mod } 8, \quad I_1 \equiv 0, \text{ mod } 8,$$

qui se trouvent satisfaites.

$$(2) \quad I_1 \equiv 0, \text{ mod } 4,$$

$$\zeta = \frac{3}{2^{5+i}}[3 + (-1)^{\frac{1}{2}(I_3 f_2 + 1)}],$$

où $i = 0, 1, 2$, représente le nombre des congruences

$$I_1 \equiv 0, \text{ mod } 8, \quad I_2 \equiv 0, \text{ mod } 8,$$

qui se trouvent satisfaites.

$$(b) \quad [\theta_1, \theta_2, \theta_3] \equiv [1, 1, 0], \text{ mod } 2.$$

$$(1) \quad I_2 \equiv 2, \text{ mod } 4,$$

$$\zeta = \frac{2}{2^{2+i}},$$

$$(2) \quad I_2 \equiv 0, \text{ mod } 4,$$

$$\zeta = \frac{1}{2^{4+i}}[2 + (-1)^{\frac{1}{8}(I_3 f_2^2 - 1)}].$$

Dans ces deux formules (b), (1) et (2), $i = 1$ ou $= 0$, selon que a congruence $I_2 \equiv 0$, mod 8, est satisfaite ou non.

$$D'. \quad [I_1, I_2, I_3] \equiv [1, 0, 0], \text{ mod } 2.$$

C'est le cas réciproque de *D*.

$$E. \quad [I_1, I_2, I_3] \equiv [0, 1, 0], \text{ mod } 2.$$

$$(a) \quad [\theta_1, \theta_2, \theta_3] \equiv [1, 1, 1], \text{ mod } 2.$$

(1) Si les congruences

$$I_1 \equiv 0, \quad I_3 \equiv 0, \text{ mod } 4,$$

ne sont pas satisfaites en même temps,

$$\zeta = \frac{3}{2^{4+i}},$$

$i = 0, 1, 2$, étant le nombre des congruences

$$I_1 \equiv 0, \text{ mod } 4, \quad I_1 \equiv 0, \text{ mod } 8,$$
$$I_2 \equiv 0, \text{ mod } 4, \quad I_2 \equiv 0, \text{ mod } 8,$$

qui sont satisfaites en même temps.

(2) Si l'on a à la fois

$$I_1 \equiv I_3 \equiv 0, \text{ mod } 4,$$
$$\zeta = \frac{3}{2^{5+i}} \left[3 + (-1)^{\frac{1}{2}(I_2\theta_1\theta_3+1)}\right],$$

où $i = 0, 1, 2$, est le nombre des congruences

$$I_1 \equiv 0, \text{ mod } 8, \quad I_3 \equiv 0, \text{ mod } 8,$$

qui sont satisfaites en même temps.

F. $[I_1, I_2, I_3] \equiv [0, 0, 0]$, mod 2.

(1) Soit $I_2 \equiv 2$, mod 4,

$$\zeta = \frac{3}{2^{2+i}},$$

$i = 0, 1, 2, 3, 4$, étant le nombre des congruences

$$I_1 \equiv 0, \text{ mod } 4, \quad I_1 \equiv 0, \text{ mod } 8,$$
$$I_3 \equiv 0, \text{ mod } 4, \quad I_3 \equiv 0, \text{ mod } 8,$$

qui sont satisfaites en même temps.

(2) Soit $I_2 \equiv 0$, mod 4,

$$\zeta = \frac{3}{2^{4+i}},$$

$i = 0, 1, 2, 3$, étant le nombre des congruences

$$I_1 \equiv 0, \text{ mod } 8, \quad I_2 \equiv 0, \text{ mod } 8, \quad I_3 \equiv 0, \text{ mod } 8,$$

qui se trouvent satisfaites en même temps.

15. Parmi ces formules, il suffira de démontrer celles dont nous aurons besoin pour la théorie de la décomposition des nombres en cinq carrés.

Démonstration de la formule (A, *b*).—On peut prendre pour M tout nombre

impair ; donc $\eta' = \frac{1}{2}$, et par le théorème (IV) de l'article (8),

$$\eta = \tfrac{1}{2}\Big[1 - \tfrac{1}{4}\Big(\frac{2}{I_1 I_3}\Big)\Big].$$

La valeur de λ étant $\frac{1}{3}$, il vient

$$\zeta = \tfrac{1}{15}\Big[4 + \Big(\frac{2}{I_1 I_3}\Big)\Big]$$

Démonstration des formules B.—(*a*). $I_3 \equiv 2$, mod 4. Nous prendrons pour M tous les nombres congrus, pour le module 8, à un seulement des résidus 1, 3, 5, 7 ; on aura ainsi $\eta' = \frac{1}{8}$,

$$\lambda = \tfrac{1}{3}[2 + (-1)^{\frac{1}{4}(I_1+1)(MI_2+1)+\sigma_1}],$$

en désignant par $(-1)^{\sigma_1}$ le caractère simultané des formes ternaires (I_1, MI_2), qui peuvent être représentées par les formes du genre quaternaire donné. D'ailleurs on a, d'après le théorème (V) de l'article (8),

$$\eta = \tfrac{1}{16}[2 + (-1)^{\rho+\sigma}],$$

en désignant par $(-1)^{\sigma}$ le caractère simultané du genre quaternaire et par ρ le nombre

$$\tfrac{1}{4}(I_1+1)(I_2+1) + \tfrac{1}{4}(M-1)(I_1 I_3 - 1) + \tfrac{1}{8}(M^2-1).$$

Les indices σ_1 et σ sont donnés par les formules

$$\sigma_1 = \tfrac{1}{4}(\theta_1-1)(\theta_2-1) + \tfrac{1}{4}(I_1+1)(\theta_1-1) + \tfrac{1}{4}(I_2 M+1)(\theta_2-1), \text{ mod } 2,$$

$$\sigma = \tfrac{1}{4}(\theta_1-1)(\theta_2-1) + \tfrac{1}{4}(\theta_2-1)(\theta_3-1) + \tfrac{1}{4}(I_1+1)(\theta_1-1) + \tfrac{1}{4}(I_2+1)(\theta_2-1) + \tfrac{1}{4}(I'_3+1)(\theta_3-1) + \tfrac{1}{8}(\theta_3^2-1), \text{ mod } 2.$$

Mais si (m_1, m_2) est un système caractéristique pour une forme ternaire représentée par θ_1, (m_1, m_2, M) est un système caractéristique pour θ_1 ; on peut donc écrire m_1, m_2, M pour $\theta_1, \theta_2, \theta_3$ dans les deux formules précédentes. On trouve ainsi

$$\tfrac{1}{4}(I_1+1)(MI_2+1) + \sigma_1 \equiv \rho + \sigma, \text{ mod } 2,$$

et, par conséquent,

$$\zeta = 1.$$

(*b*). $I_3 \equiv 4$, mod 8. La forme θ_3 aura le caractère $(-1)^{\frac{1}{2}(\theta_3-1)}$, qui appar-

tiendra aussi à tous les nombres M. En effet, la forme canonique équivalente satisfera à la congruence

$$\theta_3 \equiv Mx^2, \text{ mod } 4.$$

Il y aura aussi pour (θ_1, θ_2) un caractère simultané que nous désignerons par $(-1)^\sigma$, et qui appartiendra en même temps aux formes ternaires représentées par θ_3. Donc on aura

$$\eta = \tfrac{1}{2}, \quad \eta' = \tfrac{1}{4},$$

$$\lambda = \tfrac{1}{3}[2 + (-1)^{\frac{1}{4}(I_1+1)(I_2\theta_3+1)+\sigma}],$$

$$\zeta = \tfrac{1}{4}[2 + (-1)^{\frac{1}{4}(I_1+1)(I_2\theta_3+1)+\sigma}].$$

(*c*). $I_3 \equiv 0$, mod 8. Ici l'on a

$$\theta_3 \equiv Mx^2, \text{ mod } 8.$$

Tout est comme dans le cas précédent, seulement il faut écrire $\eta' = \frac{1}{8}$. On trouve par conséquent

$$\zeta = \tfrac{1}{8}[2 + (-1)^{\frac{1}{4}(I_1+1)(I_2\theta_3+1)+\sigma}].$$

16. Puisque la forme

$$\sum_{s=1}^{s=5} x_s^2,$$

qui est elle-même sa contravariante, représente la seule classe qui existe de forme quinaire de déterminant $+1$*, il est évident qu'on peut déduire des théorèmes précédents une expression pour le nombre des décompositions d'un nombre donné M en cinq carrés. Pour cela, on déterminera la densité du genre unique, ou des deux genres, des formes quaternaires $(1, 1, M)$, qui peuvent être représentées par cinq carrés. Soient γ cette densité, μ le nombre des diviseurs premiers impairs de M, $j = 0, 1, 2$, selon qu'on a

(1) M impair ou impairement pair ; (2) $M \equiv 4$, mod 8 ; (3) $M \equiv 0$, mod 8 ;

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* Cette proposition peut se démontrer, soit par des considérations tout à fait élémentaires, soit en déterminant la densité des classes quinaires de déterminant $+1$. Il n'y a évidemment qu'un genre, et la densité de ce genre, par une application très simple de la méthode que nous avons employée pour les formes quaternaires, se trouve être égale à $\frac{1}{2^4 \times \Pi 5}$.

N le nombre des représentations primitives de M par cinq carrés. La densité de la classe quinaire étant $\frac{1}{n^4 \times \Pi 5}$, on aura pour N l'expression suivante :

$$N = 2^4 \times \Pi 5 \times 2^{\mu+j} \times \gamma,$$

$$= 2^4 \times 5\Pi \times \frac{1}{12} \times \zeta' \times M^{\frac{3}{2}} \times \frac{1}{\pi^2} \Sigma \left(\frac{M}{m}\right) \frac{1}{m^2},$$

dans laquelle $\zeta' = 2^j \zeta$, ζ étant un des coefficients de la table ou la somme de deux de ces coefficients. Tout se réduit donc à la détermination du coefficient ζ'. Il y a six cas à considérer :

I. $M \equiv 3$, mod 4 ;

II. $M \equiv 5$, mod 8 ;

III. $M \equiv 1$, mod 8 ;

IV. $M \equiv 2$, mod 4 ;

V. $M \equiv 4$, mod 8 ;

VI. $M \equiv 0$, mod 8.

I. Soit $M \equiv 3$, mod 4. Il n'existe pas de formes paires ayant $(1, 1, M)$ pour invariants, et il n'y a, par conséquent, qu'un seul genre qui admette une représentation par une somme de cinq carrés. On a donc $\zeta' = \zeta$, c'est-à-dire $\zeta' = 1$, puisque $\zeta = 1$ pour tous les genres.

II. Soit $M \equiv 5$, mod 8. Il existe un ordre pair aussi bien qu'un ordre impair de formes quaternaires aux invariants $(1, 1, M)$, et, dans chacun de ces deux ordres, il y a un genre de formes capables de représentation par une somme de cinq carrés.

Dans l'ordre pair, la condition de possibilité, à laquelle tout caractère générique doit satisfaire, s'exprime par l'équation

$$\left(\frac{\theta_3'}{M}\right) = (-1)^\sigma,$$

ou

$$\sigma \equiv \tfrac{1}{4}(\theta_1' - 1)(\theta_2 - 1) + \tfrac{1}{4}(\theta_2 - 1)(\theta_3' - 1) + \tfrac{1}{2}(\theta_1' - 1) + \tfrac{1}{2}(\theta_2 - 1) + \tfrac{1}{2}(\theta_3 - 1), \text{ mod } 2 ;$$

mais on a $\frac{1}{2}(\theta_2 - 1) \equiv 1$, mod 2, parce que θ_1 est une forme paire dont le premier

invariant est $\equiv 1$, mod 4 ; donc on a $\sigma \equiv 1$, mod 2, les formes θ_1', θ_3' disparaissant d'elles-mêmes de l'expression de σ (ce qui d'ailleurs est d'accord avec la théorie générale de l'article 7), et la condition de possibilité devient

$$\left(\frac{\theta_3'}{M}\right) = -1 \quad \text{ou bien} \quad \left(\frac{\theta_3}{M}\right) = -\left(\frac{2}{M}\right) = +1.$$

D'autre part, la congruence

$$-\theta_3 \equiv \square, \text{ mod } M,$$

qui exprime la condition nécessaire et suffisante pour que θ_1 soit capable de représentation par une somme de cinq carrés, conduit à la même valeur de $\left(\frac{\theta_3}{M}\right)$; d'où l'on conclut qu'il existe un genre pair qui admet une telle représentation. Pour ce genre la valeur du coefficient ζ sera $\frac{1}{5}$.

Quant à l'ordre impair, le genre défini par les équations

$$\left(\frac{\theta_3}{q}\right) = (-1)^{\frac{1}{2}(q-1)}, \quad (-1)^{\sigma} = \left(\frac{\theta_3'}{M}\right) = -1,$$

où q est un diviseur premier de M, satisfera à la condition de possibilité et sera en même temps capable de représentation par une somme de cinq carrés ; le coefficient ζ sera $\frac{1}{2}$.

Donc on aura, en ajoutant,

$$\zeta' = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}.$$

III. Soit $M \equiv 1$, mod 8. L'ordre pair existe toujours, mais aucune forme de cet ordre ne peut être représentée par cinq carrés ; en effet, puisque $\left(\frac{2}{M}\right) = +1$, $\left(\frac{-1}{M}\right) = +1$, la congruence

$$-\theta_3 \equiv \square, \text{ mod } M$$

est incompatible avec la condition de possibilité

$$\left(\frac{\theta_3'}{M}\right) = -1.$$

Dans l'ordre impair il y aura, comme dans le cas II, un genre capable de

représentation par cinq carrés. Pour ce genre on aura

$$(-1)^{\sigma} = -1, \quad \zeta = +\tfrac{1}{2},$$

et, par conséquent,

$$\zeta' = \zeta = \tfrac{1}{2}.$$

IV. Soit $M \equiv 2$, mod 4. Ici il n'y a qu'un seul genre impair qui soit capable de représentation, et l'on a

$$\zeta' = \zeta = 1.$$

V. Soit $M \equiv 4$, mod 8 ; $M = 4M'$. De la congruence

$$-\theta_3 \equiv \square, \text{ mod } M,$$

on tire

$$(-1)^{\frac{1}{2}(\theta_3 - 1)} = -1, \quad \left(\frac{\theta_3}{M'}\right) = (-1)^{\frac{1}{2}(M'-1)};$$

donc, en désignant par $(-1)^{\sigma}$ le caractère simultané des formes θ_1, θ_2, on a

$$(-1)^{\sigma} = -1,$$

puisque, par la condition de possibilité,

$$(-1)^{\sigma} \times (-1)^{\frac{1}{4}(M'+1)(\theta_3 - 1)} \times \left(\frac{\theta_3}{M'}\right) = +1.$$

On a ainsi

$$\zeta = \tfrac{1}{4}.$$

Mais il faut doubler cette valeur, parce que $j = 1$. Donc on aura

$$\zeta' = 2\zeta = \tfrac{1}{2}.$$

VI. Soit $M \equiv 0$, mod 8. On aura, pour le genre de formes capables de représentation par une somme de cinq carrés,

$$(-1)^{\frac{1}{2}(\theta_3 - 1)} = -1, \quad (-1)^{\frac{1}{8}(\theta_3^2 - 1)} = +1,$$

$$\left(\frac{\theta_3}{M'}\right) = (-1)^{\frac{1}{2}(M'-1)}, \quad (-1)^{\sigma} = 1,$$

en désignant toujours par M' le plus grand diviseur impair de M. La valeur de ζ sera $\frac{1}{8}$; mais puisqu'il faudra la quadrupler, le résultat sera le même que dans le cinquième cas.

17. On obtient sous forme finie les sommes des suites infinies qui figurent dans les formules des articles 13 et 16, en employant les méthodes de Dirichlet et de Cauchy*. Comme on sait, on peut donner aux résultats un assez grand nombre de formes différentes. Celle que nous avons préférée est loin d'être la plus simple, mais elle se présente comme cas particulier d'une forme qu'on peut donner à la somme de la série

$$\frac{1}{\pi^\nu}\Sigma\left(\frac{D}{m}\right)\frac{1}{m^\nu}$$

pour toute valeur de ν†.

Soit $D=\omega^2\times\delta$, δ n'étant divisible par aucun carré ; soit aussi q un nombre premier impair qui divise ω mais ne divise pas δ ; on aura les expressions suivantes pour la valeur de la série

$$S=\frac{1}{\pi^2}\Sigma\left(\frac{D}{m}\right)\frac{1}{m^2}.$$

I. Si $D\equiv 1$, mod 4,

$$S=\left[1-\left(\frac{2}{\delta}\right)\tfrac{1}{4}\right]\times\sum_{s=1}^{s=\delta}\left(\frac{s}{\delta}\right)\frac{s}{\delta}\left(\frac{s}{\delta}-1\right)\times\frac{1}{\sqrt{\delta}}\Pi\left[1-\left(\frac{\delta}{q}\right)\frac{1}{q^2}\right],$$

s étant supposé premier à δ.

II. Dans tous les autres cas,

$$S=\sum_{s=1}^{s=4\delta}\left(\frac{\delta}{s}\right)\frac{s}{4\delta}\left(\frac{s}{4\delta}-1\right)\times\frac{1}{\sqrt{\delta}}\Pi\left[1-\left(\frac{\delta}{q}\right)\frac{1}{q^2}\right],$$

le nombre s étant premier à 4δ.

La première formule est en défaut lorsque $\delta=1$; mais dans ce cas

$$\Sigma\frac{1}{m^2}=\frac{\pi^2}{8}.$$

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* Lejeune-Dirichlet, *Recherches sur l'application de l'analyse infinitésimale à la théorie des nombres* (*Journal de Crelle*, vol. XXI) ; Cauchy, *Mémoire sur la théorie des nombres* (*Mémoires de l'Académie*. vol. xvii, note 12).

† Voir la seconde note sur les formes quadratiques dans les *Proceedings de la Société royale*, année 1867 [vol. i. p. 510].

En employant ces formules, on trouve les expressions suivantes pour le nombre N des représentations du nombre $\omega^2 \times \delta$ par une somme de cinq carrés :

I. Si $\delta \equiv 1$, mod 4,

$$N = 5\eta \times \frac{\omega^3}{\delta} \times \Pi\left[1 - \left(\frac{\delta}{q}\right)\frac{1}{q^2}\right] \times \sum_1^{\delta}\left(\frac{s}{\delta}\right) s\,(s-\delta);$$

$\eta = 12$, si $\delta \equiv 1$, mod 8 ;

$\eta = 28$, si $\delta \equiv 5$, mod 8 ; $\omega \equiv 1$, mod 2 ;

$\eta = 20$, si $\delta \equiv 5$, mod 8 ; $\omega \equiv 0$, mod 2.

Si $\delta = 1$, il faut remplacer les facteurs $\eta \times \Pi\left[1 - \left(\frac{\delta}{q}\right)\frac{1}{q^2}\right]$ par 2.

II. Dans tous les autres cas,

$$N = 5\eta \times \frac{\omega^3}{\delta} \times \Pi\left[1 - \left(\frac{\delta}{q}\right)\frac{1}{q^2}\right] \times \sum_1^{4\delta}\left(\frac{\delta}{s}\right) s\,(s-4\delta);$$

$\eta = 1$, si $\omega \equiv 1$, mod 2 ; $\eta = \frac{1}{2}$, si $\omega \equiv 0$, mod 2.

En faisant $\omega = 1$, on déduit facilement de ces formules les expressions plus simples données par Eisenstein.

A. Soit en premier lieu $\delta = 2m$; on aura

$$N = \frac{5}{2} \times \frac{1}{m} \times \sum_1^{8m} (-1)^{\frac{1}{4}(m-1)(s-1)}\left(\frac{2}{s}\right) s\,(s-8m).$$

En faisant successivement $s = 4m \pm p$, les limites de p étant 1 et $4m$; $p = 2m \pm r$, les limites de r étant 1 et $2m$; enfin $r = 2m - t$, les limites du nombre impair t étant 1 et m ; on obtient

$$N = 40\,\Sigma\,(-1)^{\frac{1}{4}(t-1)(m+1)}\left(\frac{2}{t}\right)\left(\frac{t}{m}\right) \times \left[t + (-1)^{\frac{1}{2}(t-1)}(2m-t)\right],$$

ce qui est bien la formule d'Eisenstein.

B. Pareillement, si $\delta = m \equiv 1$, mod 5, nous tirons de la formule ci-dessus

$$N = 140 \times \frac{\Sigma a^2 - \Sigma b^2}{m},$$

en désignant par a et b les nombres moindres que m qui satisfont respectivement aux équations

$$\left(\frac{a}{m}\right)=1, \quad \left(\frac{b}{m}\right)=-1.$$

Si α et β sont des valeurs de a et b qui ne surpassent pas $\frac{1}{2}m$, on aura, en observant que $\left(\frac{2}{m}\right)=-1$,

$$\begin{aligned}
5\Sigma\alpha^2 &= \Sigma\alpha^2+\Sigma(2\alpha)^2,\\
&= \Sigma\alpha^2+\Sigma(2\alpha)^2+\Sigma(2m-2\alpha)^2,\\
&= \Sigma\alpha^2+\Sigma(2\alpha)^2+\Sigma(m-2\alpha)^2+3\mu m^2-4m\Sigma\alpha,\\
&= \Sigma\alpha^2+\Sigma b^2+3\mu m^2-4m\Sigma\alpha,
\end{aligned}$$

le coefficient μ étant le nombre des résidus α ou β. Le même raisonnement donne aussi

$$5b^2=\Sigma\alpha^2+\Sigma b^2+3\mu m^2-4m\Sigma\beta,$$

d'où l'on tire

$$N=140\,\frac{\Sigma\alpha^2-\Sigma b^2}{m}=112\,(\Sigma\beta-\Sigma\alpha),$$

d'accord avec le résultat d'Eisenstein.

C. Si $M\equiv 1$, mod 8, on a

$$N=60\,\frac{\Sigma\alpha^2-\Sigma b^2}{m};$$

mais on trouve facilement

$$\begin{aligned}
3\Sigma\alpha^2 &= \Sigma(2\alpha)^2-\Sigma\alpha^2 = -\Sigma\alpha^2+\Sigma(2\alpha)^2\\
&\qquad +\Sigma(m-2\alpha)^2+3\mu m^2-4m\Sigma\alpha\\
&= 3\mu m^2-4m\Sigma\alpha\,;\\
3\Sigma b^2 &= 3\mu m^2-4m\Sigma\beta.
\end{aligned}$$

Donc

$$N=80\,(\Sigma\beta-\Sigma\alpha).$$

D. Si $\Delta\equiv 3$, mod 4, il s'agit de transformer la somme

$$S=5\times\frac{1}{m}\sum_{1}^{4m}(-1)^{\frac{1}{2}(s-1)}\left(\frac{s}{m}\right)s^2.$$

Faisons $s = 2m \pm p$; on aura d'abord

$$S = -10 \times \frac{1}{m} \sum_{1}^{2m} (-1)^{\frac{1}{2}(p-1)} \left(\frac{p}{m}\right) p^2,$$

puisque

$$\sum_{1}^{2m} (-1)^{\frac{1}{2}(p-1)} \left(\frac{p}{m}\right) = 0 ;$$

ensuite écrivons $p = m \pm 2t$; on aura aussitôt la formule d'Eisenstein

$$80 \times \left(\frac{+2}{m}\right) \sum_{1}^{\frac{1}{2}m} (-1)^{t} \left(\frac{t}{m}\right) t.$$

Eisenstein a aussi indiqué pour le même cas les formules suivantes :

$$\sum_{1}^{\frac{1}{2}(m-1)} (-1)^{s} \left(\frac{s}{m}\right) s = 4 \sum_{1}^{\frac{1}{4}(m-3)} \left(\frac{s}{m}\right) s ; \quad m \equiv 7, \text{ mod } 8. \tag{1}$$

$$\left.\begin{aligned} \sum_{1}^{\frac{1}{2}(m-1)} (-1)^{s} \left(\frac{s}{m}\right) s &= -4 \sum_{1}^{\frac{1}{4}(m-3)} \left(\frac{s}{m}\right) s \\ &\quad - \tfrac{1}{3} m \times \sum_{1}^{\frac{1}{2}(m-1)} \left(\frac{s}{m}\right) ; \quad m \equiv 3, \text{ mod } 8. \end{aligned}\right\} \quad \ldots\ldots \tag{2}$$

Puisque les transformations de cette espèce, quoique très élémentaires, sont quelquefois assez difficiles à retrouver, nous en donnerons une démonstration ici. Les lettres a, b, α, β ayant la même signification, désignons par A le nombre des α et par B celui des β. Étant donné un des nombres b, il y aura toujours parmi ces mêmes nombres un autre b', qui satisfera à la congruence $2b - b' \equiv 0$, mod m ; de plus, si $b < \frac{1}{2}m$, on aura $2b - b' = 0$, si $b' > \frac{1}{2}m$, $2b - b' = m$; donc l'identité $\frac{\Sigma b'}{m} = \Sigma \frac{2b - b'}{m}$ donne immédiatement

$$\frac{\Sigma b}{m} = A.$$

On trouve pareillement

$$\frac{\Sigma a}{m} = B,$$

et, par conséquent,

$$\Sigma \frac{b - a}{m} = A - B. \quad \ldots\ldots\ldots \tag{3}$$

D'un autre côté, tout nombre b satisfait, soit à l'équation $b = m - 2a$, soit à l'équation $m = 2\beta$; donc $\Sigma b = mA + 2\Sigma(\beta - a)$, et pareillement $\Sigma a = mB + 2\Sigma(a - \beta)$, d'où l'on tire

$$\Sigma(b-a) = m(A-B) + 4\Sigma(\beta - a); \quad \ldots\ldots\ldots \quad (4)$$

et ayant égard à (3),

$$\Sigma(a - \beta) = 0 \quad \ldots\ldots\ldots\ldots \quad (5)$$

On a donc, en désignant par $2a'$ et $2\beta'$ ceux des nombres a et β qui sont pairs.

$$\sum_{1}^{\frac{1}{2}(m-1)} (-1)^s \left(\frac{s}{m}\right) s = \Sigma[(-1)^a a - (-1)^\beta \beta] + \Sigma(a - \beta)$$

$$= 4\Sigma(a' - \beta') = \sum_{1}^{\frac{1}{4}(m-3)} \left(\frac{s}{m}\right) s.$$

C. Q. F. D.

Pour démontrer l'équation (2), on se sert des formules

$$\Sigma \frac{b-a}{m} = \tfrac{1}{3}(A - B), \quad \ldots\ldots\ldots \quad (6)$$

$$\Sigma(a - b) = m(A - B) + 4\Sigma(\beta - a), \quad \ldots\ldots\ldots \quad (7)$$

qui donnent

$$\Sigma(\beta - a) + \tfrac{1}{3}\Sigma(A - B) = 0,$$

et qu'on établit de la même manière que les équations (3) et (4).

Les équations (3) et (6) avaient été déjà données par Lejeune-Dirichlet (*Journal de Crelle*, vol. xxi, p. 152).

APPENDIX.

I.

ADDRESS TO THE MATHEMATICAL AND PHYSICAL SECTION OF THE BRITISH ASSOCIATION.

[Bradford, Sept. 18, 1873, Report of the British Association for 1873, Transactions of the Sections, pp. 1–11.]

For several years past it has been the custom for the president of this Section, as of the other Sections of the Association, to open its proceedings with a brief address. I am not willing upon this occasion to deviate from the precedent set by my predecessors, although I feel that the task presents peculiar difficulties to one who is by profession a pure mathematician, and who, in other branches of science, can only aspire to be regarded as an amateur.

But, although I thus confess myself a specialist, and a specialist it may be said of a narrow kind, I shall not venture, in the few remarks which I now propose to make, to indulge my own specialty too far.

I am well aware that we are certain, in this Section, to have a sufficient number of communications, which of necessity assume a special and even an abstruse character, and which, whatever pains may be taken to give them clearness, and however valuable may be the results to which they lead, are nevertheless extremely difficult to follow, not only for a popular audience, but even for men of science whose attention has not been specially, and recently, directed to the subject under discussion. I should think it, therefore, almost unfair to the Section if, at the very commencement of its proceedings, I were to attempt to direct its attention in any exclusive manner to the subject which, I confess, if I were left to myself, I should most naturally have chosen—the history of the advances that have been made during the last ten or twenty years in mathematical science. Instead, therefore, of adventuring myself on this difficult course, which, however, I strongly recommend to some successor of mine less scrupulous than myself, I propose, though at the risk of repeating what has been better said by others before me, to offer some general considerations which may have a more equal interest for all those who take part in the proceedings of this Section, and which appear to me at the present time to be more than usually deserving of the notice of those who desire to promote the growth of the scientific spirit in this country. I intend, therefore, while confining myself as strictly as I can to the range of subjects belonging to this Section, to point out one or two, among many, of the ways in which sectional meetings, such as ours, may contribute to the advancement of science.

We all know that Section A of the British Association is the Section of Mathematics and Physics; and I daresay that many of us have often thought how astonishingly vast is the range of subjects which we slur over, rather than sum up, in this brief designation. We include the most abstract speculations of pure mathematics, and we come down to the most concrete of all phenomena—the most every-day of all experiences. I think I have heard in this Section a discussion on spaces of five dimensions, and we know that one of our committees, a committee which is of long standing, and which has done much useful work, reports to us annually on the Rainfall of the British Isles. Thus our wide range covers the mathematics of number and quantity in their most abstract forms, the mathematics of space, of time, of matter, of motion, and of force, the many sciences which we comprehend under the name of astronomy, the theories of sound, of light, heat, electricity; and besides the whole physics of our earth, sea, and atmosphere, the theory of earthquakes, the theory of tides, the theory of all the movements of the air, from the lightest ripple that affects the barometer up to a cyclone. As I have already said, it is impossible that communications on all these subjects should be interesting, or indeed intelligible to all our members; and, notwithstanding the pains taken by the committee and by the secretaries, to classify the communications offered to us, and to place upon the same days those of which the subjects are cognate to one another, we cannot doubt that the disparateness of the material which comes before us in this Section is a source of serious inconvenience to many members of the Association. Occasionally, too, the pressure upon our time is very great, and we are obliged to hurry over the discussions on communications of great importance, the number of papers submitted to us being, of course, in a direct proportion to the number of the subjects included in our programme. It has again and again been proposed to remedy these admitted evils by dividing the Section, or at least by resolving it into one or more sub-sections. But I confess that I am one of those who have never regretted that this proposal has not commended itself to the Association, or indeed to the Section itself. I have always felt that by so subdividing ourselves we should run the risk of losing one or two great advantages which we at present possess; and I will briefly state what, in my judgment, these advantages are.

I do not wish to undervalue the use to a scientific man of listening to and taking part in discussions on subjects which lie wholly in the direction in which his own mind has been working. But I think, nevertheless, that most men who have attended a meeting of this Association, if asked what they have chiefly gained by it, would answer in the first place that they have had opportunities of forming or of renewing those acquaintances or intimacies with other scientific men which, to most men engaged in scientific pursuits, are an indispensable condition of successful work; and, in the second place, that while they may have heard but little relating to their own immediate line of inquiry which they might not as easily have found in journals or transactions elsewhere, they have learned much which might otherwise have never come to their knowledge of what is going on in other directions of scientific inquiry, and that they have carried away many new conceptions, many fruitful germs of thought, caught perhaps from a discussion turning upon questions apparently very remote from their own pursuits. An object just perceptible on a distant horizon is sometimes better descried by a careless sideward glance than by straining the sight directly at it; and so capricious a gift is the

inventive faculty of the human mind that the clue to the mystery hid beneath some complicated system of facts will sometimes elude the most patient and systematically conducted search, and yet will reveal itself all of a sudden upon some casual suggestion arising in connexion with an apparently remote subject. I believe that the mixed character and wide range of our discussions has been most favourable to such happy accidents. But even apart from these, if the fusion in this Section of so many various branches of human knowledge tends in some degree to keep before our minds the essential oneness of Science, it does us a good service. There can be no question that the increasing specialization of the sciences, which appears to be inevitable at the present time, does nevertheless constitute one great source of danger for the future progress of human knowledge. This specialization is inevitable, because the further the boundaries of knowledge are extended in any direction, the more laborious and time-absorbing a process does it become to travel to the frontier; and thus the mind has neither time nor energy to spare for the purpose of acquainting itself with regions that lie far away from the track over which it is forced to travel. And yet the disadvantages of excessive specialization are no less evident, because in natural philosophy, as indeed in all things on which the mind of man can be employed, a certain wideness of view is essential to the achievement of any great result, or to the discovery of anything really new. The two-fold caution so often given by Lord Bacon against over-generalization on the one hand, and against over-specialization on the other, is still as deserving as ever of the attention of mankind. But, in our time, when vague generalities and empty metaphysics have been beaten once, and we may hope for ever, out of the domain of exact science, there can be but little doubt on which side the danger of the natural philosopher at present lies. And perhaps in our Section, as at present constituted, there is a freer and fresher air—we are, perhaps, a less inadequate representation of 'that greater and common world,' of which Lord Bacon speaks, than if we were subdivided into as many parts as we include—I will not say sciences—but groups of sciences. Perhaps there is something in the very diversity and multiplicity of the subjects which come before us which may serve to remind us of the complexity of the problems of science—of the diversity and multiplicity of nature.

On the other hand, it is not, as it seems to me, difficult to assign the nature of the unity which underlies the diversity of our subjects, and which justifies to a very great extent the juxtaposition of them in our Section. That unity consists not so much in the nature of the subjects themselves, as in the nature of the methods by which they are treated. A mathematician, at least—and it is as a mathematician I have the privilege of addressing you—may be excused for contending that the bond of union among the physical sciences is the mathematical spirit and the mathematical method which pervade them. As has been said with profound truth by one of my predecessors in this chair, our knowledge of nature, as it advances, continuously resolves differences of quality into differences of quantity. All exact reasoning—indeed all reasoning—about quantity, is mathematical reasoning; and thus as our knowledge increases that portion of it which becomes mathematical increases at a still more rapid rate. Of all the great subjects which belong to the province of this Section, take that which at first sight is the least within the domain of mathematics—I mean meteorology. Yet the part which mathematics bears in meteorology increases every year, and seems destined to increase. Not only is the theory of the simplest instruments of meteorology essentially mathematical, but the discussion of the observations

—upon which, be it remembered, depend the hopes which are already entertained with increasing confidence of reducing the most variable and complex of all known phenomena to exact laws—is a problem which not only belongs wholly to mathematics, but which taxes to the utmost the resources of the mathematics which we now possess. So intimate is the union between mathematics and physics that probably by far the larger part of the accessions to our mathematical knowledge have been obtained by the efforts of mathematicians to solve the problems set to them by experiment, and to create 'for each successive class of phenomena, a new calculus or a new geometry, as the case might be, which might prove not wholly inadequate to the subtlety of nature.' Sometimes, indeed, the mathematician has been before the physicist, and it has happened that when some great and new question has occurred to the experimentalist or the observer, he has found in the armoury of the mathematician the weapons which he has needed ready made to his hand. But much oftener the questions proposed by the physicist have transcended the utmost powers of the mathematics of the time, and a fresh mathematical creation has been needed to supply the logical instrument requisite to interpret the new enigma. Perhaps I may be allowed to mention an example of each of these two ways in which mathematical and physical discovery have acted and reacted on each other. I purposely choose examples which are well known, and belong, the one to the oldest, the other to the latest times of scientific history.

The early Greek geometers, considerably before the time of Euclid, applied themselves to the study of the various curve lines in which a conical figure may be cut by a plane—curve lines to which they gave the name, never since forgotten, of conic sections. It is difficult to imagine that any problem ever had more completely the character of a 'problem of mere curiosity,' than this problem of the conic sections must have had in those earlier times. Not a single natural phenomenon which in the state of science at that time could have been intelligently observed was likely to require for its explanation a knowledge of the nature of these curves. Still less can any application to the arts have seemed possible; a nation which did not even use the arch was not likely to use the ellipse in any work of construction. The difficulties of the inquiry, the pleasure of grappling with the unknown, the love of abstract truth, can alone have furnished the charm which attracted some of the most powerful minds in antiquity to this research. If Euclid and Apollonius had been told by any of their contemporaries that they were giving a wholly wrong direction to their energies, and that, instead of dealing with the problems presented to them by nature, they were applying their minds to inquiries, which not only were of no use, but which never could come to be of any use, I do not know what answer they could have given which might not now be given with equal, or even greater, justice to the similar reproaches which it is not uncommon to address to those mathematicians of our own day who study quantities of n-indeterminates, curves of the nth order, and, it may be, spaces of n-dimensions. And not only so, but for pretty nearly two thousand years the experience of mankind would have justified the objection; for there is no record that during that long period which intervened between the first invention of the conic sections and the time of Galileo and Kepler the knowledge of these curves possessed by geometers was of the slightest use to natural science. And yet, when the fullness of time was come, these seeds of knowledge, that had waited so long, bore splendid fruit in the discoveries of Kepler. If we may use the great names of Kepler and Newton to signify stages in the progress of

human discovery, it is not too much to say that without the treatises of the Greek geometers on the conic sections there could have been no Kepler, without Kepler no Newton, and without Newton no science in our modern sense of the term, or at least no such conception of nature as now lies at the basis of all our science, of nature as subject in its smallest as well as in its greatest phenomena, to exact quantitative relations, and to definite numerical laws.

This is an old story, but it has always seemed to me to convey a lesson, occasionally needed even in our own time, against a species of scientific utilitarianism which urges the scientific man to devote himself to the less abstract parts of science, as being more likely to bear immediate fruit in the augmentation of our knowledge of the world without. I admit, however, that the ultimate good fortune of the Greek geometers can hardly be expected by all the abstract speculations which, in the form of mathematical memoirs, crowd the transactions of the learned societies; and I would venture to add that, on the part of the mathematician, there is room for the exercise of good sense, and, I would almost say, of a kind of tact, in the selection of those branches of mathematical inquiry which are likely to be conducive to the advancement of his own or any other science.

I pass to my second example, of which I may treat very briefly. In the course of the present year a treatise on electricity has been published by Professor Maxwell, giving a complete account of the mathematical theory of that science, as we owe it to the labours of a long series of distinguished men, beginning with Colomb, and ending with our own contemporaries, including Professor Maxwell himself. No mathematician can turn over the pages of these volumes without very speedily convincing himself that they contain the first outlines (and something more than the first outlines) of a theory which has already added largely to the methods and resources of pure mathematics, and which may one day render to that abstract science services no less than those which it owes to astronomy. For electricity now, like astronomy of old, has placed before the mathematician an entirely new set of questions, requiring the creation of entirely new methods for their solution, while the great practical importance of telegraphy has enabled the methods of electrical measurement to be rapidly perfected to an extent which renders their accuracy comparable to that of astronomical observations, and thus makes it possible to bring the most abstract deductions of theory at every moment to the test of fact. It must be considered fortunate for the mathematicians that such a vast field of research in the application of mathematics to physical inquiries should be thrown open to them at the very time when the scientific interest in the old mathematical astronomy has for the moment flagged, and when the very name of physical astronomy, so long appropriated to the mathematical development of the theory of gravitation, appears likely to be handed over to that wonderful series of discoveries which have already taught us so much concerning the physical constitution of the heavenly bodies themselves.

Having now stated, from the point of view of a mathematician, the reasons which appear to me to justify the existence of so composite an institution as Section A, and the advantages which that very compositeness sometimes brings to those who attend its meetings, I wish to refer very briefly to certain definite services which this Section has rendered and may yet render to Science. The improvement and extension of scientific education is to many of us one of the most urgent questions of the day; and the British Association has already exerted itself more than once to press the question on the public

attention. Perhaps the time has arrived when some further efforts of the same kind may be desirable. Without a rightly organized scientific education we cannot hope to maintain our supply of scientific men; since the increasing complexity and difficulty of science renders it more and more difficult for untaught men, by mere power of genius, to force their way to the front. Every improvement, therefore, which tends to render scientific knowledge more accessible to the learner, is a real step towards the advancement of science, because it tends to increase the number of well qualified workers in science.

For some years past this Section has appointed a committee to aid in the improvement of geometrical teaching in this country. The report of this committee will be laid before the Section in due course; and without anticipating any discussion that may arise on that report, I think I may say that it will show that we have advanced at least one step in the direction of an important and long-needed reform. The action of this Section led to the formation of an Association for the improvement of geometrical teaching, and the members of that Association have now completed the first part of their work. They seem to me, and to other judges much more competent than myself, to have been guided by a sound judgment in the execution of their difficult task, and to have held, not unsuccessfully, a middle course between the views of the innovators who would uphold the absolute monarchy of Euclid, or, more properly, of Euclid as edited by Simpson, and the radicals who would dethrone him altogether. One thing at least they have not forgotten, that geometry is nothing if it be not rigorous, and that the whole educational value of the study is lost if strictness of demonstration be trifled with. The methods of Euclid are, by almost universal consent, unexceptional in point of rigour. Of this perfect rigorousness his doctrine of parallels, and his doctrine of proportion, are perhaps the most striking examples. That Euclid's treatment of the doctrine of parallels is an example of perfect rigorousness, is an assertion which sounds almost paradoxical, but which I, nevertheless, believe to be true. Euclid has based his theory on an axiom (in the Greek text it is one of the postulates, but the difference for our purpose is immaterial) which, it may be safely said, no unprejudiced mind has ever accepted as self-evident. And this unaxiomatic axiom Euclid has chosen to state, without wrapping it up or disguising it, not, for example, in the plausible form in which it has been stated by Playfair, but in its crudest shape, as if to warn his reader that a great assumption was being made. This perfect honesty of logic, this refusal to varnish over a weak point, has had its reward; for it is one of the triumphs of modern geometry to have shown that the eleventh axiom is so far from being an axiom, in the sense which we usually attach to the word, that we cannot at this moment be sure whether it is absolutely and rigorously true, or whether it is a very close approximation to the truth. Two of those whose labours have thrown much light on this difficult theory are present at this meeting—Prof. Cayley, and a distinguished German mathematician, Dr. Felix Klein; and I am sure of their adherence when I say that the sagacity and insight of the old geometer are only put in a clearer light by the success which has attended the attempt to construct a system of geometry, consistent with itself, and not contradicted by experience, upon the assumption of the falsehood of Euclid's eleventh axiom.

Again, the doctrine of proportion, as laid down in the fifth book of Euclid, is, probably, still unsurpassed as a masterpiece of exact reasoning; although the cumbrousness of the forms of expression which were adopted in the old geometry, has led to the total exclusion of this part of the elements from the ordinary course of geometrical education. A zealous

defender of Euclid might add with truth that the gap thus created in the elementary teaching of mathematics has never been adequately supplied.

But after all has been said that can be said in praise of Euclid, the fact remains that the form in which the work is composed renders it unsuitable for the earlier stages of education. Euclid wrote for men; whereas his work has been used for children, and it is surely no disparagement to the great geometer to suppose that after more than 2000 years the experience of generations of teachers can suggest changes which may make his Elements, I will not say more perfect as a piece of geometry, but more easy for very young minds to follow. The difficulty of a book or subject is indeed not in itself a fatal objection to its use in education, for to learn how to overcome difficulties is one great part of education: Geometry is hard, just as Greek is hard, and one reason why Geometry and Greek are such excellent educational subjects is precisely that they are hard. But in a world in which there is so much to learn, we must learn everything in the easiest way in which it can be learnt; and after we have smoothed the way to the utmost of our power there is sure to be enough of difficulty left. I regard the question of some reform in the teaching of elementary geometry as so completely settled by a great concurrence of opinion on the part of the most competent judges, that I should hardly have thought it necessary to direct the attention of the Section to it, if it were not for the following reasons:—

First, that the old system of geometrical instruction still remains (with but few exceptions) paramount in our schools, colleges, and universities, and must remain so until a very great *consensus* of opinion is obtained in favour of some one definite text-book. It appears to me, therefore, that the duty will eventually devolve upon this Section of the British Association, of reporting on the attempts that have been made to frame an improved system of geometrical education; and if it should be found that these attempts have been at last successful, I think that the British Association should lend the whole weight of its authority to the proposed change. I am far from suggesting that any such decision should be made immediately. The work undertaken by the Association for the improvement of geometrical teaching is still far from complete; and even when it is complete it must be left to hold its own against the criticism of all comers before it can acquire such an amount of public confidence as would justify us in recommending its adoption by the great teaching and examining bodies of the country.

Secondly, I have thought it right to remind the Section of the part it has taken with reference to the reform of geometrical teaching, because it appears to me that a task, at once of less difficulty, and of more immediate importance, might now be undertaken by it with great advantage. There is at the present moment a very general agreement that a certain amount of natural science ought to be introduced into school education; and many schools of the country have already made most laudable efforts in this direction. As far as I can judge, there is further a general agreement that a good school course of natural science ought to include some part or parts of physics, of chemistry, and of biology; but I think it will be found that while the courses of chemistry given at our best schools are in the main identical, there is the greatest diversity of opinion as to the parts of physics and of biology which should be selected as suitable for a school education, and a still greater diversity of opinion as to the methods which should be pursued in teaching them. Under these circumstances it is not surprising to find that the masters of those schools into which natural science has hardly as yet found its way (and some of the largest

and most important schools in the country are in this class), are doubtful as to the course which they should take; and from not knowing precisely what they should do, have not as yet made up their minds to do anything of importance. There can be no doubt that the masters of such schools would be glad on these points to be guided by the opinion of scientific men, and I cannot help thinking that this opinion would be more unanimous than is commonly supposed, and further that no public body would be so likely to elicit an expression of it as a Committee appointed by the British Association. I believe that if such an expression of the opinion of scientific men were once obtained, it would not only tend to give a right direction to the study of natural science in schools, but might also have the effect of inducing the public generally to take a higher and more truthful view of the objects which it is sought to attain by introducing natural science as an essential element into all courses of education. All knowledge of natural science that is imparted to a boy, is, or may be, useful to him in the business of his after-life; but the claim of natural science to a place in education cannot be rested upon its practical usefulness only. The great object of education is to expand and to train the mental faculties, and it is because we believe that the study of natural science is eminently fitted to further these two objects, that we urge its introduction into school studies. Science expands the minds of the young, because it puts before them great and ennobling objects of contemplation; many of its truths are such as a child can understand, and yet such, that, while in a measure he understands them, he is made to feel something of the greatness, something of the sublime regularity, and of the impenetrable mystery, of the world in which he is placed. But science also trains the growing faculties, for science proposes to itself truth as its only object, and it presents the most varied, and at the same time the most splendid examples of the different mental processes which lead to the attainment of truth, and which make up what we call reasoning. In science error is always possible, often close at hand; and the constant necessity for being on our guard against it is one important part of the education which science supplies. But in science sophistry is impossible; science knows no love of paradox; science has no skill to make the worse appear the better reason; science visits with a not long deferred exposure all our fondness for preconceived opinions, all our partiality for views that we have ourselves maintained, and thus teaches the two best lessons that can well be taught—on the one hand the love of truth, and on the other sobriety and watchfulness in the use of the understanding.

In accordance with these views I am disposed to insist very strongly on the importance of assigning to physics, that is to say to those subjects which we discuss in this Section, a very prominent place in education. From the great sciences of observation, such as botany or zoology or geology, the young student learns to observe, or more simply, to use his eyes; he gets that education of the senses which is after all so important, and which a purely grammatical and literary education so wholly fails to give. From chemistry he learns, above all other things, the art of experimenting, and of experimenting for himself. But from physics, better as it seems to me than from any other part of science, he may learn to reason with consecutiveness and precision, from the data supplied by the immediate observation of natural phenomena. I hope we shall see the time when each successive portion of mathematical knowledge acquired by the pupil will be made immediately available for his instruction in physics; and when everything that he learns in the physical laboratory will be made the subject of mathematical reasoning and calculation.

In some few schools I believe that this is already the case, and I think we may hope well for the future both of mathematics and physics in this country when the practice becomes universal. In one respect the time is favourable for such a revolution in the mode of teaching physical science. During the past few years a number of text-books have been made available to the learner, which far surpass anything that was at the disposal of former generations of pupils, and which are probably as completely satisfactory as the present state of science will admit. It is pleasant to record that these text-books are the work of distinguished men who have always taken a prominent part in the proceedings of this Section. We have Deschanel's Physics, edited, or rather re-written, by Prof. Everett, a book remarkable alike for the clearness of its explanations and for the beauty of the engravings with which it is illustrated; and, passing to works intended for students somewhat further advanced, we have the treatises of Prof. Balfour Stewart on Heat, of Prof. Clerk Maxwell on the Theory of Heat, of Prof. Fleeming Jenkin on Electricity, and we expect a similar treatise on Light from another of our most distinguished members.

These works breathe the very spirit of the method which should guide both research and education in physics. They express the most profound and far-reaching generalizations of science in the simplest language, and yet with the utmost precision. With the most sparing use of mathematical technicalities, they are a perfect storehouse of mathematical ideas and mathematical reasonings. An old French geometer used to say that a mathematical theory was never to be considered complete till you had made it so clear that you could explain it to the first man you met in the street. This is of course a brilliant exaggeration, but it is no exaggeration to say that the eminent writers to whom I have referred have given something of this clearness and completeness to such abstract mathematical theories as those of the electrical potential, the action of capillary forces, and the definition of absolute temperature. A great object will have been attained when an education in physical science on the basis laid down in these treatises has become generally accepted in our schools.

I do not wish to close this address without adverting, though only for one moment, to a question which occupies the minds of many of the friends of science at the present time—the question, What should be the functions of the State in supporting or in organizing scientific inquiry? I do not mean to touch on any of the difficulties which attend this question, or to express any opinion as to the controversies to which it has given rise. But I do not think it can be out of place for the president of this Section to call your attention to the inequality with which, as between different branches of science, the aid of Government is afforded. National observations for astronomical purposes are maintained by this as by every civilized country. Large sums of money are yearly expended, and most rightly expended, by the Government for the maintenance of museums and collections of mineralogy, botany, and zoology. At a very recent period an extensive chemical laboratory, with abundant appliances for research as well as for instruction, has been opened at South Kensington. But for the physical sciences—such sciences as those of heat, light, and electricity—nothing has been done; and I confess I do not think that any new principle would be introduced, or any great burden incurred, capable of causing alarm to the most sensitive Chancellor of the Exchequer, if it should be determined to establish, at the national cost, institutions for the prosecution of these branches of knowledge, so vitally important to the progress of science as a whole. Perhaps, also, upon this

general ground of fairness, even the pure mathematicians might prefer a modest claim to be assisted in the calculation and printing of a certain number of Tables, of which even the physical applications of their science are beginning to feel the pressing need.

One word further on this subject of State assistance to science, and I have done. It is, no doubt, true that for a great, perhaps an increasing, number of purposes science requires the assistance of the State; but is it not nearer to the truth to say that the State requires the assistance of science? It is my conviction that if the true relations between science and the State are not recognized, it is the State, rather than science, that will be the great loser. Without science the State may build a ship that cannot swim, and may waste a million or two on experiments, the futile result of which science could have foreseen. But without the State science has done very well in the past, and may do very well in time to come. I am not sure that we should know more of pure mathematics, or of heat, of light, or electricity than we do at this moment if we had had the best help of the State all the time. There are, however, certain things which the State might do, and ought to do, for science. It, or corporations created by it, ought to undertake the responsibility of carrying on those great systems of observation which, having a secular character, cannot be completed within the lifetime of a single generation, and cannot, therefore, be safely left to individual energy. One other thing the State ought to do for science. It ought to pay scientific men properly for the services which they render directly to the State, instead of relying, as at present, on their love for their work as a means of obtaining their services on lower terms. If any one doubts the justice of this remark, I would ask him to compare the salaries of the officers in the British Museum with those which are paid in other departments of the Civil Service.

But what the State cannot do for science is to create the scientific spirit, or to control it. The spirit of scientific discovery is essentially voluntary; voluntary, and even mutinous, it will remain: it will refuse to be bound with red tape, or ridden by officials, whether well-meaning or perverse. You cannot have an Established Church in science, and, if you had, I am afraid there are many scientific men who would turn scientific nonconformists.

I venture upon these remarks because I cannot help feeling that the great desire which is now manifesting itself on the part of some scientific men to obtain for science the powerful aid of the State may perhaps lead some of us to forget that it is self-reliance and self-help which have made science what it is, and that these are qualities the place of which no Government help can ever supply.

II.

ARITHMETICAL INSTRUMENTS.

[South Kensington Museum Handbook to the Special Loan Collection of Scientific Apparatus, 1876, pp. 22–33.]

Of all those branches of human knowledge which are comprehended under the name of Science, Arithmetic is that which has the most abstract character, and which, at the same time, is of the most universal application in the study of natural phenomena. The art of counting, or of numeration, is one of the earliest, if not the earliest product of nascent civilization; and, in the case of the savage races of mankind, the greater or less progress which has been made towards the acquisition of this art affords no unfair measure of the degree of culture and of intellectual development which has been attained. It is said that there are races whose scale of numeration is limited to two or three; others can go to five, or ten, or twenty. And we may be sure that no tribe of men, untaught by a superior race, ever acquired the art of counting by hundreds or thousands, without possessing a high average of mental capacity, and without sharing in the privilege, accorded only to certain nations, of occasionally producing men of inventive genius, and real leaders of thought.

The more favoured branches of the Semitic and Aryan families—the Jews, the Egyptians, the Greeks, the Sanskrit-speaking nations of India—must have reached this, comparatively speaking. advanced standard of culture at a very remote period. But it is remarkable that the real extent of the domain of arithmetic—a domain in a certain sense coequal with that of exact science—was not perceived till a much later epoch.

The Greek philosophers, at least as early as the time of Aristotle, had learned to distinguish between *discrete*, or *discontinuous*, and *continuous* quantity. All counting, properly so called, is of discontinuous quantity; all measurement is of continuous quantity. To use a simple illustration: if we are counting points or dots on a line we can say, 'two dots and one dot make three dots'; if we are measuring inches we can equally say, 'two inches and one inch make three inches.' But in the latter case we can, if we please, pass by insensible degrees, and through every intermediate gradation of magnitude, from two inches to three inches: in the former case we can only pass abruptly from counting two dots to counting three dots; there is no such thing as half a dot, and no intermediate stage is conceivable.

But while this important distinction was clearly seen in very ancient times, being indeed of a nature to commend itself specially to the philosophical spirit of classical antiquity, there was not an equally distinct apprehension of the truth that continuous quantity, no less than discontinuous, appertains to the domain of arithmetic. By whom

the first dim perception, or by whom the first vivid realization, of this truth was attained, we have no means of ascertaining with precision. It must have been gradually impressed on the minds of men by the growth of science. It is, perhaps, hardly discernible in the writings of Plato and Aristotle: it underlies, but is carefully excluded from, the fifth book of the Elements of Euclid. It must have been present to the mind of Archimedes when he measured the proportions to one another of the sphere, the cylinder, and the cone; it must have forced itself on the notice of the Greek astronomers, whose business it was to record numerically at discontinuous intervals the phases of continuous phenomena; and it became firmly established as an axiomatic principle by the development of that mode of arithmetic which is called algebra; by the great invention of Descartes which reduced geometry to algebra; and, last of all, by the creation of those arithmetical methods which are briefly described as the infinitesimal calculus.

Although this conception of the absolute continuity of arithmetical magnitude is of a very abstract character, it has exercised a prepondering influence over scientific thought. That, on the one hand, all natural phenomena take place by a continuous process, and that they are all measurable quantitatively: that, on the other hand, the law of any continuous process can be expressed by an arithmetical formula, and the amount of any quantitative measurement can be stated in arithmetical figures, are propositions which are admitted by every one who understands them, and which, indeed, are in some instances believed with a more unlimited faith than is warranted by the evidence, strong as it is, which can be brought in support of them. Nor is this all; for if there be any one opinion concerning nature at the present time universally accepted by scientific men, it is that the minutest as well as the greatest phenomena are subject to a 'reign of law.' And if we ask for the strongest reasons which can be given for this belief, they may be summed up by saying that, so far as our measurements are exact, and so far as our arithmetic has been able to cope with the arithmetic of nature, we have uniformly found our observations of continuous phenomena to be in strict accordance with our deductions from the abstract science of continuous number.

We proceed to offer a few observations with reference to each of the two branches of arithmetic—that of discontinuous and that of continuous quantity. The course of these remarks will make it clear why it is that a science of incalculable importance to other sciences does not, nevertheless, make any considerable display of its pretensions in an exhibition of scientific apparatus.

(1) The simple operations of counting, and of recording numbers counted, and of comparing them with one another, which constitute the main business of practical arithmetic, have been so facilitated by the two great inventions of the decimal system of notation, and of logarithms, that, in many cases, but little inducement has existed to supersede the labour involved in such calculations by means of mechanical appliances.

Counting machines, however, for certain purposes have been found indispensably necessary. A clock is defined by Sir John Herschel as a machine for counting and recording the number of the oscillations of a pendulum; though to this definition we are obliged to add that every clock must also contain a mechanism adapted to maintain the state of oscillation of the pendulum against friction and the resistance of the air. A pedometer is an instrument for counting and recording the number of steps taken by the person carrying it. Distances along a road are approximately measured by rolling

a wheel along the road, an apparatus being annexed to the wheel which counts and records its revolutions. In the same way a turnstile may be made to record the number of its own revolutions, *i.e.* the number of persons admitted through it.

The above are simple instances of counting machines employed for the common purposes of life; but the construction of calculating engines, adapted to more varied and complicated purposes than that of simple counting, is to be reckoned among the great achievements of mathematical and mechanical skill. The first idea of such a machine appears to have been due to the celebrated Blaise Pascal; the apparatus constructed by him was arranged for the addition and subtraction of sums of money. Two calculating machines, constructed in 1775 and 1777 by James Bullock for Viscount Mahon, are included in the Exhibition. But the idea of a difference engine, which should serve to calculate tables of analytical functions, was first successfully realized by Charles Babbage; the analogous contrivances which had previously been proposed having been designed merely for the performance of single arithmetical operations, such as addition, subtraction, multiplication, and division. The later years of Babbage's life were devoted to the construction, or rather to the design, of a great analytical engine, which was intended to possess a range of calculating power, far exceeding that of the difference engine, and, in fact, extending over the whole field of arithmetical analysis.

An article on Babbage's Difference Engine, in the *Edinburgh Review* for 1834, suggested to George Scheutz, of Stockholm, the idea of constructing a machine for simultaneously calculating and printing arithmetical tables. After many discouragements, which were overcome by the indefatigable perseverance of George Scheutz and his son Edward, this machine was at last completed in October, 1853. The 'Specimens of Tables, calculated, stereo-moulded, and printed by Machinery,' published by them in London in 1857, afford a convincing proof of the completeness and utility of their invention. Its originality was gladly recognized by Babbage; and indeed two things only are common to the engines of Babbage and Scheutz; the principle of calculation by differences, and the contrivance by which the computed results are conveyed to the printing apparatus.

Several arithmetical machines, on a smaller scale and of simpler construction, have been produced in recent years. Some of these are actually in use in the public offices of this country. We may mention especially the calculating machine of M. Thomas, of Colmar, and the panometer of Edward Grohmann, of Vienna.

In the ancient world, and before the invention of the decimal notation, the common operations of arithmetic were carried on with the aid of a 'counting board,' or *abacus*, the units being represented by counters, or pebbles (*calculi*, whence the word calculation). The authorities are not entirely agreed as to the precise arrangement of the ancient abacus, which, probably, was not always the same in all instances. It would seem certain, however, that the principle of decimal arrangement was to some extent adopted; counters in one compartment being valued as units, in that to the left of it as tens, and so on. It may seem strange that this partial introduction of a decimal system should not have led sooner to the invention of a decimal notation such as we now employ. The transition would probably have been instantaneous if the idea of employing a distinct symbol for zero had occurred to those who used the abacus. But it is precisely the introduction of this symbol which forms the central point of the whole decimal notation; and it may be admitted that in the abacus itself there was nothing to suggest its introduction. The

nearest approximations in the modern world to the ancient abacus are the bean-tables, multiplication boards, and other similar appliances employed in elementary education;* and the marking boards in use in certain games.

The monetary transactions of the ancient world were occasionally on a scale approaching those of our own times. When Vespasian became emperor he found that, after the profligate expenditure of Nero, and the subsequent civil wars, the indebtedness and pressing requirements of the imperial and public treasuries amounted to no less a sum than about three hundred and thirty millions sterling. However rudely the accounts of these vast liabilities may have been kept, they must have required an enormous amount of calculation; and all this calculation must have been performed with the abacus; for it would have been almost impossible with the written characters of the Roman system of numeration. Perhaps no single instance could better serve to show the great saving of human labour which has been effected by the use of a decimal notation.

The arithmetic of whole numbers, of which we are here speaking, has its theoretical as well as its practical part. This theoretical part is called the Theory of Numbers, and is perhaps the only branch of pure mathematics against which the charge of uselessness has ever been seriously alleged. Nevertheless, at all periods of the history of mathematical science it has excited a keen interest, and to it, rather than to researches of more obvious utility, we owe the development of the practical branch of arithmetic. As early as the second or third century before Christ, the Indian ritualists were led to the problem 'To find two square numbers of which the sum shall also be a square' by the existence of a religious feeling which required that altars of different shapes should have the same superficial area. By these and similar inquiries they were brought into contact with many questions of mensuration, and learned to solve them by approximate methods of considerable exactness; the value, for example, which they obtained for the side of a square equal to a circle of given diameter is correct as far as the third decimal place inclusively. Contemporary records of these researches still exist, and though they tell us of a time when science was in its infancy, they bear emphatic testimony to the genius and patient industry of the ancient workers. They are further characterized by that predominance of the arithmetical above the geometrical spirit, which forms so marked a contrast between the mathematical tendencies of India and of Greece. But while in these earlier treatises we can watch the growth of mathematical conceptions, called forth and fostered by the practical requirements of the old Vedic ceremonial, the purely scientific study of geometry and arithmetic in India belongs to a later period, probably to the fourth century after our era. Even then, the Hindus were the first to discover the method of solving indeterminate equations of the first degree, a method which was not known in Europe till the seventeenth century, and perhaps not demonstrated till the eighteenth. But the crowning achievement of Indian mathematical genius was the solution of the problem known as the Pellian Equation, upon which the analysis of indeterminate equations of the second degree may be said entirely to depend. The Indian mathematicians gave no demonstration of their solution. That demonstration was first given, at least fourteen hundred years later, by Lagrange, one of the greatest of European mathematicians, and the memoir in which he has recorded this discovery has always been regarded as one of the principal monuments of his genius. The

* A series of these appliances have been contributed to the Exhibition by the Committee of the Russian Pædagogical Museum.

indeterminate equation of the first degree, to which we have above referred, is specially deserving of mention in this place, because it admits of mechanical applications to the theory of wheel-work, and also because it can be represented by a simple geometrical illustration.

We may mention, for a similar reason, another important research connected with the theory of numbers, viz. the calculation of Tables giving for each number the least number by which it is divisible; or, if it is a prime number, indicating that it is so. Such tables (which considerably abbreviate certain computations) have been constructed for the first nine millions; the tables of the fourth, fifth, and sixth millions exist, however, in manuscript only, and have never been published. The first attempt to form a Table of Primes was made by Eratosthenes, and the partly mechanical method adopted by him (and called after its inventor, 'the sieve of Eratosthenes') has been adopted in principle, though with appropriate modifications, by his successors.

There is in general so little appearance in those laws of nature with which we are acquainted of any adherence to integral or whole numbers, that we may be allowed to call attention to two important classes of phenomena which form an exception to this remark. We refer to the laws of chemical combination, and to the laws of crystallography.

If we imagine chemical substances existing in the ideal condition of perfect gases, the law of chemical combination may be expressed in its most abstract form by saying that if two perfect gases combine chemically, and form a compound which is also supposed to exist in a perfectly gaseous condition, the volumes of the two gases before combination and of the gas resulting from their combination are to one another as three whole numbers.

The law of integral numbers to which the faces of a crystal are subject is sufficiently illustrated by the models in the Section of Mineralogy; and it would be out of place to discuss it here.

It may, however, be proper to remark that the whole numbers which present themselves in the formulæ whether of chemistry or of crystallography are never very considerable. In the case of some organic bodies the number of equivalents that enter into the formula has to be counted by hundreds; but in all such instances, owing to the imperfection of the methods of chemical analysis, the determinations that have been given must be regarded as open to correction. The 'indices' of any face of a crystal actually occurring in nature rarely exceed ten.

(2) The geometrical and mechanical appliances for aiding in the operations of arithmetic, as applied to continuous magnitude, are not very numerous, and possess in most cases a theoretical rather than a practical importance.

A very ingenious instrument of this kind, and one that has been extensively used, is the slide rule, which may be described as an apparatus for effecting multiplications and divisions by means of a logarithmic scale; the requisite additions and subtractions being performed without calculation by a proper adjustment of the instrument itself. The principle on which it depends admits of being applied in various ways, and thus there are slide rules of very various forms, and adapted to very different purposes. But the card of four figure logarithms is a formidable competitor to any logarithmic scale, and it may be doubted whether at the present time these really beautiful contrivances are in as common use as they deserve to be. The Exhibition contains a complete series of them by Messrs. Aston and Mander; besides the Estimator of Dr. F. M. Stapff, and the Pocket Calculator of General de Lisle.

Instruments for solving triangles and for finding the roots of quadratic and higher equations may next be noticed. Some of them are remarkable for their ingenuity; some are useful as educational appliances, because they serve to illustrate, in a very beautiful way, the connexion between arithmetic, or algebra, and geometry. Others again are of great interest from the difficulty of the problems which they propose to solve, and the profound character of the principles which they employ in the solution. To this last class belongs the interesting application by Professor Sylvester of the Peaucellier movement to the extraction of the roots of numbers.

With regard to all these arrangements it must be observed that the solutions which they afford are only approximate, and that the degree of the approximation cannot be carried beyond a certain point. This arises, not from any imperfection in the theory of the instruments, but from the circumstance that the solution of the problem is given by measurement; and that all measurements are necessarily approximate, and subject to errors which cannot be reduced beyond a certain point.

In this respect the analytical solution of a problem possesses a great theoretical advantage above a solution obtained by geometrical or mechanical means. The analytical solution is indeed in general approximate, no less than the geometrical or mechanical one; but the degree of the approximation is no longer limited; for if we are dissatisfied with the degree of approximation we have obtained, we can go back and repeat the process over again, retaining small terms which we before omitted, until we arrive at a result as near to the truth as we please. Of course this theoretical advantage ceases to have any practical importance whenever the degree of approximation attainable by the mechanical appliance is sufficient for the purpose in view.

At an earlier stage of the development of analytical science, graphical methods for the solution of analytical problems were of more importance than they are at present. When Descartes showed that the solution of a biquadratic equation could be made to depend on the determination of the intersection of a parabola by a circle, it is possible that, at least in certain cases, the very best method of finding the roots of a proposed biquadratic equation, which the resources of mathematics could then supply, was to describe the parabola and the circle, and actually to measure the ordinates of the points common to the two. But the continual progress of improvement in analytical processes, coupled with the greatly increased facility in calculating the arithmetical values of analytical expressions, which was obtained by the invention of logarithms, has completely outrun the present capabilities of geometrical methods; and these methods are now seldom used, except for obtaining rough first approximations. Thus it is that the degree of perfection which has been given to analysis has enabled it to dispense with mechanical aids; although instances are not wanting which may serve to show that the mechanical methods may yet receive a great future development. In addition to the applications of the Peaucellier movement, to which we have already referred, we may also mention that Sir William Thomson has recently planned an integrating machine which will integrate mechanically any differential equation, or set of simultaneous differential equations, containing only one independent variable. In the important case of the linear equation of the second order with variable coefficients, the actual construction of the requisite mechanism would, in the opinion of Sir William Thomson, present no insuperable difficulty. The kinematical principle employed in this integrating machine is due to Professor James Thomson, and consists in the transmission

of rotation from a disk or cone to a cylinder by the intervention of a loose sphere, which presses by its weight on the disk and cylinder, or on the cone and cylinder, as the case may be; the pressure being sufficient to give the necessary frictional coherence at each point of rolling contact. Sir William Thomson proposes to apply this principle to the construction of a machine adapted to calculate the harmonic constituents of any given function; and he believes that by employing such a machine in the analysis of the tides a single operator will be enabled to find, in an hour or two, any one of the simple harmonic elements of a year's tides recorded in curves by an ordinary tide gauge in the usual manner, a result which hitherto has required not less than twenty hours' computation by skilled arithmeticians. As another indication of the same tendency to substitute (wherever it may be found possible) mechanical or graphical contrivances for abstract calculations, we may refer to an excellent German treatise, the 'Graphische Statik' of Professor Culmann, which has already exercised a powerful influence upon the course of technical education in Germany, and of which it is the object to solve important engineering problems, relating to the stability of constructions, by mere geometrical drawing without the use of analytical formulæ.

III.

GEOMETRICAL INSTRUMENTS AND MODELS.

[South Kensington Museum Handbook to the Special Loan Collection of Scientific Apparatus, 1876, pp. 34–54.]

NEXT to the science of number, the science of space is that which is at once the most abstract, and admits of the most universal application to the study of natural phenomena. Everything that takes place takes place in space; and thus Geometry, or the science of space, necessarily intervenes in all exact observation of events.

When we begin to think about space at all, the properties of it which first impress the mind are its continuity, and its apparently indefinite extent, our imaginations being perhaps unable to conceive the absence of either of these two properties. Probably we next notice the existence of three dimensions of space (as seen in the length, breadth, and height of any object), and we cannot conceive it to possess more or fewer. We further observe, (1) that at any two different points space is exactly similar to itself, and (2) that in all the directions which exist at any one point it has identical properties.

These general assertions, if not really of themselves evident, are at least readily admitted as being in accordance with universal experience. They are all assumed in, and may be said to form the basis of, that analytical representation of space which we owe to Descartes, and which justly entitles him to be regarded as the founder of modern geometry. In accordance with this representation we regard space as a complex (if we may use this word as a translation of the German *Mannigfaltigkeit*) of three indeterminate quantities corresponding to its three dimensions; the surfaces, lines, and points which exist in space, being, in technical phraseology, the 'loci' obtained by imposing one, two, or three restrictive conditions upon these indeterminates. As often happens in similar cases, the mode of representation thus introduced is capable of being extended so as to apply to other objects or conceptions beside that for which it was first employed; and thus mathematicians have been led to consider complexes of more than three indeterminates, or, again, complexes not possessing the properties which we have enumerated as characteristic of space. This is the origin of such phrases as 'a space of four dimensions,' or of such assertions as 'it is conceivable that a space may not be exactly similar to itself at all its points.' These speculations are perhaps not calculated directly to promote our knowledge of the space in which we live and move, and to which they seem entirely inapplicable; but they have had the effect of advancing our knowledge of the relations of quantity, and have thus had an indirect, but not unimportant, influence upon the recent progress of geometrical science.

So great has been the influence of the Cartesian mode of representation upon geometrical speculation that it has perhaps, to a certain extent, and in certain cases,

unduly led away the minds of geometricians from that direct intuition of space upon which geometry must after all be founded. And there can be no doubt that an Exhibition of models such as those included in the present Catalogue is calculated to render a great service to geometrical science by calling attention to the concrete shapes of objects, which are too apt, even in the mind of the serious student, to exist only as conceptions very imperfectly realized.

We may for the purposes of this introduction adopt a threefold classification of the properties of space, as being either, 1. Properties of Situation; or, 2. Graphical Properties; or, 3. Metrical Properties. Of each of these three classes of properties we shall here say a few words to illustrate their importance and meaning.

1. The Properties of Situation of a figure in space are those which exist irrespectively of the magnitude and even of the shape of its parts, depending solely on the connexion of the parts, and on their situation with reference to one another. As neither the term 'properties of situation,' nor the description which we have just given of these properties, can be regarded as conveying a distinct image, a few very simple examples of what is meant may not be out of place.

If we draw, upon any surface such as a plane, two closed contours of however complicated an outline, it is quite possible that they may never meet one another, or that they meet in one or more points, and do not traverse one another. But if they traverse one another at all, they must do so an *even* number of times; *i.e.* twice, or four, or six times, &c. The truth of this proposition will be easily admitted, and it will be seen that, to understand the assertion made, we require no conception of magnitude, nor even the conception of the straight line or plane. All that we require is the idea of a continuous closed curve, and of a surface upon which it is drawn.

Again: conceive of two bodies, one a hollow sphere, the other a hollow anchor ring, and let a person imagine himself placed successively in the interior of each of these two hollow bodies. The two closed spaces in which he will thus successively find himself differ from one another at least in one remarkable respect. There is but one way of travelling from one point A inside the sphere, to another point B, also inside it; we might, of course, trace any number of routes we please from A to B, but all these routes are really reducible to one and the same route; and an elastic thread connecting A and B might be stretched so as to assume the shape of any one of them. But now take two points A and B inside the hollow anchor ring, and it will be seen at once that there are two different ways, irreducible to one another, of travelling from A to B. We have thus before us an example of a *singly connected* space (the interior of the sphere), and a *doubly connected* space (the interior of the anchor ring). The distinction depends entirely on the properties of situation of the two bounding surfaces; and it is one which has been found to be of some importance in the theories of the motion of fluids, and of electricity.

As a third example, take an oblong strip of paper, and fasten its two ends together so as to form a portion of a cylindrical surface. Take another similar piece of paper, and again fasten its two ends together, but give the paper a half twist so as to bring the upper surfaces of the two ends in contact with one another. Between the surface thus formed, and the cylindrical surface at first obtained, an important distinction will be found to subsist; viz. the cylindrical surface has an outside and inside surface, and there is no way of passing from one to the other except by penetrating the paper or crossing its edge:

whereas the two sides of the second surface form one perfectly continuous sheet; so that by travelling once along the whole length of the oblong strip, we should pass from a point on the surface to the point exactly corresponding to it on the other side of the surface; and we should not return again to the point from which we set out, until we had completed the tour a second time. The distinction which we thus learn to draw between surfaces which have two sides, and surfaces which have but one, is fundamental, and depends solely on the properties of situation of the figure, as we have now defined them.

No complete *corps de doctrine* has yet been formed of the properties of situation of figures. This is partly owing to the great difficulty of the inquiry, partly to the fact that it is only in very recent times that the attention of mathematicians has been called to the subject, by the unexpected light which researches into it have been found to throw on some of the most obscure questions of the integral calculus. We cannot, therefore, expect to find this part of the science of geometry extensively illustrated by models, or by drawings expressly prepared for the purpose. But any great collection of geometrical objects cannot fail to supply examples of such properties; and, what is of more importance, may be expected to suggest entirely new points of view in a branch of inquiry, which, more than almost any other within the range of pure mathematics, is dependent on direct observation.

2. The Graphical Properties of space are those which involve the conceptions of the straight line and plane, but do not involve any conception of magnitude, or of measurement. The Elements of Euclid will be searched in vain for an example of a purely descriptive theorem, though it would seem that one of the lost treatises of that great geometer—the 'Porisms'—was devoted to this part of geometry. In modern times researches into the descriptive properties of figures were revived by Blaise Pascal, and his elder contemporary, Desargues. By a strange fatality, the purely geometrical works of these two eminent men were lost, or wholly neglected, for more than a century, and it is only in comparatively recent times that they have received the attention which they merited. We may take as a simple instance of a graphical theorem the proposition of Desargues: 'If two triangles lying in the same plane are such that the lines joining their vertices taken in pairs meet in a point, the three intersections of the pairs of sides opposite to these vertices lie in a straight line; and conversely.'

3. Lastly, the Metrical Properties of space are those which involve, implicitly or explicitly, the consideration of magnitude. Thus the old proposition of Pythagoras, 'The square of the hypothenuse in a right angle is equal to the squares of the sides containing the right angle': and the theorems of Archimedes, 'The surface of a sphere is equal to the curved surface of its circumscribing cylinder; the volume of the sphere is two-thirds of the volume of that cylinder,' are metrical propositions. They could not be made intelligible to a person who had not the conception of the equality of geometrical magnitudes; nor verified by any one who had not the means of making exact quantitative measurements; whereas the proposition of Desargues, above quoted, is intelligible to any one who knows what a straight line and a plane are, and may be verified by any one who has a sheet of paper, a pencil, and a straight edge.

Comte proposed to define geometry as 'la science qui a pour but la mesure des grandeurs.' As a scientific statement this definition is probably insufficient, because a great part of geometry consists, as we have seen, in propositions which have no immediate

connexion with measurement. It must, indeed, be admitted that by far the most important applications of geometry to natural science, and to the business of life, turn on the metrical properties of figures. But, in a purely theoretical point of view, there is reason to believe that the graphical properties of space are the more universal, and deeply-seated in the nature of things, *notiora naturæ*, as Lord Bacon would have said; and that the metrical properties are, in a certain sense, secondary and derivative. As an example of the character of universality, which we thus attribute to graphical properties, we may take the general principle of the *duality* (as it has been termed) of geometrical figures. This principle asserts that all purely graphical theorems are twofold; *i.e.* that any graphical proposition relating to points and planes in space give rise to another, which is correlative to it, but in which the points have been replaced by planes, and *vice versâ*: the line joining two points being replaced by the line of intersection of two planes.

We proceed to indicate the principal classes of material appliances which are of use in geometrical investigations, or in the applications of geometry to the arts, or lastly, in its employment as a means of education.

We shall mention successively—

A.—Instruments used in geometrical drawing or mapping, and in copying geometrical drawings or maps.

B.—Instruments used in tracing special curves.

C.—Models of figures in space.

D.—Modes of representing figures in space by means of plane drawings.

A.—Instruments used in Geometrical Drawing.

The Ruler and the Compasses—the two great instruments of geometrical drawing and construction—are of a very remote antiquity. Probably a stretched string—such as is still used by carpenters—was the earliest form of apparatus for obtaining a straight line; and a string attached to a peg (a contrivance still adopted by gardeners in laying out a flower-bed) afforded the earliest means of describing a circle. Compasses such as we now use, and indeed several of very different forms, have been found in the excavations of Pompeii. But it is probable that the use of the compasses, which is now universal, for transferring with exactness measured lengths from a scale to a drawing, or from one drawing to another, was hardly practised in ancient times. Had this practice prevailed, it is difficult to suppose that it would not have superseded the second and third problems of the First Book of Euclid, in which lengths are transferred by means of the actual description of circles.

Among more recent improvements in the construction of compasses we may notice (1) the arrangements adapted for very fine work, and known as hair-compasses, needlepoint compasses, and spring-dividers; (2) the proportional or reducing compasses, by which we are enabled to reduce or augment in any given ratio the distances which we transfer from one drawing to another; (3) the triangular compasses, by which the position of three points forming a triangle can be transferred from one drawing to another, and which thus serve as an instrument for transferring angles; (4) the beam compasses, consisting of a beam or

bar, along which the two points of the instrument may be moved backwards and forwards, the distance between them admitting of adjustment with great precision, by means of a micrometer screw.

Next in the universality of its employment in all geometrical plan drawing is the scale of equal parts. Let one pair of opposite sides of a rectangle be divided, say, into ten equal parts, the points of division on each being numbered 1, 2, 3, . . 9, and let the lines 11, 22, 33, . . 99, be drawn parallel to the sides of the rectangle. Let the other pair of opposite sides be similarly divided into ten equal parts, but let the points of division be joined in a slanting direction by the parallels 01, 12, 23, 34, . . .; it will be found that the first set of parallels are divided into hundredths by the second set. Such a diagonal scale is placed on every so-called *plane scale*, and serves to divide one of the primary divisions into hundredth parts. With a fine pair of compasses, we may succeed in taking off from the scale any required length with an error perhaps not exceeding one five-hundredth part of a primary division of the scale.

Besides the scale of equal parts, the plane scale usually has engraved upon it a scale of chords, and a protracting scale. These are the simplest known contrivances for setting off an angle, given in degrees and minutes, or for approximately measuring an angle already laid down. With a good scale of chords, an angle can, it is said, be set off true to the nearest minute. But, for the best and most convenient solution of the problem, 'to construct an angle equal to a given one,' we have recourse to the divided circles, or parts of circles, known as circular, or semicircular, or quadrantal, protractors.

Less elementary in their theory than the preceding simple instruments, are the arrangements called Pantographs, which enable us to copy any given plane figure upon a different scale. Of this instrument there are two principal forms, known as the older pantograph and the Milan pantograph. In each of these there is a linkage movement in which only one point is absolutely fixed. The linkage is so arranged that two points on different bars always remain in the same straight line with the fixed point, and at distances from it which are to one another in a constant ratio. It follows from this that if one of these two points be made to describe any figure, the other will describe a similar and similarly situated figure, the centre of similitude of the two figures being the fixed point. In the older pantograph the similarity is direct, in the Milan pantograph it is inverse; *i.e.* in the first case the figures are on the same side of their centres of similitude, in the second they are on opposite sides.

B.—Instruments for Tracing Special Curves.

Geometrical drawings consist very mainly, but not exclusively, of straight lines and circles. And the same limitation is observable in all the ordinary constructions of theoretical geometry. It has been ascertained that every problem which admits of one solution, and one only, can, if the data of it are given graphically, be solved with the ruler only, *i.e.* by drawing straight lines only, and without using the compasses; and, again, that every problem which is *quadratic*, *i.e.* which admits two solutions, but not more, can be solved with the ruler and compasses. Indeed, it is further known that, for this purpose, one circle, traced once for all, would, as a matter of theory, be sufficient.

These considerations are perhaps sufficient to account for the great preponderance of

importance which attaches, in theoretical geometry, to the straight line and circle. On the other hand, the straight line and circle possess in common a property which is peculiar to them among all plane curves, and which is invaluable in all the practical applications of geometry. They are the only plane (or untwisted) lines any part of which can be applied exactly to any other. In very many mechanical arrangements this property is indispensable, and it is advantageous in all cases where accuracy of form is required, because it offers a simple means of verifying that accuracy has been obtained. There is but one twisted curve which has the same property, viz. the helix, or screw-curve, and it is precisely because any part of a screw-curve can be superposed upon any other that the screw and nut arrangement is possible, which renders this curve of so much use in mechanics.

But notwithstanding these prerogatives of the straight line and circle, the tracing of other curves is occasionally indispensable both in theoretical and practical geometry. It is by no means an easy matter to invent a good method of tracing a curve. Even when the theory of a curve is pretty well known, that theory may fail to suggest any mode of describing it mechanically; and not every mode which theory suggests can be made to work accurately in practice. Of all curves, after the circle, the ellipse would seem to be the simplest and easiest to draw, but some authorities on the subject recommend the draughtsman not to attempt a true ellipse, but to put together an imitation semi-ellipse, by means of six or seven arcs of circles with centres and radii appropriately chosen. It is said that such an imitation will impose even on a well-trained eye, although it is plain that, whereas the curvature of an ellipse changes continuously, the curvature of the imitation curve changes abruptly at the points of junction of the circular arcs.

The discovery of M. Peaucellier that a linkage can be constructed capable of describing a straight line may perhaps hereafter revolutionize this part of geometry. Already it is known that any conic section, and several of the more important curves of the third and fourth orders, can be described by linkage movements, or compound compasses, as M. Peaucellier has called them, not too complicated to work steadily. Theoretically, the results obtained are of a far wider scope, and Professor Sylvester has shown good reason for believing that every geometrical curve is capable of being described by a link-movement.

We may briefly mention some other mechanical arrangements which exist for describing certain plane curves.

(1) The ellipse can hardly be described with great accuracy by means of a thread attached to its two foci and stretched by a pencil, because of the extensibility of the thread. It can be described as an hypocycloid, as in the apparatus of Mr. A. E. Donkin; or we may use the old-fashioned elliptic trammels, or some other form of elliptic compasses of more recent invention. If none of these arrangements are at hand, perhaps the best way to obtain easily a really good ellipse is to take an oblique section of a carefully turned cylinder. All the three conic sections can be described by means of the conograph of Dr. Zmurko, of Lemberg.

(2) The cycloid is the curve traced by a point in the circumference of a circle rolling on a straight line; the epicycloid and hypocycloid are traced in the same way, only that the rolling circle rolls on the outside or inside of a circle instead of on a straight line. These definitions have suggested various modes of describing these curves mechanically; an interesting cycloidograph is exhibited by Dr. Zmurko.

(3) Mr. A. G. Donkin has constructed a beautiful apparatus for tracing harmonic curves. This machine will draw as many different forms of the curve

$$y = a \sin(mx + \alpha) + b \sin(nx + \beta)$$

as there are means for varying the constants a, b, m, n, α, β: the number of variations being practically unlimited, except in the case of m and n, which are the numbers of teeth in certain wheels of which only a limited number of changes can be provided. Thus the machine exhibits to the eye the effect of the composition of two harmonic curves of any different intensities and phases, and of different intervals.

(4) Several forms of spirals (or volutes) are of use in the arts, and appropriate modes of describing them have been given. Among these curves we may select for special notice the involute of the circle, which gives the proper form for the teeth of toothed wheels.

(5) We may refer, lastly, to the epicycloidal curves of Mr. Perigal; and to the beautiful diagrams, not properly epicycloidal, but of a more complicated type, obtained by his compound geometric chuck.

C.—Models of Figures in Space.

Models of certain geometrical solids, for example, of the so-called three round bodies—the sphere, the right cone, and the right cylinder; of the five regular solids—the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron; and of the various forms of prisms, pyramids, and parallelepipeds, have been long in use as educational appliances*. All these geometrical forms have been well known to mathematicians from very ancient times; surprising as it may appear that researches into the nature of such solids as the dodecahedron and icosahedron should have preceded other inquiries of a more elementary and at the same time of a more important character.

In very recent times the great development of geometrical speculation has led to the effort to reproduce in a material shape a much greater variety of geometrical conceptions. We proceed to enumerate some of the more important classes of objects of this kind.

(1) Models of surfaces of the second order.

Among these, two of the central surfaces, the ellipsoid and the hyperboloid of one sheet, have occupied a great place in the recent history of geometry. We may mention a few important theoretical considerations which models of these services have served to illustrate.

The curvature of a surface at any point is either such that the surface at that point is entirely concave on one side and convex on the other, or else it is such that the surface on each side alike is partly convex and partly concave. A sphere is an example of a surface of the first kind; the upper surface of a saddle may serve as an instance of curvature of the second kind. But the ellipsoid and the hyperboloid furnish perfect typical examples of these two kinds of curvature.

At any point of a surface there are always two directions at right angles to one another, along one of which the curvature is the greatest, and along the other the least. At special points, called umbilics, the greatest and least curvatures (and therefore all the curvatures) are equal to one another. The sphere has the peculiarity that every point on

* A complete collection of such models, made by Stronkoff, is exhibited by the Committee of the Russian Pedagogical Museum.

it is an umbilic; on the sphere, therefore, there are no directions of greatest and least curvature; but on every other surface two series of curves can be drawn, cutting one another at right angles, and indicating for each point of the surface the directions of the greatest and least curvature. These lines are called the lines of curvature; they are, it may be said, detected by the eye itself on any surface. The ellipsoid has four umbilics; if a thread, attached to any two of these, be strained along the surface by a moving pencil, the pencil will describe a line of curvature, so that when the ellipsoid has once been modelled, and its umbilics determined, it is easy to draw its lines of curvature with sufficient approximation. It was suggested long ago by Monge that a semi-ellipsoidal vault would form the most appropriate covering for an oval room, that the natural lines of the vaulting would be the lines of curvature, and that the umbilics would be the proper centres of illumination.

Every surface of the second order is either *umbilical* or *rectilineal; i.e.* it either possesses umbilics, or it is capable of being generated by the motion of a straight line: it cannot unite both properties. The ellipsoid, as we have just seen, is umbilical; the hyperboloid of one sheet is rectilineal; and the two systems of straight lines which lie upon this surface are very conspicuous in any model of it. It will be seen that one line of each system passes through each point on the surface; but that no two lines of the same system ever meet one another. The hyperbolic paraboloid, which may be regarded as a variety of the hyperboloid of one sheet, is characterized by similar properties: the Exhibition includes a beautiful series of models of these two surfaces made by M. Fabre de Lagrange, in 1872, for the South Kensington Museum. A series of cardboard models of surfaces of the second order (the cone, the cylinder, the ellipsoid, the hyperboloids, and both the paraboloids) is also exhibited by Professor Henrici, of University College, and by Professor Brill, of Munich. These models exhibit very clearly the circular sections of the various surfaces, having, indeed, been constructed by means of them; with the exception of the hyperbolic paraboloid, in which (as well as in the hyperbolic and parabolic cylinders) such sections (strictly speaking) do not exist.

(2) Models of surfaces of the third order.

Nothing like a complete series of models of surfaces of the third order has as yet been attempted; and indeed it may be said that our knowledge of these surfaces is still too imperfect to justify such an attempt. Nevertheless, models of certain of these surfaces have been made, and we may mention a few important properties which they serve to illustrate.

(*a*) As we have already said, the two curvatures of an ellipsoid at any point upon the surface are turned the same way, whereas the two curvatures of an hyperboloid are everywhere turned opposite ways. But, generally speaking, a curve surface consists of two regions, on one of which its two curvatures are turned the same way, and on the other opposite ways. These two regions are separated by a bounding line, technically called the parabolic curve. Surfaces of the third order offer typical examples of these general geometrical facts. But an example, though of a less perfect kind, is afforded by the figure of a ring (such as an anchor ring, or a wedding ring); the parabolic curve here consisting of either of the two circles on which the ring would rest if placed lying on a horizontal plane.

(*b*) Wherever the two curvatures of a surface are opposite to one another, there

always exist upon the surface two sets of lines along which the surface is inflected; *i.e.* at any point of the surface these two lines separate the directions in which the curvature is turned one way from those directions in which it is turned the other way. On the hyperboloid these 'curves of principal tangents,' as they are called by the German writers, are represented by the rectilinear generators; on surfaces of the third order we have the simplest examples of the general case in which they are not straight, but curved. The eye can just recognize these curves upon a surface, though of course in an approximate manner. The two which pass through any point are equally inclined to the lines of curvature at that point; they have a theoretical importance, even greater than that of the lines of curvature, because their definition is graphical, and not metrical.

(*c*) As a general rule, three lines of curvature pass through an umbilic. On the surfaces of the second order this property of the umbilics does not exist; and the first example of it occurs on surfaces of the third order. It is clearly seen in Professor Henrici's model of the cubic surface $xyz = k^3(x+y+z-1)^3$.

(*d*) A model of a surface of the third order ought to exhibit to the eye one of the most characteristic properties of these surfaces; *i.e.* that of containing a certain finite number of straight lines. The maximum number of these lines is twenty-seven; and the model exhibited by Dr. Wiener has this maximum number. A model, showing the distribution in space of the lines themselves, unaccompanied by the surface on which they lie, has been constructed by Professor Henrici.

In the model (*c*) to which we have just referred, the twenty-seven lines are all real, but are coincident in sets of nine; so that there appear to be three only.

(3) Models of ruled, or rectilinear surfaces.

Ruled, or rectilinear surfaces are those which may be generated by the continuous motion of a straight line in space, and which therefore may be said to consist of an infinite number of straight lines. Models of these surfaces exist in great variety, because they can be constructed with threads or wires, instead of being carved out of a solid material, or moulded out of a plastic one. They are, of course, only approximate or diagrammatic, the generating lines being represented in such number only as may suffice to convey to the eye an accurate impression of the form of the surface. Rectilinear surfaces are of two very different kinds, being termed skew, or developable, according as the successive generating lines intersect or not. Of skew surfaces the hyperboloid may be taken as an example: it may be defined as the surface generated by the motion of a straight line, which moves so as always to intersect three fixed straight lines which do not meet in space. In the hyperbolic paraboloid the generating line always intersects two fixed straight lines, and is always parallel to a fixed plane: this surface is the simplest example of the family of skew surfaces called conoids. The series of M. Fabre de Lagrange contains several models of conoidal surfaces; they are all generated by the motion of a straight line, which (1) continually remains parallel to a fixed plane, and which also (2) continually intersects a fixed straight line, and (3) some other fixed line in space. The surface of the thread of a square cut screw (or, more precisely, the surface formed by drawing lines from all the points of a screw curve perpendicular to the axis of the screw), affords a familiar instance of a conoidal surface. In the 'Skew Helixoid' of M. Fabre de Lagrange the lines drawn from the points of the screw curve to the axis are not perpendicular

to the axis, but are inclined to it at a constant angle. The recent progress of geometry has led to a careful study of the skew surfaces of the third, fourth, and fifth orders: of some of these, models have been already made; one of the cubic surface, called the cylindroid, is exhibited by Dr. Ball, Royal Astronomer of Ireland.

Developable surfaces form a class of surfaces entirely *sui generis*. They are called *developable* because, if such a service consist of a flexible and inextensible membrane, it can be 'developed,' or flattened out, upon a plane, without any tearing or crumpling. Cones and cylinders are the simplest instances of developable surfaces, but they are far from giving a complete idea of the general character of these formations. A more typical instance may be obtained by considering all the lines tangent to any twisted curve. It will be found that these tangents all form a developable surface, and that the twisted curve is an edge-curve or cuspidal line upon the surface. This edge-curve is characteristic of a developable: in the cone it dwindles to a point, and in the cylinder this point lies at an infinite distance. Since a developable surface, when made to roll on a plane or on another developable surface, has a *line* of contact with it, whereas in general two surfaces made to roll on one another have only a *point* of contact, it is easy to see that these surfaces are of great importance in the arts of construction. But, in a theoretical point of view, it is even more important to notice what is rendered visible by any model of such a surface:—(1) that at each point of the surface one of the two curvatures is infinite, the generating line being always one of the lines of curvature; (2) that, whereas in other surfaces each tangent plane has an infinite number of tangent planes lying near to it (because we can travel from any point on the surface to an adjacent point in an infinite number of different directions), in the case of developable surfaces this is not so, but each tangent plane is preceded and followed by only one other tangent plane; these planes, in fact, forming a singly, instead of a doubly, infinite series. It follows from this, that in the duality of space there answers to a developable surface a curve line, whereas to any surface not developable there answers a surface not developable. As an example, easy to understand and to remember, of a developable surface, we may mention the 'Developable Helixoid' of M. Fabre de Lagrange.

Most of the models of ruled surfaces to which we have referred are so arranged as to be capable of *deformation; i.e.* their shape can be changed by altering the form, or the relative position, of the director curves or straight lines, which serve to regulate the motion of the generating straight line. Thus the same model is rendered capable of assuming in succession the forms of several different surfaces; and the study of the transformations by which we pass from one of these forms to another is of great interest and importance. The cardboard models of Professor Henrici admit of deformation in a similar manner, since the angle at which the two planes of circular section are inclined to one another may be changed.

(4) Models of certain Special Surfaces.

(*a*) The Surfaces of Plücker.

In very recent times the properties of systems of lines in space have been studied with great ardour, chiefly by the German geometers. The consideration of such systems is suggested by many geometrical problems (for example, the study of the straight lines which cut a given surface at right angles, the so-called normals of the surface), but

it is also forced upon our notice in all those optical researches in which the properties of systems of rays form the subject of inquiry. A further interest attaches to the study of systems of straight lines in space, inasmuch as an examination of their properties would seem to be an indispensable preliminary to the more general inquiry into the properties of systems of curves; such systems of curves meeting us at every turn in Physics; as stream lines in Hydrodynamics, as lines of flow in the theory of the conduction of Heat, as lines of force in the theories of Electricity and Magnetism.

The present Collection contains a beautiful series of models of a class of surfaces which are found to play an important part in the theory of rectilinear systems. A certain historical interest attaches to these models; they are copies of those exhibited in 1866, at the Nottingham Meeting of the British Association, by the celebrated mathematician, Julius Plücker, to whom, more than to any other single person, we are indebted for our knowledge of the geometry of systems of lines. They were made by Epkens, of Bonn, and were presented by Dr. Hirst to the London Mathematical Society. Professor Hennessy, of the Royal College of Science for Ireland, also exhibits a series of models illustrative of the researches of Plücker.

(*b*) The Wave Surface.

The importance of this surface in the undulatory theory of Light forms its principal claim to attention. But, even apart from any physical interpretation, its geometrical properties entitle it to a place in a collection of geometrical models. It furnishes us with an instance of a closed surface of two sheets, one of them lying inside the other; it is an *apsidal* surface, and its reciprocal surface is a surface of the same nature as itself; finally, it offers typical examples of the singularity termed a conical node: and of the correlative singularity of a tangent plane touching a surface along a conic section (in the case of the wave surface the conic section is a circle). The geometrical study of these singularities led Sir William Rowan Hamilton to his celebrated discovery of the optical phenomena of external and internal conical refraction.

(*c*) The Surface of Steiner.

Professor Cayley exhibits a rough model of this surface, which has attracted considerable attention among mathematicians, from its being the polar reciprocal of the cubic surface with four conical nodes, and from its having the property that every one of its tangent planes cuts it in two conic sections.

(*d*) The Amphigenous Surface of Professor Sylvester.

This surface is of great importance in the theory of equations of the fifth order. A model of it has been prepared by Professor Henrici.

(*e*) Surfaces of constant curvature.

The total curvature of a surface at any point is the product of its two principal curvatures at that point; and the total curvature is positive or negative, according as the two principal curvatures are in the same direction or in opposite directions. It is an important geometrical theorem, that if two inextensible and flexible surfaces have at corresponding points the same total curvature, either of them can be 'developed' upon the other without tearing or crumpling. Thus every surface of constant positive curvature can be developed upon a sphere. Surfaces of constant negative curvature

cannot, of course, be developed upon a sphere; but they possess a great theoretical interest of their own, since it has been ascertained that the geometry of figures traced upon these surfaces is precisely that which the geometry of Euclid would become, if we were to erase from it the assumption known as the eleventh axiom. Models of both these classes of surfaces are contributed by Professor Henrici.

(5) Crystallographic Models.

These are either models of those geometric polyhedra (solids bounded by plane faces) which actually occur in nature; or they are intended to serve as illustrations of the theory of crystallography, and to exhibit the relations of the plane faces of a crystal to its crystallographic axis.

D.—Representations of Figures in Space by means of Drawings on a Plane.

We have lastly to refer to modes of representation by drawings on a plane of figures in space. We have here at our disposal the ordinary methods of perspective; but the practical use of these is too dependent on the hand and eye of the artist to satisfy the requirements of geometrical rigour. The method known as 'Descriptive Geometry' is therefore preferred for the purposes of geometrical drawing. This method consists in representing an object in space, by means of its orthogonal projections on two planes at right angles to one another. It therefore can hardly be said to be of recent date, as, in principle, it comes to representing a figure in space by a plan and elevation. But to Monge is perhaps due the idea of placing the plan and elevation on one sheet, and treating them (for the purpose of geometrical construction) as one plane figure; and to him is certainly due the development of the principles of the method into a complete and scientific system. The volumes of plates which accompany works on descriptive geometry (of which, since the time of Monge, there have been a considerable number) offer copious and varied illustrations of the method. But models have also been constructed, which serve to exhibit to the student the relations of the object represented to its two projections, and of these to one another. We may refer, among others, to the diagrams and models exhibited by Professor Franz Tilser, of Prague, by Professor O. Reynolds, of the Owens College, Manchester, by the Committee of the Russian Pædagogical Museum, and by Professor Pigot, of the Royal College of Science for Ireland.

The *épures* of descriptive geometry, however accurate, and however useful for constructive purposes, do not offer much assistance to the imagination in conceiving complicated geometrical figures. Such assistance, however, is abundantly afforded by stereoscopic representations; and it is earnestly to be hoped that the applications of stereoscopy to geometry may hereafter receive a much greater development than has been the case as yet. Any polyhedron can (as is well known) be represented with extraordinary beauty by stereoscopy; the edges only of the polyhedron being drawn on the two faces of the stereoscopic slide. It ought in the same way to be possible to represent any twisted curve; and further, any developable surface, by representing first the twisted curve which forms its cuspidal line, and then a sufficient number of the straight lines tangent to that curve. Similarly to represent a skew rectilineal surface, it would be sufficient to exhibit stereoscopically a certain number of its generating lines. Surfaces which are not rectilinear

could, theoretically at least, be represented by a sufficient number of their lines of curvature, or by means of their curves of principal tangents, when those curves exist. It must be admitted, however, that the accurate tracing of the plane diagrams which would be required for such representations would be subject to very serious practical difficulties, which it would be desirable to avoid by using special methods adapted to each particular surface; for example, in the case of the ellipsoid and the other umbilical surfaces of the second order, by employing the two systems of circular sections.

IV.

INTRODUCTION TO THE MATHEMATICAL PAPERS OF WILLIAM KINGDON CLIFFORD.

[London, 1882, pp. xxxiii–xlviii.]

It will be generally admitted that the publication in a collected form of the works of the eminent men, who have moulded the mathematical sciences into their present form, has become little less than a necessity to those who desire to follow in their footsteps, and to advance, if possible, beyond the limits attained by them. And mathematicians will gratefully acknowledge that no inconsiderable progress has already been made towards satisfying this requirement. To the Academy of Sciences of Paris we are indebted for magnificent editions of the complete works of Laplace and Lagrange; the government of Norway has given to the world the collected memoirs of Abel; the Academy of Goettingen has fulfilled the same duty towards the great names of Gauss and Riemann; and the Academy of Berlin has followed the example by undertaking editions of the works of Steiner and Jacobi. In our own country we have collected the works of Green, of Mac Cullagh, of Gregory, of Leslie Ellis, and of Macquorn Rankine; not to mention the volumes of reprinted memoirs which we owe to living writers; for example, to Sir William Thomson and to Professor Stokes. Such collections, we may hope, will form an increasing portion of every scientific Library. At the present time the results of mathematical research almost always appear in the Transactions of Societies, or in periodicals specially devoted to the mathematical writings; the contents even of the most original treatises being generally anticipated by their authors in memoirs which are often not wholly superseded by the works themselves. But the number of the periodical repositories of mathematical literature has become so great, that papers consigned to them, although preserved, as we may hope, for all time, are in imminent danger of passing out of sight within a few years after their first appearance. They are preserved from destruction, but not from oblivion; they share the fate of manuscripts hidden in the archives of some great library from which it is in itself a work of research to disinter them. This 'mislaying,' if it may be so termed, of important memoirs is not only a loss to the history of science, but interferes seriously with the discovery of new knowledge. For notwithstanding the ardour with which mathematical investigation is at present pursued in every direction, a much longer time than is perhaps sometimes supposed elapses before a mathematical work of genuine originality, be it a brief note, or an elaborate treatise, becomes antiquated. It would be out of place in this connexion to mention the Principia of Newton, which stands apart by itself, and of which not only the methods and results, but even the very words have become the common

property of all men of science. Nor need we even refer to works which have marked the beginning of a new epoch in their respective departments, such as the Mécanique Céleste, the Mécanique Analytique, the Disquisitiones Arithmeticæ, the Traité Analytique de la Chaleur, the Fundamenta Nova, or the Systematische Entwickelung der Geometrischen Gestalten—the freshness of which time has hardly impaired, while the superstructures which have been based upon them have added incalculably to their importance. But leaving out of count these and other great classics of the science, the trains of thought hidden in the opuscula of Euler, of Lagrange, of Gauss, of Poisson, of Cauchy, of Abel, and Jacobi, are still unexhausted; and, far from having lost their value by the lapse of time, have in many cases acquired an increased suggestiveness from the light which the more extended knowledge of recent times has thrown upon them. Such a prospect of future and long-continued usefulness, we may venture to hope, awaits many of Clifford's memoirs; and, even more than the immediate interest attaching to them, justifies, if any justification be needed, their appearance in their present collected form.

It might be interesting to inquire why it is that mathematical writings retain a scientific (as opposed to a merely historical) value for a longer time than memoirs recording researches in the sciences of experiment and observation. Among many partial answers which might be given to this question, one is suggested by the character of many of Clifford's papers, and has its foundation in the nature of the subjects with which they deal. Speculation in pure mathematics resembles metaphysical speculation in this, that the whole universe of thought to which it refers is so closely interdependent, that a clear-sighted and powerful thinker cannot fix his mental vision (however keen his effort after concentration may be) on any one region in it, without catching glimpses of something that lies beyond, and without discerning, more or less dimly, new relations to be examined, and new lines of research, which may perhaps have no immediate relevancy to the particular inquiry in which he is engaged. And these glimpses, if recorded, or even if only half unconsciously indicated, in the account which he afterwards gives of his work, are not unlikely to suggest a wholly new departure to some kindred spirit in a future time. On this ground, more strongly perhaps than on any other, we may venture to commend the present volume to the rising generation of English mathematicians. The collection includes papers—some of them youthful efforts—some suggested, one might say casually, by the researches of scientific friends—which relate to special problems, and which nevertheless would be sufficient of themselves to establish a considerable mathematical reputation. There are others, again, the work of a maturer time, and planned with a wider scope, which are models of artistic perfection, in respect both of the clearness and depth of the thought, and of the manner of its presentation. Lastly, besides these finished pieces there are others, rough-hewn and imperfect in execution, but conveying a still stronger impression of the fertility of the invention; and of the far-reaching power of mental vision, with which Clifford was endowed. Some of these fragmentary records are full of great ideas, shadowed forth in outlines, not always free from indistinctness, but always suggesting long vistas of future discovery, the path to which seems for the moment to lie clear before his mind. Their very incompleteness reminds us how much the world has lost by losing him; and brings home to us the melancholy feeling that, however highly we may estimate the work

which he actually accomplished during his brief lifetime, he must nevertheless be counted among 'the inheritors of unfulfilled renown.'

But if the republication of Clifford's papers stands in no need of any justification, some apology is wanted for an Introduction which can offer but little interest to those who do not intend to study the volume itself, and which to those who do, must seem at the best superfluous. Perhaps, however, even in these days of increasing specialization, there may still be found an intermediate class of readers, who are not mathematicians by profession, who nevertheless do not regard analysis and geometry as volumes sealed except to the initiated few, but as belonging, in their results at least, to the whole world of science. Some persons, one is willing to believe, partly from a recollection of their own early studies, and partly from a general sympathy with all branches of intellectual activity, are disposed to follow with an appreciative, or at least an indulgent curiosity, the exposition of new mathematical ideas. For such friendly readers these pages are intended; and it would be strange if the class of persons, among whom they are to be found, has not been considerably increased by the admirable lectures in which Clifford has himself analyzed, in popular phraseology but with the utmost scientific precision, the fundamental principles of geometry and arithmetic, as they appear in the 'fierce light' which has been turned upon them by a controversy in which both metaphysicians and mathematicians have taken part.

All then that can with any propriety be attempted here is, in the first place, to characterize some one or more of the principal trains of thought which seem to have exercised an abiding influence on Clifford's mind; and then to classify his memoirs in a few main groups, pointing out the central ideas of each group, and showing, as far as possible, the interdependence of these ideas upon one another.

Clifford was above all and before all a geometer. Nine-tenths, and more, of the contents of this volume, including nearly all the systematic researches recorded in it, are geometrical. It is true that in the treatment of geometrical questions he shows a marked preference for symbolical methods; and, as might be expected, a marvellous command over analytical expression. It may even be true that the limitations involved in a scrupulous adherence to the methods of pure geometry would have been distasteful to him. Of his skill in the use of these special methods the 'Problems and Solutions' so liberally contributed by him to the *Educational Times* afford abundant proof. But among his more elaborate papers there is perhaps but one, the 'Geometry on an Ellipsoid,' which would satisfy purists of the school of Poncelet and Chasles, as being wholly free from the contamination of analytical methods; and even in this beautiful application of the method of the stereographic projection—in the generalized sense in which that method is used in modern pure geometry—the 'imaginary circle at infinity' occurs in the first sentence. But, whatever the method employed—and in variety of method Clifford takes an evident pleasure—the properties of space remain the perpetual subject of his contemplation. Purely analytical researches, undertaken without any impulse from or reference to geometry, are few and far between. For even the Elliptic and Abelian functions were approached by Clifford from the side of Geometry. His early note 'On some Porismatic Problems' relating to the theorem of Poncelet, which asserts that 'given two conics, a polygon of a given number of sides, all whose vertices shall lie on one of them, and all whose sides shall touch

the other, can either not be drawn at all, or else can be drawn in an infinite number of different ways' led him to the study of the connexion, established by Professor Cayley, between this theorem and the addition of elliptic functions of the first species; and thus to the discovery of a geometrical theory of the transformation of the elliptic functions, which forms the subject of one of his most brilliant investigations (XXII). But in his further prosecution of the subject the elliptic functions again disappear from view, and he returns to the geometrical doctrine of correspondences, and to the theory of the polyhedra, of which the faces osculate a skew cubic curve.

In like manner it would appear that he was attracted to the consideration of the Abelian Integrals by their relation to another and widely different part of geometry, the Geometry of Situation, as it has been termed. His memoir on the 'Canonical Dissection of a Riemann Surface,' founded on the researches of Clebsch and Lüroth, contains the simplest account which has yet been given of this important chapter of a great theory; and the reduction of a Riemann surface to the surface of a solid having a certain number of holes through it, presents to the mind what is perhaps the clearest image which it is possible to obtain of the space of two dimensions upon which a many-valued algebraical function can be mapped with the same distinctness with which a one-valued function can be mapped on a plane. But the study of the Abelian Integrals—however geometrical may have been the form in which he first envisaged this theory—led him by an inevitable sequence of ideas to the Theta functions, which form the indispensable basis for a study of the relations subsisting between the upper limits of a system of algebraical integrals, and the values of the integrals themselves. His posthumous and unfinished memoirs on the 'Double and Multiple Theta functions' (XXXVIII, XXXIX, and XL) form a not unimportant contribution to pure analysis; and, though now published long after their proper date, are, it may be hoped, neither too late, nor too incomplete, to exercise some influence on the development of this rapidly growing theory. The 'Algebraical Introduction to Elliptic functions,' which is in fact a treatise on the single Theta functions, probably took its rise in connexion with these ulterior researches; Clifford desiring to obtain a complete command over the manipulation of these series in the simplest cases, before proceeding to apply them to the general problem of the inversion of algebraical integrals.

Enough has been said to show that Clifford's predilection for geometry lay deep. But to this his favourite science he attributed the widest imaginable scope, and at times regarded it as so co-extensive with the whole domain of nature. He was a metaphysician (though he would only have accepted the name subject to an interpretation) as well as a mathematician; and geometry was to him an important factor in the problem of 'solving the universe.' Thus he was a geometer of a type peculiarly his own; and his dealings with the science were characterized by an amount of scepticism and an amount of faith which one would hardly expect to find combined in a mathematician. He had early read and translated Riemann's celebrated discourse (IX) 'On the Hypotheses which lie at the Basis of Geometry,' and had imbibed the views set forth in it as a part of his intellectual nature. Some men who have an ardent love for new knowledge find it difficult to maintain an unflagging interest in geometry, because they regard it as a purely deductive science, of which the first principles (axioms, postulates, and definitions), whether derived from experience or not, are unquestionable,

and contain implicitly in themselves all possible propositions concerning space. Thus the unknown, or at least the unforeseen, seems to be excluded from geometry, because whatever may be found out hereafter must be latent in what is already known. But upon the view put forward by Riemann and adopted by Clifford, the essential properties of space have to be regarded as things still unknown, which we may one day hope to find out by closer observation and more patient reflection, and not as axioms to be accepted on the authority of universal experience, or of the inner consciousness.

These speculations had so much influence on a great part of Clifford's work that it may not be out of place to pursue the subject a little further. In this lecture, 'On the Postulates of the Science of Space,' he has stated his own views on the question with singular clearness and brilliancy; and the pages in which he has expressed them are likely to be remembered, as marking an important moment in the controversy concerning the nature of space and the origin of our knowledge of it, which is likely to last as long as metaphysical inquiries have any interest for mankind. In this lecture he enumerates four fundamental postulates on which the ordinary conception of space is founded, (1) its continuity, (2) its flatness in its smallest parts, (3) its similarity to itself at every point, or, which is the same thing, the possibility of the existence of the same figure in any two different places, (4) the possibility of the existence of figures similar to one another, but on different scales of magnitude. The second of these postulates requires some comment to make it intelligible. Perhaps the simplest account that can be given of a space which is flat in its smallest parts is, that if anywhere in it we take three points very near to one another and join them by the shortest lines that can possibly be drawn, the triangular figure so formed will lie very nearly in a plane; the mathematical equivalent of this statement being that the square of the distance between any given point and any other infinitely near to it can always be expressed as a homogeneous quadratic form, in which the indeterminates are the infinitesimal differences between the co-ordinates of the two points, and the coefficients are functions of the co-ordinates of the given point. If we go further and join one of the infinitely near points by the shortest lines possible to every point on the line already joining the other two, the assemblage of these lines will form a triangular surface-element: if this surface-element is absolutely plane, whatever be the three infinitely near points which we have taken, the space is flat; if the surface-element has a finite curvature, the space, while retaining the property of elementary flatness, is said to have curvature; and this curvature is measured, for the surface direction determined by three points, by the curvature of the surface-element which we have constructed.

As to the first postulate, Clifford indicates his readiness to adopt either of the two opposite hypotheses that space is continuous or that it is discontinuous, while admitting fully that no phenomena have yet been observed which point to its discontinuity.

Of the second postulate, in this respect following Riemann, he speaks in the same general terms; we must not shrink from rejecting it, if its rejection should be found to assist us in the explanation of natural phenomena. The postulate is not inconsistent with a hypothesis which at one time was a great favourite with him, and which he has described in a remarkable communication (V) presented to the Cambridge Philosophical Society in 1870. In this brief note comprised within a single page, he appears

to adopt the hypothesis (for his language on the point is not quite free from ambiguity), that space has everywhere a finite curvature, but that this curvature is continually changing, and that all the phenomena of the universe may possibly consist in changes of the curvature of space. A finite curvature, it will be remembered, is consistent with, and indeed implies, elementary flatness. Unfortunately, Clifford, though in earlier days he was fond of discussing this theory, no doubt as one possible mode of 'solving the universe,' has left no memoranda relating to it, perhaps because the efforts which he made to work it out in detail led him to no satisfactory conclusion. In the note of 1870 he speaks of it with a confidence which must not be taken too literally. He would probably have allowed that Lord Bacon's criticism on Gilbert, 'postquam in contemplationibus magnetis se laboriosissime exercuisset, confinxit statim philosophiam consentaneam rei apud ipsum præpollenti,' admitted of an application, *mutatis mutandis*, to his own effort to resolve all philosophy into geometry; though he would no doubt have maintained with the utmost depth of conviction that, for aught we know to the contrary, the properties of space may change with time.

But whatever importance he may have temporarily attached to the opinion that space may not be independent of time, this idea has left no other perceptible traces in his mathematical writings. Very different is the case with another hypothesis as to the nature of space, which is somewhat less widely divergent from ordinary conceptions, and to which Clifford appears at all times to have turned with peculiar favour.

This hypothesis admits the first three of the postulates enumerated above, as expressing true properties of space, but rejects the fourth, substituting for it the new postulate that space has a finite but very small curvature, which is approximately the same for any two points, and for any two surface directions at the same point. Admitting this postulate we find ourselves in the presence of two alternatives, between which we have to choose. For we may imagine either that the curvature of all surface-elements, constructed in the manner above described, has the same positive value, or that it has the same negative value; understanding by a positive curvature a curvature such as that of the outer portion of the surface of an anchor-ring, where the tangent plane at any point just meets the surface and does not cut it; and by a negative curvature a curvature such as that of the inner portion of the same surface, where the tangent plane cuts the surface at the point of contact. To the hypothesis that space has a constant negative curvature considerable historical interest attaches. For this hypothesis was first arrived at, not by following out such general views as those indicated by Riemann, but in a more elementary manner. The celebrated twelfth axiom, as is well known, is the basis of Euclid's theory of parallel lines; and the assertion made in it is in fact equivalent to an assumption of the fundamental proposition of plane geometry, that the three angles of a triangle are equal to two right angles. It is now universally allowed that all efforts to demonstrate Euclid's axiom have failed; but the Russian mathematician, Lobatchewsky, appears to have been the first person to whom the idea occurred of dispensing with the axiom altogether, and trying to see what would become of geometry without it. The idea was obvious, but it was also profound; and Lobatchewsky was rewarded by the discovery that it is possible to construct a consistent and complete system of geometry upon the hypothesis that the three angles of a triangle are less than two right angles. Till the discovery of

Lobatchewsky, the only substantial addition that had been made to Euclid's theory of parallel lines was a demonstration by Legendre, that the angles of a triangle cannot be greater than two right angles. As a matter of fact the demonstration of Legendre depends on the assumption that space is infinite; an assumption which, from the point of view taken by Riemann, cannot be regarded as justified by experience: but the considerations upon which the demonstration rests decided Lobatchewsky, as between the two alternative hypotheses that the angles of a triangle are less, and that they are greater, than two right angles, to adopt the former. This was in effect to adopt the hypothesis (though it does not appear to have occurred to Lobatchewsky in that light) that a plane has negative curvature. It was reserved for an Italian mathematician, Beltrami, to show that the plane geometry of Lobatchewsky is identical with the geometry of a pseudo-spherical surface, *i. e.* of a surface of constant negative curvature.

What Clifford thought of the philosophical importance of the work of Lobatchewsky the following quotation may serve to show:—

'Each of them [Copernicus and Lobatchewsky] has brought about a revolution in scientific ideas so great that it can be only compared with that wrought by the other. And the reason of the transcendent importance of these two changes is that they are changes in the conception of the Cosmos. Before the time of Copernicus men knew all about the universe. They would tell you in the schools, pat off by heart, all that it was, and what it had been, and what it would be ... In any case the universe was a known thing. Now the enormous effect of the Copernican system, and of the astronomical discoveries that have followed it, is that, in place of this knowledge of a little, which was called knowledge of the universe, of Eternity and Immensity, we have now got knowledge of a great deal more; but we only call it the knowledge of Here and Now... This then was the change effected by Copernicus in the idea of the universe. But there was left another to be made. For the laws of space and motion implied an infinite space and an infinite duration, about whose properties as space and time everything was accurately known. The very constitution of those parts of it which are at an infinite distance from us, 'geometry upon the plane at infinity,' is just as well known, if the Euclidean assumptions are true, as the geometry of any portion of this room ..., so that here we have real knowledge of something at least that concerns the Cosmos; something that is true of the Immensities and the Eternities. That something Lobatchewsky and his successors have taken away. The geometer of to-day knows nothing about the nature of actually existing space at an infinite distance; he knows nothing about the properties of this present space in a past or future eternity. He knows, indeed, that the laws assumed by Euclid are true with an accuracy that no direct experiment can approach ...; but he knew this as of Here and Now; beyond his range is a There and Then, of which he knows nothing at present but may ultimately come to know more. So, you see, there is a real parallel between the work of Copernicus and his successors on the one hand, and the work of Lobatchewsky and his successors on the other.'—*Lectures and Essays*, vol. I, pp. 298–300.

But in spite of this eulogium, the conception of space which has left the deepest traces in Clifford's writing is not that of Lobatchewsky, but that founded on the alternative hypothesis (rejected by the Russian geometer) of a constant positive curvature. This conception lies at the bottom of Clifford's theory of biquaternions, to which he

devoted much continuous thought, and which was the origin of his researches into the classification of geometric algebras. A space of constant positive curvature is most easily represented to the mathematician (in the absence of any possibility of imaging it to the mind) as the locus of an equation of the form

$$x^2+y^2+z^2+w^2 = \text{constant}$$

in a flat space of four dimensions in which *xyzw* are rectangular co-ordinates. It is related to the two dimensional surfaces of a sphere, just as in ordinary geometry space of three dimensions is related to a plane surface. The following description of a space of this kind is taken from the lecture 'On the Postulates of the science of Space.' It can hardly be necessary to point out, that in the last sentence Clifford is half laughing at himself.

'I cannot perhaps do better than conclude by describing to you as well as I can what is the nature of things on the supposition that the curvature of all space is uniform and positive.

'In this case the universe, as known, becomes again a valid conception; for the extent of space is a given number of cubic miles. And this comes about in a curious way. If you were to start in any direction whatever, and move in that direction in a perfect straight line according to the definition of Leibnitz; after travelling a most prodigious distance, to which parallactic unit 200,000 times the diameter of the earth's orbit would be only a few steps, you would arrive at—this place... Upon this supposition of a positive curvature the whole of geometry is far more complete and interesting, the principle of duality, instead of half breaking down over metrical relations, applies to all propositions without exception. In fact I do not mind confessing that I personally have often found relief from the dreary infinities of homaloidal space in the consoling hope that, after all, this other may be the true state of things.'—*Lectures and Essays*, vol. I, pp. 322–3.

A third line of thought, different from those followed by Lobatchewsky and by Riemann, had no doubt a large share in determining Clifford to regard the hypothesis of constant positive curvature with special favour. One of the earliest geometrical inquiries of wide scope which interested him was the connexion between the descriptive and metrical properties of figures. In two unfinished memoirs, 'On Analytical Metrics' (XI) and 'On the Theory of Distances' (XVI), he applied himself to work out the conception, which he justly attributed to Poncelet, that the metrical properties of any figure are in reality descriptive properties of the figure considered in relation to certain fixed geometrical elements, which Professor Cayley has termed the Absolute. In the ordinary geometry of a plane, the Absolute consists of two fixed imaginary points, and of the real straight line containing them [the imaginary circular points, and the straight line, at an infinite distance]. For these two imaginary points, in the geometry of a spherical surface, we have to substitute the imaginary circle in which the sphere is cut by the plane at an infinite distance. In ordinary space of three dimensions the Absolute is the same imaginary circle and the plane at an infinite distance in which it lies. Professor Cayley in his celebrated 'Sixth Memoir of Quantics,' generalized this conception by substituting any quadric whatever for the imaginary circle at an infinite distance (which may be regarded as a quadric surface of which one dimension

has vanished). The effect of this substitution is to change the metric properties of space, the nature of the change depending on the nature of the quadric chosen as the Absolute. If we wish the space which we thus bring under contemplation to possess one of the most obvious properties which we know by experience to characterize the space in which we live (viz. that the rotation round a fixed axis, which bring a body from any given position back into the same position again, is a finite and not an infinite operation), our choice of the form of the Absolute is limited to three hypotheses, (1) the Absolute is an imaginary quadric, (2) the Absolute is a real umbilical quadric (i. e. a quadric not having real right lines on it) and the space considered is internal to the quadric, (3) the Absolute is an imaginary quadric which has degenerated into a conic section by losing one of its dimensions. Of these three hypotheses the last corresponds to the ordinary conception of space: the spaces characterized by the suppositions (1) and (2) have been termed elliptic and hyperbolic respectively by Professor Klein, who succeeded in showing that in each of them the curvature is constant, being positive in the elliptic, and negative in the hyperbolic space. Thus the geometry of Lobatchewsky is the geometry of hyperbolic space; and Professor Klein's discovery of the identity of the two has thrown a wholly new light upon the researches of the former geometer. Of Clifford's study of the details of the system of Lobatchewsky only one brief note is preserved (Appendix, p. 531). Indeed he seems to have quickly abandoned hyperbolic for elliptic geometry, influenced no doubt by the reason indicated in the passage which we have quoted—the perfect duality of the properties of elliptic space.

In the geometry of Lobatchewsky every straight line has two real points on it at an infinite distance, viz. the two real points in which it intersects the Absolute. Again, among the planes which pass through a given straight line there are two which belong to the Absolute, and which therefore are to be regarded as planes at an infinite distance. But these two planes are imaginary, being the two planes which can be drawn through the given line to touch the Absolute. Thus in the hyperbolic geometry there is no perfect duality, because when we compare the points which lie along a line, and the planes which pass through it, the absolute elements are real in the one case, and imaginary in the other: in fact, the space which is the dual correlative of an hyperbolic space is not itself a similar space, but is analogous to the space outside the Absolute of the hyperbolic space. On the other hand, in elliptic geometry all the elements of the Absolute, whether points or planes, are imaginary, and the duality is as perfect as it is on the surface of a sphere. It follows at the same time that all distances as well as all rotations are finite, and that a point moving on a straight line (or more properly on a shortest line) will come round after a finite journey to the point from which it set out, just as a plane revolving round a straight line returns after describing a finite angle of 360° to its original position.

[*The rest of the introduction*, pp. xlviii—lxx, *which relates to the separate memoirs, is not reprinted.*]

THE END.

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